Éléments de géométrie algébrique

A. Grothendieck and J. Dieudonné
Publications mathématiques de l'I.H.É.S

Contributors
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What this is

This whole chapter is written by the translators.

This is a community translation of Alexander Grothendieck’s and Jean Dieudonné’s *Éléments de géométrie algébrique* (EGA). As it is a work in progress by multiple people, there will probably be a few mistakes—if you spot any then please do let us know. To contribute, please visit https://github.com/ryankeleti/ega.

On est désolés, Grothendieck.

IN DEFENSE OF A TRANSLATION

From Wikipedia:

In January 2010, Grothendieck wrote the letter “Déclaration d’intention de non-publication” to Luc Illusie, claiming that all materials published in his absence have been published without his permission. He asks that none of his work be reproduced in whole or in part and that copies of this work be removed from libraries.

It is a matter of often heated contention as to whether or not any translation of Grothendieck’s work should take place, given his extremely explicit views on the matter. By no means do we mean to argue that somehow Grothendieck’s wishes should be invalidated or ignored, nor do we wish to somehow twist his earlier words around in order to justify what we have done: we fully accept that he himself would probably have branded this project “an abomination”. With this in mind, it remains to explain why we have gone ahead anyway.

First, and possibly foremost, it does not make sense (to us) for an individual to own the rights to knowledge. Arguments can be made about how the EGA is the product of years and years of intense work by Grothendieck, and so this is something that he ‘owns’ and has full control over. Indeed, it is true that there are almost innumerable many sentences in these works that only Grothendieck himself could have engineered, but, in translation, we have never improved anything, but only (regrettably, but almost certainly) worsened. The work in these pages is that of Grothendieck; we have been not much more than typesetters and eager readers. However, there is some important point to be made about the fact that Grothendieck collaborated and worked with many other incredibly proficient mathematicians during the writing of this treatise; although it is impossible to pinpoint which parts exactly others may have contributed (and by no means do we wish to imply that any of this work is derivative or fraudulent in any way whatsoever—EGA was written by Grothendieck) it seems fair that, in some amount, there are bits of the EGAs that ‘belong’ to a broader collection of minds.

It is a very good idea here to repeat the oft-quoted aphorism: “the work here is not ours, but any mistakes are”—it is very understandable for an author to not want their name on something that they have not themselves written, or, at the very least, read. This may be, in part, a reason for Grothendieck’s wishes, but that is pure speculation. Even so, we include this above disclaimer.

1 https://github.com/ryankeleti/ega/issues
2 https://en.wikipedia.org/wiki/Alexander_Grothendieck#Retirement_into_reclusion_and_death
Secondly, then, we note that the French version of EGA is still entirely readily accessible. Anybody reading these copies who is not a native French speaker, will probably be translating at least some part of EGA into English in their head, or into their notebooks, as they read. This document is just the product of a few people doing exactly that, but then passing on their efforts to make things just that little bit easier for anyone else who follows.

Lastly, to quote another adage, “the guilty person is often the loudest”. If it seems like we are over-eager to defend ourselves because we know that we are somehow in the wrong, it is because we are, at least partially. Working on this translation has meant going against Grothendieck’s explicit requests, and for that we are sorry. We only hope that the freedom of knowledge is an excusable defense.

**Notes from the Translators**

Grothendieck’s writing style in EGA is quite particular, most notably for its long sentence structure. As translators, we have tried to give the best possible approximation of this style in English, resisting the temptation to “streamline” things in places where the language is more dense than usual.

***

Any translations about which we are not entirely sure will be marked with a (?)

***

Whenever a note is made by the translators, it will be prefaced by “[Trans.]”.

***

Along the margins we have provided the page numbers corresponding to the original text (of the first edition), as published by *Publications mathématiques de l’I.H.É.S.*, where the EGA were published as the following volumes:

- EGA I (*tome 4, 1960*)
- EGA II (*tome 8, 1961*)
- EGA III, part 1 (*tome 11, 1961*)
- EGA III, part 2 (*tome 17, 1963*)
- EGA IV, part 1 (*tome 20, 1964*)
- EGA IV, part 2 (*tome 24, 1965*)
- EGA IV, part 3 (*tome 28, 1966*)
- EGA IV, part 4 (*tome 32, 1967*).

Due to EGA being a collection of volumes (one non-preliminary chapter, or part of a chapter, per volume), the page numbers reset at every new chapter. In addition, the preliminary section is stretched out over multiple volumes. To combat this, we label the pages as

\[ X | p, \]

referring to Chapter X, page p. For EGA III and IV, which are split across multiple chapters, we label the pages as

\[ X-n | p, \]

referring to Chapter X, part n, page p. In the case of the preliminaries (which are often collectively referred to as EGA 0), the preliminaries from volume Y are denoted as \( 0_Y \).

***

Later volumes (EGA II, III, and IV) include errata for earlier chapters. Where possible, we have used these to ‘update’ our translation, and entirely replace whatever mistakes might have been in the original copies of EGA I and II. If the change is minor (e.g. ‘intersection’ replacing ‘inter-section’) then we will not mention it; if it is anything more fundamental (e.g. \( X' \) replacing \( X \)) then we will include some margin note on the relevant line detailing the location of the erratum (e.g. **Err**II to denote that the correction is listed in the Errata section of EGA II).

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5PDFs of which can be found online, hosted by the *Grothendieck circle*.  

EGA IV sections

In EGA IV-1, the summary included a tentative list of section that EGA IV would contain. As EGA IV was written, §5, §7, and the §§11–21 would contain different sections than initially envisaged. We include the original listing here:

§5. Dimension and depth for preschemes.
§7. Application to the relations between a local Noetherian ring and its completion. Excellent rings.
§11. Topological properties of finitely presented flat morphisms. Local flatness criteria.
§12. Study of fibres of finitely presented flat morphisms.
§15. Study of fibres of a universally open morphism.
§17. Smooth morphisms, unramified morphisms, and étale morphisms.
§19. Regular immersions and transversely regular immersions.
§20. Hyperplane sections; generic projections.

MATHEMATICAL WARNINGS

EGA uses prescheme for what is now usually called a scheme, and scheme for what is now usually called a separated scheme.
In some cases, we (the translators) have changed “→” to “↪” where appropriate.
Introduction

To Oscar Zariski and André Weil.

This memoir, and the many others will undoubtedly follow, are intended to form a treatise on the foundations of algebraic geometry. They do not, in principle, presume any particular knowledge of the subject, and it has even been recognised that such knowledge, despite its obvious advantages, could sometimes (because of the much-too-narrow interpretation—through the birational point of view—that it usually implies) be a hindrance to the one who wants to become familiar with the point of view and techniques presented here. However, we assume that the reader has a good knowledge of the following topics:

(a) **Commutative algebra**, as it is laid out, for example, in the volumes (in progress of being written) of the *Éléments* of N. Bourbaki (and, pending the publication of these volumes, in Samuel–Zariski [SZ60] and Samuel [Sam53b, Sam53a]).

(b) **Homological algebra**, for which we refer to Cartan–Eilenberg [CE56] (cited as (M)) and Godement [God58] (cited as (G)), as well as the recent article by A. Grothendieck [Gro57] (cited as (T)).

(c) **Sheaf theory**, where our main references will be (G) and (T); this theory provides the essential language for interpreting, in “geometric” terms, the essential notions of commutative algebra, and for “globalizing” them.

(d) Finally, it will be useful for the reader to have some familiarity with **functorial language**, which will be constantly used in this treatise, and for which the reader may consult (M), (G), and especially (T); the principles of this language and the main results of the general theory of functors will be described in more detail in a book currently in preparation by the authors of this treatise.

***

It is not the place, in this introduction, to give a more or less summary description from the “schemes” point of view in algebraic geometry, nor the long list of reasons which made its adoption necessary, and in particular the systematic acceptance of nilpotent elements in the local rings of “manifolds” that we consider (which necessarily shifts the idea of rational maps into the background, in favor of those of regular maps or “morphisms”). To be precise, this treatise aims to systematically develop the language of schemes, and will demonstrate, we hope, its necessity. Although it would be easy to do so, we will not try to give here an “intuitive” introduction to the notions developed in Chapter I. For the reader who would like to have a glimpse of the preliminary study of the subject matter of this treatise, we refer them to the conference by A. Grothendieck at the International Congress of Mathematicians in Edinburgh in 1958 [Gro58], and the exposé [Gro] of the same author. The work [Ser55a] (cited as (FAC)) of J.-P. Serre can also be considered as an intermediary exposition between the classical point of view and the schemes point of view in algebraic geometry, and, as such, its reading may be an excellent preparation for the reading of our *Éléments.*

***

We give below the general outline planned for this treatise, subject to later modifications, especially concerning the later chapters.
Chapter I. — The language of schemes.
— II. — Elementary global study of some classes of morphisms.
— III. — Cohomology of algebraic coherent sheaves. Applications.
— IV. — Local study of morphisms.
— V. — Elementary procedures of constructing schemes.
— VII. — Group schemes, principal fibre bundles.
— VIII. — Differential study of fibre bundles.
— IX. — The fundamental group.
— X. — Residues and duality.
— XI. — Theories of intersection, Chern classes, Riemann–Roch theorem.
— XII. — Abelian schemes and Picard schemes.
— XIII. — Weil cohomology.

In principle, all chapters are considered open to changes, and supplementary sections could always be added later; such sections would appear in separate fascicles in order to minimize the inconveniences accompanying whatever mode of publication adopted. When the writing of such a section is foreseen or in progress during the publication of a chapter, it will be mentioned in the summary of the chapter in question, even if, owing to certain orders of urgency, its actual publication clearly ought to have been later. For the convenience of the reader, we give in “Chapter 0” the necessary tools in commutative algebra, homological algebra, and sheaf theory, that will be used throughout this treatise, that are more or less well known but for which it was not possible to give convenient references. It is recommended for the reader to not read Chapter 0 except whilst reading the actual treatise, when the results to which we refer seem unfamiliar. Besides, we think that in this way, the reading of this treatise could be a good method for the beginner to familiarize themselves with commutative algebra and homological algebra, whose study, when not accompanied with tangible applications, is considered tedious, or even depressing, by many.

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It is outside of our capabilities to give a historic overview, or even a summary thereof, of the ideas and results described herein. The text will contain only those references considered particularly useful for comprehension, and we indicate the origin of only the most important results. Formally, at least, the subjects discussed in our work are reasonably new, which explains the scarcity of references made to the fathers of algebraic geometry from the 19th to the beginning of the 20th century, whose works we know only by hear-say. It is suitable, however, to say some words here about the works which have most directly influenced the authors and contributed to the development of scheme-theoretic point of view. We absolutely must mention the fundamental work (FAC) of J.-P. Serre first, which has served as an introduction to algebraic geometry for more that one young student (the author of this treatise being one), deterred by the dryness of the classic Foundations of A. Weil [Wei46]. It is there that it is shown, for the first time, that the “Zariski topology” of an “abstract” algebraic variety is perfectly suited to applying certain techniques from algebraic topology, and notably to be able to define a cohomology theory. Further, the definition of an algebraic variety given therein is that which translates most naturally to the idea that we develop here. Just as J.-P. Serre informed us, it is right to note that the idea of defining the structure of a manifold by the data of a sheaf of rings is due to H. Cartan, who took this idea as the starting point of his theory of analytic spaces. Of course, just as in algebraic geometry, it would be important in “analytic geometry” to give the allow the use of nilpotent elements in local rings of analytic spaces. This extension of the definition of H. Cartan and J.-P. Serre has recently been broached by H. Grauert [Gra60], and there is room to hope that a systematic report of analytic geometry in this setting will soon see the light of day. It is also evident that the ideas and techniques developed in this treatise retain a sense of analytic geometry, even though one must expect more considerable technical difficulties in this latter theory. We can foresee that algebraic geometry, by the simplicity of its methods, will be able to serve as a sort of formal model for future developments in the theory of analytic spaces.
of the same author [Ser57]. We have also vastly profited from the Séminaire de géométrie algébrique de C. Chevalley [CC]; in particular, the systematic usage of “constructible sets” introduced by him has turned out to be extremely useful in the theory of schemes (cf. Chapter IV). We have also borrowed from him the study of morphisms from the point of view of dimension (Chapter IV), that translates with negligible change to the framework of schemes. It also merits noting that the idea of “schemes of local rings”, introduced by Chevalley, naturally lends itself to being extended to algebraic geometry (not having, however, all the flexibility and generality that we intend to give it here); for the connections between this idea and our theory, see Chapter I, §8. One such extension has been developed by M. Nagata in a series of memoirs [Nag58a], which contain many special results concerning algebraic geometry over Dedekind rings.***

It goes without saying that a book on algebraic geometry, and especially a book dealing with the fundamentals, is of course influenced, if only by proxy, by mathematicians such as O. Zariski and A. Weil. In particular, the Théorie des fonctions holomorphes by Zariski [Zar51], reasonably flexible thanks to the cohomological methods and an existence theorem (Chapter III, §§4 and 5), is (along with the method of descent described in Chapter VI) one of the principal tools used in this treatise, and it seems to us one of the most powerful at our disposal in algebraic geometry.

The general technique in which it is employed can be sketched as follows (a typical example of which will be given in Chapter XI, in the study of the fundamental group). We have a proper morphism (Chapter II) \( f : X \rightarrow Y \) of algebraic varieties (or, more generally, of schemes) that we wish to study on the neighborhood of a point \( y \in Y \), with the aim of resolving a problem \( P \) relative to a neighborhood of \( y \). We proceed step by step:

1st We can suppose that \( Y \) is affine, so that \( X \) becomes a scheme defined on the affine ring \( A \) of \( Y \), and we can even replace \( A \) by the local ring of \( y \). This reduction is always easy in practice (Chapter V) and brings us to the case where \( A \) is a local ring.

2nd We study the problem in question when \( A \) is a local Artinian ring. So that the problem \( P \) still makes sense when \( A \) is not assumed to be integral, we sometimes have to reformulate \( P \), and it appears that we often obtain a better understanding of the problem in doing so, in an “infinitesimal” way.

3rd The theory of formal schemes (Chapter III, §§3, 4, and 5) lets us pass from the case of an Artinian ring to a complete local ring.

4th Finally, if \( A \) is an arbitrary local ring, considering “multiform (?) sections” of suitable schemes over \( X \), approximating a given “formal” section (Chapter IV), will let us pass, by extension of scalars, to the completion of \( A \), from a known result (about the scheme induced by \( X \) by extension of scalars to the completion of \( A \)) to an analogous result for a finite simple (e.g. unramified) extension of \( A \).

This sketch shows the importance of the systematic study of schemes defined over an Artinian ring \( A \). The point of view of Serre in his formulation of the theory of local class fields, and the recent works of Greenberg, seem to suggest that such a study could be undertaken by functorially attaching, to some such scheme \( X \), a scheme \( X' \) over the residue field \( k \) of \( A \) (assumed perfect) of dimension equal (in nice cases) to \( n \dim X \), where \( n \) is the height of \( A \).

As for the influence of A. Weil, it suffices to say that it is the need to develop the tools necessary to formulate, with full generality, the definition of “Weil cohomology”, and to tackle the proof of all the formal properties necessary to establish the famous conjectures in Diophantine geometry [Wei49], that has been one of the principal motivations for the writing of this treatise, as well as the desire to find the natural setting of the usual ideas and methods of algebraic geometry, and to give the authors the chance to understand said ideas and methods.

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7Among the works that come close to our point of view of algebraic geometry, we pick out the work of E. Kähler [Käh58] and a recent note of Chow and Igusa [CI58], which go back over certain results of (FAC) in the context of Nagata–Chevalley theory, as well as giving a Künneth formula.

8To avoid any misunderstanding, we point out that this task has barely been undertaken at the moment of writing this introduction, and still hasn’t led to the proof of the Weil conjectures.
Finally, we believe it useful to warn the reader that they, as did all the authors themselves, will almost certainly have difficulty before becoming accustomed to the language of schemes, and to convince themselves that the usual constructions that suggest geometric intuition can be translated, in essentially only one sensible way, to this language. As in many parts of modern mathematics, the first intuition seems further and further away, in appearance, from the correct language needed to express the mathematics in question with complete precision and the desired level of generality. In practice, the psychological difficulty comes from the need to replicate some familiar set-theoretic constructions to a category that is already quite different from that of sets (the category of preschemes, or the category of preschemes over a given prescheme): Cartesian products, group laws, ring laws, module laws, fibre bundles, principal homogeneous fibre bundles, etc. It will most likely be difficult for the mathematician, in the future, to shy away from this new effort of abstraction (maybe rather negligible, on the whole, in comparison with that supplied by our fathers) to familiarize themselves with the theory of sets.

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The references are given following the numerical system; for example, in III, 4.9.3, the III indicates the volume, the 4 the chapter, the 9 the section, and the 3 the paragraph. If we reference a volume from within itself then we omit the volume number.

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[Trans] This is not a direct translation of the original, but instead uses the language more familiar to modern book (and \LaTeX document) layouts.
CHAPTER 0

Preliminaries (EGA 0)

§1. RINGS OF FRACTIONS

1.0. Rings and Algebras

(1.0.1). All the rings considered in this treatise will have a unit element; all the modules over such a ring will be assumed to be unitary; the ring homomorphisms will always be assumed to send the unit element to the unit element; unless otherwise stated, a subring of a ring \( A \) will be assumed to contain the unit element of \( A \). We will focus in particular on commutative rings, and when we speak of a ring without specifying any details, it will be implied that it is commutative. If \( A \) is a not-necessarily-commutative ring, by \( A \)-module we will we mean a left module unless stated otherwise.

(1.0.2). Let \( A \) and \( B \) be not-necessarily-commutative rings and \( \phi : A \to B \) a homomorphism. Any left (resp. right) \( B \)-module \( M \) can be provided with a left (resp. right) \( A \)-module structure by \( a \cdot m = \phi(a) \cdot m \) (resp. \( m \cdot a = m \cdot \phi(a) \)); when it will be necessary to distinguish \( M \) as an \( A \)-module or a \( B \)-module, we will denote by \( M_{[\phi]} \) the left (resp. right) \( A \)-module defined as such. If \( L \) is an \( A \)-module, then a homomorphism \( u : L \to M_{[\phi]} \) is a homomorphism of abelian groups such that \( u(a \cdot x) = \phi(a) \cdot u(x) \) for \( a \in A, x \in L \); we will also say that it is a \( \phi \)-homomorphism \( L \to M \), and that the pair \( (\phi, u) \) (or, by abuse of language, \( u \)) is a di-homomorphism from \( (A, L) \) to \( (B, M) \). The pairs \( (A, L) \) consisting of a ring \( A \) and an \( A \)-module \( L \) thus form a category whose morphisms are di-homomorphisms.

(1.0.3). Under the hypotheses of (1.0.2), if \( J \) is a left (resp. right) ideal of \( A \), we denote by \( BJ \) (resp. \( JB \)) the left (resp. right) ideal \( B\phi(J) \) (resp. \( \phi(J)B \)) of \( B \) generated by \( \phi(J) \); it is also the image of the canonical homomorphism \( B \otimes_A J \to B \) (resp. \( J \otimes_A B \to B \)) of left (resp. right) \( B \)-modules.

(1.0.4). If \( A \) is a (commutative) ring, and \( B \) a not-necessarily-commutative ring, then the data of a structure of an \( A \)-algebra on \( B \) is equivalent to the data of a ring homomorphism \( \phi : A \to B \) such that \( \phi(A) \) is contained in the center of \( B \). For all ideals \( J \) of \( A \), \( JB = BJ \) is then a two-sided ideal of \( B \), and for every \( B \)-module \( M \), \( JM \) is then a \( B \)-module equal to \( BM \).

(1.0.5). We will not dwell much on the notions of modules of finite type and (commutative) algebras of finite type; to say that an \( A \)-module \( M \) is of finite type means that there exists an exact sequence \( A^p \to M \to 0 \). We say that an \( A \)-module \( M \) admits a finite presentation if it is isomorphic to the cokernel of a homomorphism \( A^p \to A^q \), or, in other words, if there exists an exact sequence \( A^p \to A^q \to M \to 0 \). We note that for a Noetherian ring \( A \), every \( A \)-module of finite type admits a finite presentation.

Let us recall that an \( A \)-algebra \( B \) is said to be integral over \( A \) if every element in \( B \) is a root in \( B \) of a monic polynomial with coefficients in \( A \); equivalently, if every element of \( B \) is contained in a subalgebra of \( B \) which is an \( A \)-module of finite type. When this is so, and if \( B \) is commutative, the subalgebra of \( B \) generated by a finite subset of \( B \) is an \( A \)-module of finite type; for a commutative algebra \( B \) to be integral and of finite type over \( A \), it is necessary and sufficient that \( B \) be an \( A \)-module of finite type; we also say that \( B \) is an integral \( A \)-algebra of finite type (or simply finite, if there is no chance of confusion). It should be noted that in these definitions, it is not assumed that the homomorphism \( A \to B \) defining the \( A \)-algebra structure is injective.

(1.0.6). An integral ring (or an integral domain) is a ring in which the product of a finite family of elements \( \neq 0 \) is \( \neq 0 \); equivalently, in such a ring, we have \( 0 \neq 1 \), and the product of two elements \( \neq 0 \) is \( \neq 0 \). A prime ideal of a ring \( A \) is an ideal \( p \) such that \( A/p \) is integral; this implies that \( p \neq A \). For a ring \( A \) to have at least one prime ideal, it is necessary and sufficient that \( A \neq \{0\} \).
A local ring is a ring $A$ in which there exists a unique maximal ideal, which is thus the complement of the invertible elements, and contains all the ideals $\neq A$. If $A$ and $B$ are local rings, and $m$ and $n$ their respective maximal ideals, then we say that a homomorphism $\phi : A \to B$ is local if $\phi(m) \subset n$ (or, equivalently, if $\phi^{-1}(n) = m$). By passing to quotients, such a homomorphism then defines a monomorphism from the residue field $A/m$ to the residue field $B/n$. The composition of any two local homomorphisms is a local homomorphism.

1.1. Radical of an ideal. Nilradical and radical of a ring

1.1.1. Let $a$ be an ideal of a ring $A$; the radical of $a$, denoted by $\sqrt{a}$, is the set of $x \in A$ such that $x^n \in a$ for an integer $n > 0$; it is an ideal containing $a$. We have $\sqrt{\sqrt{a}} = \sqrt{a}$; the relation $a \subset b$ implies $\sqrt{a} \subset \sqrt{b}$; the radical of a finite intersection of ideals is the intersection of their radicals. If $\phi$ is a homomorphism from another ring $A'$ to $A$, then we have $\sqrt{\phi^{-1}(a)} = \phi^{-1}(\sqrt{a})$ for any ideal $a \subset A$. For an ideal to be the radical of an ideal, it is necessary and sufficient that it be an intersection of prime ideals. The radical of an ideal $a$ is the intersection of the minimal prime ideals which contain $a$; if $A$ is Noetherian, there are finitely many of these minimal prime ideals.

The radical of the ideal $(0)$ is also called the nilradical of $A$; it is the set $\text{nil}$ of the nilpotent elements of $A$. We say that the ring $A$ is reduced if $\text{nil} = (0)$; for every ring $A$, the quotient $A/\text{nil}$ of $A$ by its nilradical is a reduced ring.

1.1.2. Recall that the nilradical $\text{nil}(A)$ of a (not-necessarily-commutative) ring $A$ is the intersection of the maximal left ideals of $A$ (and also the intersection of maximal right ideals). The nilradical of $A/\text{nil}(A)$ is $(0)$.

1.2. Modules and rings of fractions

1.2.1. We say that a subset $S$ of a ring $A$ is multiplicative if $1 \in S$ and the product of two elements of $S$ is in $S$. The examples which will be the most important in what follows are: 1st, the set $S_f$ of powers $f^n$ ($n \geq 0$) of an element $f \in A$; and 2nd, the complement $A - p$ of a prime ideal $p$ of $A$.

1.2.2. Let $S$ be a multiplicative subset of a ring $A$, and $M$ an $A$-module; on the set $M \times S$, the relation between pairs $(m_1, s_1)$ and $(m_2, s_2)$:

$$\text{"there exists an } s \in S \text{ such that } \frac{s_1 m_2 - s_2 m_1}{s_1 s_2} = 0\text{"}$$

is an equivalence relation. We denote by $S^{-1}M$ the quotient set of $M \times S$ by this relation, and by $m/s$ the canonical image of the pair $(m, s)$ in $S^{-1}M$; we call $i^S_M : m \mapsto m/1$ (also denoted $i^S$) the canonical map from $M$ to $S^{-1}M$. This map is, in general, neither injective nor surjective; its kernel is the set of $m \in M$ such that there exists an $s \in S$ for which $sm = 0$.

On $S^{-1}M$ we define an additive group law by setting

$$(m_1/s_1) + (m_2/s_2) = (s_2 m_1 + s_1 m_2)/(s_1 s_2)$$

(one can check that it is independent of the choice of representative of the elements of $S^{-1}M$, which are equivalence classes). On $S^{-1}A$ we further define a multiplicative law by setting $(a_1/s_1)(a_2/s_2) = (a_1 a_2)/(s_1 s_2)$, and finally an exterior law on $S^{-1}M$, acted on by the set of elements of $S^{-1}A$, by setting $(a/s)(m/s') = (am)/(ss')$. It can then be shown that $S^{-1}A$ is endowed with a ring structure (called the ring of fractions of $A$ with denominators in $S$) and $S^{-1}M$ with the structure of an $S^{-1}A$-module (called the module of fractions of $M$ with denominators in $S$); for all $s \in S$, $s/1$ is invertible in $S^{-1}A$, its inverse being $1/s$. The canonical map $i^S_A$ (resp. $i^S_M$) is a ring homomorphism (resp. a homomorphism of $A$-modules, $S^{-1}M$ being considered as an $A$-module by means of the homomorphism $i^S_A : A \to S^{-1}A$).

1.2.3. If $S_f = \{f^n\}_{n \geq 0}$ for a $f \in A$, we write $A_f$ and $M_f$ instead of $S^{-1}_f A$ and $S^{-1}_f M$; when $A_f$ is considered as algebra over $A$, we can write $A_f = A[1/f]$. $A_f$ is isomorphic to the quotient algebra $A[T]/(fT - 1)A[T]$. When $f = 1$, $A_f$ and $M_f$ are canonically identified with $A$ and $M$; if $f$ is nilpotent, then $A_f$ and $M_f$ are 0.

When $S = A - p$, with $p$ a prime ideal of $A$, we write $A_p$ and $M_p$ instead of $S^{-1}A$ and $S^{-1}M$; $A_p$ is a local ring whose maximal ideal $q$ is generated by $i^S_A(p)$, and we have $(i^S_A)^{-1}(q) = p$; by passing to quotients, $i^S_A$ gives a monomorphism from the integral ring $A/p$ to the field $A_p/q$, which can be identified with the field of fractions of $A/p$. 

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The ring of fractions $S^{-1}A$ and the canonical homomorphism $i_A^S$ are a solution to a universal mapping problem: any homomorphism $u$ from $A$ to a ring $B$ such that $u(S)$ is composed of invertible elements in $B$ factors uniquely as

$$
u : A^S \rightarrow S^{-1}A \xrightarrow{u^*} B$$

where $u^*$ is a ring homomorphism. Under the same hypotheses, let $M$ be an $A$-module, $N$ a $B$-module, and $v : M \rightarrow N$ a homomorphism of $A$-modules (for the $B$-module structure on $N$ defined by $u : A \rightarrow B$); then $v$ factors uniquely as

$$v : M^S \rightarrow S^{-1}M \xrightarrow{v^*} N$$

where $v^*$ is a homomorphism of $S^{-1}A$-modules (for the $S^{-1}A$-module structure on $N$ defined by $u^*$).

We define a canonical isomorphism $S^{-1}A \otimes_A M \simeq S^{-1}M$ of $S^{-1}A$-modules, sending the element $(a/s) \otimes m$ to the element $(am)/s$, with the inverse isomorphism sending $m/s$ to $(1/s) \otimes m$.

For every ideal $a'$ of $S^{-1}A$, $a = (i_A^S)^{-1}(a')$ is an ideal of $A$, and $a'$ is the ideal of $S^{-1}A$ generated by $i_A^S(a)$, which can be identified with $S^{-1}a$ (1.3.2). The map $p' \mapsto (i_A^S)^{-1}(p')$ is an isomorphism, for the structure given by ordering, from the set of prime ideals of $S^{-1}A$ to the set of prime ideals $p$ of $A$ such that $p \cap S = \emptyset$. In addition, the local rings $A_p$ and $(S^{-1}A)_{S^{-1}p}$ are canonically isomorphic (1.5.1).

When $A$ is an integral ring, for which $K$ denotes its field of fractions, the canonical map $i_A^S : A \rightarrow S^{-1}A$ is injective for any multiplicative subset $S$ not containing 0, and $S^{-1}A$ is then canonically identified with a subring of $K$. In particular, for every prime ideal $p$ of $A$, $A_p$ is a local ring containing $A$, with maximal ideal $pA_p$, and we have $pA_p \cap A = p$.

If $A$ is a reduced ring (1.1.1), so is $S^{-1}A$: indeed, if $(x/s)^n = 0$ for $x \in A$, $s \in S$, then this means that there exists an $s' \in S$ such that $s'x^n = 0$, hence $(s'x)^n = 0$, which, by hypothesis, implies $s'x = 0$, so $x/s = 0$.

### 1.3. Functorial properties

(1.3.1). Let $M$ and $N$ be $A$-modules, and $u$ an $A$-homomorphism $M \rightarrow N$. If $S$ is a multiplicative subset of $A$, we define a $S^{-1}A$-homomorphism $S^{-1}M \rightarrow S^{-1}N$, denoted by $S^{-1}u$, by setting $S^{-1}u(m/s) = u(m)/s$; if $S^{-1}M$ and $S^{-1}N$ are canonically identified with $S^{-1}A \otimes_A M$ and $S^{-1}A \otimes_A N$ (1.2.5), then $S^{-1}u$ is considered as $1 \otimes u$. If $P$ is a third $A$-module, and $v$ an $A$-homomorphism $N \rightarrow P$, we have $S^{-1}(v \circ u) = (S^{-1}v) \circ (S^{-1}u)$; in other words, $S^{-1}M$ is a covariant functor in $M$, from the category of $A$-modules to that of $S^{-1}A$-modules ($A$ and $S$ being fixed).

(1.3.2). The functor $S^{-1}M$ is exact; in other words, if the sequence

$$M \xrightarrow{u} N \xrightarrow{v} P$$

is exact, then so is the sequence

$$S^{-1}M \xrightarrow{S^{-1}u} S^{-1}N \xrightarrow{S^{-1}v} S^{-1}P.$$
(1.3.4). Let $M$ and $N$ be $A$-modules; there is a canonical functorial (in $M$ and $N$) isomorphism

$$(S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N) \simeq S^{-1}(M \otimes_{A} N)$$

which sends $(m/s) \otimes (n/t)$ to $(m \otimes n)/st$.

(1.3.5). We also have a functorial (in $M$ and $N$) homomorphism

$$S^{-1}\text{Hom}_{A}(M,N) \longrightarrow \text{Hom}_{S^{-1}A}(S^{-1}M,S^{-1}N)$$

which sends $u/s$ to the homomorphism $m/t \mapsto u(m)/st$. When $M$ has a finite presentation, the above homomorphism is an isomorphism: it is immediate when $M$ is of the form $A^t$, and we pass to the general case by starting with the exact sequence $A^p \rightarrow A^q \rightarrow M \rightarrow 0$ and using the exactness of the functor $S^{-1}M$ and the left-exactness of the functor $\text{Hom}_{A}(M,N)$ in $M$. Note that this is always the case when $A$ is Noetherian and the $A$-module $M$ is of finite type.

### 1.4. Change of multiplicative subset

(1.4.1). Let $S$ and $T$ be multiplicative subsets of a ring $A$ such that $S \subset T$; there exists a canonical homomorphism $\rho^{T,S}_{A}$ (or simply $\rho^{T,S}$) from $S^{-1}A$ to $T^{-1}A$, sending the element denoted $a/s$ to the element denoted $a/t$ in $T^{-1}A$; we have $i^{T}_{A} = \rho^{T,S}_{A} \circ i^{S}_{A}$. For every $A$-module $M$, there exists, in the same way, an $S^{-1}A$-linear map from $S^{-1}M$ to $T^{-1}M$ (the latter considered as an $S^{-1}A$-module by the homomorphism $\rho^{T,S}_{A}$), which sends the element $m/s$ of $S^{-1}M$ to the element $m/t$ of $T^{-1}M$; we denote this map by $p^{T,S}_{M}$, or simply $\rho^{T,S}$, and we still have $i^{T}_{M} = p^{T,S}_{M} \circ i^{S}_{M}$, by the canonical identification (1.2.5), $\rho^{T,S}_{M}$ is identified with $\rho^{T,S}_{A} \otimes_{A} 1$. The homomorphism $\rho^{T,S}_{A}$ is a functorial morphism (or natural transformation) from the functor $S^{-1}M$ to the functor $T^{-1}M$, in other words, the diagram

$$
\begin{array}{ccc}
S^{-1}M & \overset{S^{-1}u}{\longrightarrow} & S^{-1}N \\
\rho^{T,S}_{M} \downarrow & & \downarrow \rho^{T,S}_{N} \\
T^{-1}M & \overset{T^{-1}u}{\longrightarrow} & T^{-1}N
\end{array}
$$

is commutative, for every homomorphism $u : M \rightarrow N$; $T^{-1}u$ is entirely determined by $S^{-1}u$, since, for $m \in M$ and $t \in T$, we have

$$(T^{-1}u)(m/t) = (t/1)^{-1} \rho^{T,S}((S^{-1}u)(m/1)).$$

(1.4.2). With the same notation, for $A$-modules $M$ and $N$, the diagrams (cf. (1.3.4) and (1.3.5))

$$
\begin{array}{ccc}
(S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N) & \overset{\sim}{\longrightarrow} & S^{-1}(M \otimes_{A} N) \\
\downarrow & & \downarrow \\
(T^{-1}M) \otimes_{T^{-1}A} (T^{-1}N) & \overset{\sim}{\longrightarrow} & T^{-1}(M \otimes_{A} N) \\
\downarrow & & \downarrow \\
S^{-1}\text{Hom}_{A}(M,N) & \longrightarrow & \text{Hom}_{S^{-1}A}(S^{-1}M,S^{-1}N) \\
\downarrow & & \downarrow \\
T^{-1}\text{Hom}_{A}(M,N) & \longrightarrow & \text{Hom}_{T^{-1}A}(T^{-1}M,T^{-1}N)
\end{array}
$$

are commutative.

(1.4.3). There is an important case, in which the homomorphism $\rho^{T,S}$ is bijective, when we then know that every element of $T$ is a divisor of an element of $S$; we then identify the modules $S^{-1}M$ and $T^{-1}M$ via $\rho^{T,S}$. We say that $S$ is saturated if every divisor in $A$ of an element of $S$ is in $S$; by replacing $S$ with the set $T$ of all the divisors of the elements of $S$ (a set which is multiplicative and saturated), we see that we can always, if we wish, consider only modules of fractions $S^{-1}M$, where $S$ is saturated.

(1.4.4). If $S$, $T$, and $U$ are three multiplicative subsets of $A$ such that $S \subset T \subset U$, then we have

$$\rho^{U,S} = \rho^{U,T} \circ \rho^{T,S}.$$
Consider an increasing filtered family \((S_a)\) of multiplicative subsets of \(A\) (we write \(a \leq \beta\) for \(S_a \subset S_\beta\)), and let \(S\) be the multiplicative subset \(\bigcup_a S_a\); let us put \(\rho_a = \rho_{A, S_a}^S\) for \(a \leq \beta\); according to (1.4.4), the homomorphisms \(\rho_{a}^S\) define a ring \(A^\prime\) as the inductive limit of the inductive system of rings \((S_\alpha^{-1} A, \rho_{\alpha}^S)\). Let \(\rho_a\) be the canonical map \(\rho_a: S_a^{-1} A \to A^\prime\), and let \(\phi_a = \rho_A^S S_a\); as \(\phi_a = \phi_{\beta} \circ \rho_a\) for \(a \leq \beta\) according to (1.4.4), we can uniquely define a homomorphism \(\phi: A^\prime \to S^{-1} A\) such that the diagram

\[
\begin{array}{ccc}
S_a^{-1} A & \xrightarrow{\rho_a} & S_\beta^{-1} A \\
\downarrow{\phi_a} & & \downarrow{\phi_{\beta}} \\
A^\prime & \xrightarrow{\phi} & S^{-1} A
\end{array}
\]

is commutative. In fact, \(\phi\) is an isomorphism; it is indeed immediate by construction that \(\phi\) is surjective. On the other hand, if \(\rho_a(a/s_a) \in A^\prime\) is such that \(\phi(\rho_a(a/s_a)) = 0\), then this means that \(a/s_a = 0\) in \(S_a^{-1} A\), that is to say that there exists an \(s \in S\) such that \(sa = 0\); but there is a \(\beta \geq a\) such that \(s \in S_\beta\), and consequently, as \(\rho_a(a/s_a) = \rho_{\beta}^{S}(sa/ss_a) = 0\), we find that \(\phi\) is injective. The case for an \(A\)-module \(M\) is treated likewise, and we have thus defined canonical isomorphisms

\[
\lim \frac{S_a^{-1} A}{S_2^{-1} M} \simeq \frac{S_1^{-1} A}{(s_1 s_2)^{-1} M},
\]

the second being functorial in \(M\).

(1.4.6). Let \(S_1\) and \(S_2\) be multiplicative subsets of \(A\); then \(S_1 S_2\) is also a multiplicative subset of \(A\). Let us denote by \(S_2^\prime\) the canonical image of \(S_2\) in the ring \(S_1^{-1} A\), which is a multiplicative subset of this ring. For every \(A\)-module \(M\) there is then a functorial isomorphism

\[
S_2^{-1} (S_1^{-1}M) \simeq (S_1 S_2)^{-1} M
\]

which sends \((m/s_1)/(s_2/1)\) to the element \(m/(s_1 s_2)\).

1.5. Change of ring

(1.5.1). Let \(A\) and \(A^\prime\) be rings, \(\phi\) a homomorphism \(A^\prime \to A\), and \(S\) (resp. \(S^\prime\)) a multiplicative subset of \(A\) (resp. \(A^\prime\)), such that \(\phi(S^\prime) \subset S\); the composition homomorphism \(A^\prime \xrightarrow{\phi} A \to S^{-1} A\) factors as

\[
A^\prime \longrightarrow S^{-1} A^\prime \; \phi^S \longrightarrow S^{-1} A,
\]

by (1.2.4); where \(\phi^S(a'/s') = \phi(a')/\phi(s')\). If \(A = \phi(A^\prime)\) and \(S = \phi(S^\prime)\), then \(\phi^S\) is surjective. If \(A^\prime = A\) and \(\phi\) is the identity, then \(\phi^S\) is exactly the homomorphism \(\rho_{A}^SS^\prime\) defined in (1.4.1).

(1.5.2). Under the hypotheses of (1.5.1), let \(M\) be an \(A\)-module. There exists a canonical functorial morphism

\[
\sigma: S^{-1} (M_{[\phi^S]} A^\prime) \longrightarrow (S^{-1} M)_{[\phi^S]},
\]

of \(S^{-1} A^\prime\)-modules, sending each element \(m/s'\) of \((S^{-1} M)_{[\phi^S]}\) to the element \(m/\phi(s')\) of \((S^{-1} M)_{[\phi^S]}\); indeed, we immediately see that this definition does not depend on the representative \(m/s'\) of the element in question. When \(S = \phi(S^\prime)\), the homomorphism \(\sigma\) is bijective. When \(A^\prime = A\) and \(\phi\) is the identity, \(\sigma\) is none other than the homomorphism \( \rho_M^{SS^\prime}\) defined in (1.4.1).

When, in particular, we take \(M = A\) the homomorphism \(\phi\) defines an \(A^\prime\)-algebra structure on \(A\); \(S^{-1} (A_{[\phi]}^\prime)\) is then endowed with a ring structure, with which it can be identified with \((\phi(S^\prime))^{-1} A\), and the homomorphism \(\sigma: S^{-1} (A_{[\phi]}^\prime) \to S^{-1} A\) is a homomorphism of \(S^{-1} A^\prime\)-algebras.

(1.5.3). Let \(M\) and \(N\) be \(A\)-modules; by composing the homomorphisms defined in (1.3.4) and (1.5.2), we obtain a homomorphism

\[
(S^{-1} M \otimes_{S^{-1} A} S^{-1} N)_{[\phi^S]} \longrightarrow S^{-1} ((M \otimes A)_{[\phi]}),
\]
which is an isomorphism when $\phi(S') = S$. Similarly, by composing the homomorphisms in (1.3.5) and (1.5.2), we obtain a homomorphism

$$S'^{-1}((\text{Hom}_A(M, N))_{[\phi]}) \longrightarrow (\text{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N))_{[\phi]}$$

which is an isomorphism when $\phi(S') = S$ and $M$ admits a finite presentation.

(1.5.4). Let us now consider an $A'$-module $N'$, and form the tensor product $N' \otimes_{A'} A_{[\phi]}$, which can be considered as an $A$-module by setting $a \cdot (n' \otimes b) = n' \otimes (ab)$. There is a functorial isomorphism of $S^{-1}A$-modules

$$\tau: (S'^{-1}N') \otimes_{S'^{-1}A'} (S^{-1}A)_{[\phi]} \simeq S^{-1}(N' \otimes_{A'} A_{[\phi]})$$

which sends the element $(n'/s') \otimes (a/s)$ to the element $(n' \otimes a)/((\phi(s'))s)$; indeed, we can show that when we replace $n'/s'$ (resp. $a/s$) by another expression of the same element, $(n' \otimes a)/((\phi(s'))s)$ does not change; on the other hand, we can define a homomorphism inverse to $\tau$ by sending $(n'/1) \otimes (a/s)$ to the element $(n'/1) \otimes (a/s)$: we use the fact that $S^{-1}(N' \otimes_{A'} A_{[\phi]})$ is canonically isomorphic to $(N' \otimes_{A'} A_{[\phi]}) \otimes_{A} S^{-1}A$ (1.2.5), so also to $N' \otimes_{A'} (S^{-1}A)_{[\phi]}$, where we denote by $\psi$ the composite homomorphism $a' \mapsto \phi(\psi)_{1}/1$ from $A'$ to $S^{-1}A$.

(1.5.5). If $M'$ and $N'$ are $A'$-modules, then by composing the isomorphisms (1.3.4) and (1.5.4), we obtain an isomorphism

$$S'^{-1}M \otimes_{S'^{-1}A'} S'^{-1}N' \otimes_{S'^{-1}A'} S^{-1}A \simeq S^{-1}(M' \otimes_{A'} N' \otimes_{A'} A).$$

Likewise, if $M'$ admits a finite presentation, we have by (1.3.5) and (1.5.4) an isomorphism

$$\text{Hom}_{S'^{-1}A'}((S'^{-1}M', S'^{-1}N') \otimes_{S'^{-1}A'} S^{-1}A \simeq S^{-1}(\text{Hom}_{A'}(M', N') \otimes_{A'} A).$$

(1.5.6). Under the hypotheses of (1.5.1), let $T$ (resp. $T'$) be another multiplicative subset of $A$ (resp. $A'$) such that $S \subseteq T$ (resp. $S' \subseteq T'$) and $\phi(T') \subseteq T$. Then the diagram

$$
\begin{array}{ccc}
S'^{-1}A' & \phi' \\
\rho'^{-1} \downarrow & \downarrow \rho^{-1} \\
T'^{-1}A' & \phi'
\end{array}
$$

is commutative. If $M$ is an $A$-module, then the diagram

$$
\begin{array}{ccc}
S'^{-1}(M_{[\phi]}) & \sigma \\
\rho'^{-1} \downarrow & \downarrow \rho^{-1} \\
T'^{-1}(M_{[\phi]}) & \sigma
\end{array}
$$

is commutative. Finally, if $N'$ is an $A'$-module, then the diagram

$$
\begin{array}{ccc}
(S'^{-1}N') \otimes_{S'^{-1}A'} (S^{-1}A)_{[\phi]} & \sim \tau \\
\rho'^{-1} \downarrow & \downarrow \rho^{-1} \\
(T'^{-1}N') \otimes_{T'^{-1}A'} (T^{-1}A)_{[\phi]} & \sim \tau
\end{array}
$$

is commutative, the left vertical arrow obtained by applying $\rho'^{-1}S'$ to $S'^{-1}N'$ and $\rho^{-1}S$ to $S^{-1}A$.

(1.5.7). Let $A''$ be a third ring, $\phi': A'' \rightarrow A'$ a ring homomorphism, and $S''$ a multiplicative subset of $A''$ such that $\phi'(S'') \subseteq S'$. Let $\phi'' = \phi \circ \phi'$; then we have

$$\phi''_{S''} = \phi'_{S'} \circ \phi_{S''}.$$
Let $M$ be an $A$-module; evidently we have $M_{(p')} = (M_{(p)})_{(p')}$. If $\sigma'$ and $\sigma''$ are the homomorphisms defined by $\phi'$ and $\phi''$ in the same way as how $\sigma$ is defined in (1.5.2) by $\phi$, then we have the transitivity formula

$$\sigma'' = \sigma \circ \sigma'.$$

Finally, let $N''$ be an $A''$-module; the $A$-module $N'' \otimes_{A''} A_{(p'')} = \lim_{\to} \bigcup_{\mathfrak{p}''} \bigcup_{n} M_{(\mathfrak{p}, n)}$ is canonically identified with $\bigcup_{\mathfrak{p}', n} M_{(\mathfrak{p}, n)}$, and likewise the $S^{-1}A$-module $\bigcup_{\mathfrak{p}', n} M_{(\mathfrak{p}, n)}$ is canonically identified with $\bigcup_{\mathfrak{p}', n} M_{(\mathfrak{p}, n)}$. With these identifications, if $\tau'$ and $\tau''$ are the isomorphisms defined by $\phi'$ and $\phi''$ in the same way as how $\tau$ is defined in (1.5.4) by $\phi$, then we have the transitivity formula

$$\tau'' = \tau \circ (\tau' \otimes 1).$$

**1.5.8.** Let $A$ be a subring of a ring $B$; for every minimal prime ideal $p$ of $A$, there exists a minimal prime ideal $q$ of $B$ such that $p = A \cap q$. Indeed, $A_p$ is a subring of $B_p$ (1.3.2) and has a single prime ideal $p'$ (1.2.6); since $B_p$ is not 0, it has at least one prime ideal $q'$ and we necessarily have $q' \cap A_p = p'$; the prime ideal $q_1$ of $B$, the inverse image of $q'$, is thus such that $q_1 \cap A = p$, and a fortiori we have $q \cap A = p$ for every minimal prime ideal $q$ of $B$ contained in $q_1$.

**1.6. Identification of the module $M_f$ as an inductive limit**

**1.6.1.** Let $M$ be an $A$-module and $f$ an element of $A$. Consider a sequence $(M_n)$ of $A$-modules, all identical to $M$, and for each pair of integers $m \leq n$, let $\phi_{nm}$ be the homomorphism $z \mapsto f^{n-m}z$ from $M_m$ to $M_n$; it is immediate that $((M_n), (\phi_{nm}))$ is an inductive system of $A$-modules; let $N = \lim_{\to} M_n$ be the inductive limit of this system. We define a canonical functorial $A$-isomorphism $N \to M_f$. For this, let us note that, for all $n, \theta_n : z \mapsto z/f^n$ is an $A$-homomorphism from $M = M_n$ to $M_f$, and it follows from the definitions that we have $\theta_n \circ \phi_{nm} = \theta_m$ for $m \leq n$. As a result, there exists an $A$-homomorphism $\theta : N \to M_f$ such that, if $\phi_n$ denotes the canonical homomorphism $M_n \to N$, then we have $\theta_n = \theta \circ \phi_n$ for all $n$. Since, by hypothesis, every element of $M_f$ is of the form $z/f^n$ for at least one $n$, it is clear that $\theta$ is surjective. On the other hand, if $\theta(\phi_n(z)) = 0$, or, in other words, if $z/f^n = 0$, then there exists an integer $k > 0$ such that $f^kz = 0$, so $\phi_{n+k,n}(z) = 0$, which gives $\phi_n(z) = 0$. We can therefore identify $M_f$ with $\lim_{\to} M_n$ via $\theta$.

**1.6.2.** Now write $M_{f,n}, \phi_{nm}^f$, and $\phi_n^f$ instead of $M_n, \phi_{nm}$, and $\phi_n$. Let $g$ be another element of $A$. Since $f^n$ divides $f^n g^n$, we have a functorial homomorphism

$$\rho_{f,g,f} : M_f \to M_{f,g} \quad (\text{(1.4.1) and (1.4.3)});$$

if we identify $M_f$ and $M_{f,g}$ with $\lim_{\to} M_{f,n}$ and $\lim_{\to} M_{f,g,n}$ respectively, then $\rho_{f,g,f}$ identifies with the inductive limit of the maps $\rho_{f,g,f}^n : M_{f,n} \to M_{f,g,n}$ defined by $\rho_{f,g,f}^n(z) = g^n z$. Indeed, this follows immediately from the commutativity of the diagram

$$
\begin{array}{ccc}
M_{f,n} & \xrightarrow{\rho_{f,g,f}^n} & M_{f,g,n} \\
\phi_n^f \downarrow & & \phi_n^g \\
M_f & \xrightarrow{\rho_{f,g,f}} & M_{f,g}
\end{array}
$$
1.7. Support of a module

(1.7.1). Given an $A$-module $M$, we define the support of $M$, denoted by $\text{Supp}(M)$, to be the set of prime ideals $p$ of $A$ such that $M_p \neq 0$. For it to be the case that $M = 0$, it is necessary and sufficient that $\text{Supp}(M) = \emptyset$, because if $M_p = 0$ for all $p$, then the annihilator of an element $x \in M$ cannot be contained in any prime ideal of $A$, and so is the whole of $A$.

(1.7.2). If $0 \to N \to M \to P \to 0$ is an exact sequence of $A$-modules, then we have

$$\text{Supp}(M) = \text{Supp}(N) \cup \text{Supp}(P)$$

because, for every prime ideal $p$ of $A$, the sequence $0 \to N_p \to M_p \to P_p \to 0$ is exact (1.3.2) and in order that $M_p = 0$, it is necessary and sufficient that $N_p = P_p = 0$.

(1.7.3). If $M$ is the sum of a family $(M_{\lambda})$ of submodules, then $M_p$ is the sum of the $(M_{\lambda})_p$ for every prime ideal $p$ of $A$ (1.3.3) and (1.3.2), so $\text{Supp}(M) = \bigcup_{\lambda} \text{Supp}(M_{\lambda})$.

(1.7.4). If $M$ is an $A$-module of finite type, then $\text{Supp}(M)$ is the set of prime ideals containing the annihilator of $M$. Indeed, if $M$ is cyclic and generated by $x$, then to say that $M_p = 0$ is to say that there exists an $s \notin p$ such that $s \cdot x = 0$, and thus that $p$ does not contain the annihilator of $x$. Now if $M$ admits a finite system $(x_i)_{1 \leq i \leq n}$ of generators, and if $a_i$ is the annihilator of $x_i$, then it follows from (1.7.3) that $\text{Supp}(M)$ is the set of the $p$ containing one of the $a_i$, or equivalently, the set of the $p$ containing $a = \bigcap_i a_i$, which is the annihilator of $M$.

(1.7.5). If $M$ and $N$ are two $A$-modules of finite type, then we have

$$\text{Supp}(M \otimes_A N) = \text{Supp}(M) \cap \text{Supp}(N).$$

It is a question of seeing that, if $p$ is a prime ideal of $A$, then the condition $M_p \otimes_A N_p \neq 0$ is equivalent to “$M_p \neq 0$ and $N_p \neq 0$” (taking (1.3.4) into account). In other words, it is a question of seeing that, if $P$ and $Q$ are modules of finite type over a local ring $B \neq 0$, then $P \otimes_B Q \neq 0$. Let $m$ be the maximal ideal of $B$. By Nakayama’s Lemma, the vector spaces $P/mP$ and $Q/mQ$ are not 0, and so it is the same for the tensor product $(P/mP) \otimes_B (Q/mQ) = (P \otimes_B Q) \otimes_B (B/m)$, whence the conclusion.

In particular, if $M$ is an $A$-module of finite type, and $a$ an ideal of $A$, then $\text{Supp}(M/aM)$ is the set of prime ideals containing both $a$ and the annihilator $n$ of $M$ (1.7.4), that is, the set of prime ideals containing $a + n$. 

§2. IRREDUCIBLE SPACES. NOETHERIAN SPACES

2.1. Irreducible spaces

(2.1.1). We say that a topological space $X$ is irreducible if it is nonempty and if it is not a union of two distinct closed subspaces of $X$. It is equivalent to say that $X \neq \emptyset$ and the intersection of two nonempty open sets (and consequently of a finite number of open sets) of $X$ is nonempty, or that every nonempty open set is everywhere dense, or that any closed set is rare\(^1\), or, lastly, that all open sets of $X$ are connected.

(2.1.2). For a subspace $Y$ of a topological space $X$ to be irreducible, it is necessary and sufficient that its closure $\overline{Y}$ be irreducible. In particular, any subspace which is the closure $\overline{\{x\}}$ of a singleton is irreducible; we will express the relation $y \in \overline{\{x\}}$ (equivalent to $\overline{\{y\}} \subset \overline{\{x\}}$) by saying that $y$ is a specialization of $x$ or that $x$ is a generalization of $y$. When there exists, in an irreducible space $X$, a point $x$ such that $X = \overline{\{x\}}$, we will say that $x$ is a generic point of $X$. Any nonempty open subset of $X$ then contains $x$, and any subspace containing $x$ has $x$ as a generic point.

(2.1.3). Recall that a Kolmogoroff space is a topological space $X$ satisfying the axiom of separation:

(T\(_0\)) If $x \neq y$ are any two points of $X$, there is an open set containing one of the points $x$ and $y$, but not the other.

\(^1\)[Trans] also known as nowhere dense.
If an irreducible Kolmogoroff space admits a generic point, it admits exactly one, since a nonempty open set contains any generic point.

Recall that a topological space $X$ is said to be quasi-compact if, from any collection of open sets of $X$, one can extract a finite cover of $X$ (or, equivalently, if any decreasing filtered family of nonempty closed sets has a nonempty intersection). If $X$ is a quasi-compact space, then any nonempty closed subset $A$ of $X$ contains a minimal nonempty closed set $M$, because the set of nonempty closed subsets of $A$ is inductive under the relation $\subseteq$; if, in addition, $X$ is a Kolmogoroff space, $M$ is necessarily a single point (or, as we say by abuse of language, is a closed point).

(2.1.4). In an irreducible space $X$, every nonempty open subspace $U$ is irreducible, and if $X$ admits a generic point $x$, $x$ is also a generic point of $U$.

To prove this, let $(U_{\alpha})$ be a cover (whose set of indices is nonempty) of a topological space $X$, consisting of nonempty open sets; if $X$ is irreducible, it is necessary and sufficient that $U_{\alpha}$ is irreducible for all $\alpha$, and that $U_{\alpha} \cap U_{\beta} \neq \emptyset$ for any $\alpha, \beta$. The condition is clearly necessary; to see that it is sufficient, it suffices to prove that if $V$ is a nonempty open subset of $X$, then $V \cap U_{\alpha}$ is nonempty for all $\alpha$, since then $V \cap U_{\alpha}$ is dense in $U_{\alpha}$ for all $\alpha$, and consequently $V$ is dense in $X$. Now there is at least one index $\gamma$ such that $V \cap U_{\gamma} \neq \emptyset$, so $V \cap U_{\gamma}$ is dense in $U_{\gamma}$, and as for all $\alpha$, $U_{\alpha} \cap V_{\gamma} \neq \emptyset$, we also have $V \cap U_{\alpha} \cap U_{\gamma} \neq \emptyset$.

(2.1.5). Let $X$ be an irreducible space, and $f$ a continuous map from $X$ into a topological space $Y$. Then $f(X)$ is irreducible, and if $x$ is a generic point of $X$, then $f(x)$ is a generic point of $f(X)$ and hence also of $f(X)$. In particular, if, in addition, $Y$ is irreducible and with a single generic point $y$, then for $f(X)$ to be everywhere dense, it is necessary and sufficient that $f(x) = y$.

(2.1.6). Any irreducible subspace of a topological space $X$ is contained in a maximal irreducible subspace, which is necessarily closed. Maximal irreducible subspaces of $X$ are called the irreducible components of $X$. If $Z_1$ and $Z_2$ are two irreducible components distinct from the space $X$, then $Z_1 \cap Z_2$ is a closed rare set in each of the subspaces $Z_1$, $Z_2$; in particular, if an irreducible component of $X$ admits a generic point (2.1.2), such a point cannot belong to any other irreducible component. If $X$ has only a finite number of irreducible components $Z_i$ ($1 \leq i \leq n$), and if, for each $i$, we put $U_i = \bigcup_{j \neq i} Z_j$, then the $U_i$ are open, irreducible, disjoint, and their union is dense in $X$. Let $U$ be an open subset of a topological space $X$. If $Z$ is an irreducible subset of $X$ that intersects $U$, then $Z \cap U$ is open and dense in $Z$, thus irreducible; conversely, for any irreducible closed subset $Y$ of $U$, the closure $\overline{Y}$ of $Y$ in $X$ is irreducible and $\overline{Y} \cap U = Y$. We conclude that there is a bijective correspondence between the irreducible components of $U$ and the irreducible components of $X$ which intersect $U$.

(2.1.7). If a topological space $X$ is a union of a finite number of irreducible closed subspaces $Y_i$, then the irreducible components of $X$ are the maximal elements of the set of the $Y_i$ because if $Z$ is an irreducible closed subset of $X$, then $Z$ is the union of the $Z \cap Y_i$ from which one sees that $Z$ must be contained in one of the $Y_i$. Let $Y$ be a subspace of a topological space $X$, and suppose that $Y$ has only a finite number of irreducible components $Y_i$ ($1 \leq i \leq n$); then the closures $\overline{Y_i}$ in $X$ are the irreducible components of $Y$.

(2.1.8). Let $Y$ be an irreducible space admitting a single generic point $y$. Let $X$ be a topological space, and $f$ a continuous map from $X$ to $Y$. Then, for any irreducible component $Z$ of $X$ intersecting $f^{-1}(y)$, $f(Z)$ is dense in $Y$. The converse is not necessarily true; however, if $Z$ has a generic point $z$, and if $f(z)$ is dense in $Y$, then we must have $f(z) = y$ (2.1.5); in addition, $Z \cap f^{-1}(y)$ is then the closure of $\{z\}$ in $f^{-1}(y)$ and is therefore irreducible, and as an irreducible subset of $f^{-1}(y)$ containing $z$ is necessarily contained in $Z$ (2.1.6), $z$ is a generic point of $Z \cap f^{-1}(y)$. As any irreducible component of $f^{-1}(y)$ is contained in an irreducible component of $X$, we see that, if any irreducible component $Z$ of $X$ intersecting $f^{-1}(y)$ admits a generic point, then there is a bijective correspondence between all these components and all the irreducible components $Z \cap f^{-1}(y)$ of $f^{-1}(y)$, the generic points of $Z$ being identical to those of $Z \cap f^{-1}(y)$. 

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2.2. Noetherian spaces

(2.2.1) We say that a topological space $X$ is Noetherian if the set of open subsets of $X$ satisfies the maximal condition, or, equivalently, if the set of closed subsets of $X$ satisfies the minimal condition. We say that $X$ is locally Noetherian if each $x \in X$ admits a neighborhood which is a Noetherian subspace.

(2.2.2) Let $E$ be an ordered set satisfying the minimal condition, and let $P$ be a property of the elements of $E$ subject to the following condition: if $a \in E$ is such that for any $x < a$, $P(x)$ is true, then $P(a)$ is true. Under these conditions, $P(x)$ is true for all $x \in E$ ("principle of Noetherian recurrence"). Indeed, let $F$ be the set of $x \in E$ for which $P(x)$ is false; if $F$ were not empty, it would have a minimal element $a$, and as then $P(x)$ is true for all $x < a$, $P(a)$ would be true, which is a contradiction.

We will apply this principle in particular when $E$ is a set of closed subsets of a Noetherian space.

(2.2.3) Any subspace of a Noetherian space is Noetherian. Conversely, any topological space that is a finite union of Noetherian subspaces is Noetherian.

(2.2.4) Any Noetherian space is quasi-compact; conversely, any topological space in which all open sets are quasi-compact is Noetherian.

(2.2.5) A Noetherian space has only a finite number of irreducible components, as we see by Noetherian recurrence.

§3. SUPPLEMENT ON SHEAVES

3.1. Sheaves with values in a category

(3.1.1) Let $C$ be a category, $(A_\alpha)_{\alpha \in I}, (A_{\alpha \beta})_{(\alpha, \beta) \in I \times I}$ two families of objects of $C$ such that $A_{\beta \alpha} = A_{\alpha \beta}$, and $(\rho_{\alpha \beta})_{(\alpha, \beta) \in I \times I}$ a family of morphisms $\rho_{\alpha \beta} : A_\alpha \rightarrow A_\beta$. We say that a pair consisting of an object $A$ of $C$ and a family of morphisms $\rho_\alpha : A \rightarrow A_\alpha$ is a solution to the universal problem defined by the data of the families $(A_\alpha), (A_{\alpha \beta})$, and $(\rho_{\alpha \beta})$ if, for every object $B$ of $C$, the map which sends $f \in \text{Hom}(B, A)$ to the family $(\rho_\alpha \circ f) \in \Pi_\alpha \text{Hom}(B, A_\alpha)$ is a bijection of $\text{Hom}(B, A)$ to the set of all $(f_\alpha)$ such that $\rho_{\alpha \beta} \circ f_\alpha = \rho_{\beta \alpha} \circ f_\beta$ for any pair of indices $(\alpha, \beta)$. If such a solution exists, it is unique up to an isomorphism.

(3.1.2) We will not recall the definition of a presheaf $U \mapsto \mathcal{F}(U)$ on a topological space $X$ with values in a category $C$ (G, I, 1.9); we say that such a presheaf is a sheaf with values in $C$ if it satisfies the following axiom:

(F) For any covering $(U_\alpha)$ of an open set $U$ by open sets $U_\alpha$ contained in $U$, if we denote by $\rho_\alpha$ (resp. $\rho_{\alpha \beta}$) the restriction morphism
$$\mathcal{F}(U) \rightarrow \mathcal{F}(U_\alpha),$$
the pair formed by $\mathcal{F}(U)$ and the family $(\rho_\alpha)$ are a solution to the universal problem for $(\mathcal{F}(U_\alpha))$, $(\mathcal{F}(U_\alpha \cap U_\beta))$, and $(\rho_{\alpha \beta})$ in (3.1.1)\(^2\).

Equivalently, we can say that, for each object $T$ of $C$, that the family $U \mapsto \text{Hom}(T, \mathcal{F}(U))$ is a sheaf of sets.

(3.1.3) Assume that $C$ is the category defined by a "type of structure with morphisms" $\Sigma$, the objects of $C$ being the sets with structures of type $\Sigma$ and morphisms those of $\Sigma$. Suppose that the category $C$ also satisfies the following condition:

(E) If $(A, (\rho_\alpha))$ is a solution of a universal mapping problem in the category $C$ for families $(A_\alpha), (A_{\alpha \beta}), (\rho_{\alpha \beta})$, then it is also a solution of the universal mapping problem for the same families in the category of sets (that is, when we consider $A, A_\alpha, A_{\alpha \beta}$ as sets, $\rho_\alpha$ and $\rho_{\alpha \beta}$ as functions)\(^3\).

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\(^2\)This is a special case of the more general notion of a (non-filtered) projective limit (see (T, I, 1.8) and the book in preparation announced in the introduction).

\(^3\)It can be proved that it also means that the canonical functor $C \rightarrow \text{Set}$ commutes with projective limits (not necessarily filtered).
Under these conditions, the condition (F) gives that, when considered as a presheaf of sets, $U \mapsto \mathcal{F}(U)$ is a sheaf. In addition, for a map $u : T \rightarrow \mathcal{F}(U)$ to be a morphism of $\mathcal{C}$, it is necessary and sufficient, according to (F), that each map $\rho_a \circ u$ is a morphism $T \rightarrow \mathcal{F}(U_a)$, which means that the structure of type $\Sigma$ on $\mathcal{F}(U)$ is the initial structure for the morphisms $\rho_a$. Conversely, suppose a presheaf $U \mapsto \mathcal{F}(U)$ on $X$, with values in $\mathcal{C}$, is a sheaf of sets and satisfies the previous condition; it is then clear that it satisfies (F), so it is a sheaf with values in $\mathcal{C}$.

(3.1.4). When $\Sigma$ is a type of a group or ring structure, the fact that the presheaf $U \mapsto \mathcal{F}(U)$ with values in $\mathcal{C}$ is a sheaf of sets implies ipso facto that it is a sheaf with values in $\mathcal{C}$ (in other words, a sheaf of groups or rings within the meaning of (G)). But it is not the same when, for example, $\mathcal{C}$ is the category of topological rings (with morphisms as continuous homomorphisms): a sheaf with values in $\mathcal{C}$ is a sheaf of rings $U \mapsto \mathcal{F}(U)$ such that for any open $U$ and any covering of $U$ by open sets $U_a \subset U$, the topology of the ring $\mathcal{F}(U)$ is to be the least fine making the homomorphisms $\mathcal{F}(U) \rightarrow \mathcal{F}(U_a)$ continuous. We will say in this case that $U \mapsto \mathcal{F}(U)$, considered as a sheaf of rings (without a topology), is underlying the sheaf of topological rings $U \mapsto \mathcal{F}(U)$. Morphisms $u_V : \mathcal{F}(V) \rightarrow \mathcal{F}(V)$ ($V$ an arbitrary open subset of $X$) of sheaves of topological rings are therefore homomorphisms of the underlying sheaves of rings, such that $u_V$ is continuous for all open $V \subset X$; to distinguish them from any homomorphisms of the sheaves of the underlying rings, we will call them continuous homomorphisms of sheaves of topological rings. We have similar definitions and conventions for sheaves of topological spaces or topological groups.

(3.1.5). It is clear that for any category $\mathcal{C}$, if there is a presheaf (respectively a sheaf) $\mathcal{F}$ on $X$ with values in $\mathcal{C}$ and $U$ is an open set of $X$, the $\mathcal{F}(V)$ for open $V \subset U$ constitute a presheaf (or a sheaf) with values in $\mathcal{C}$, which we call the presheaf (or sheaf) induced by $\mathcal{F}$ on $U$ and denote it by $\mathcal{F}|U$.

For any morphism $u : \mathcal{F} \rightarrow \mathcal{G}$ of presheaves on $X$ with values in $\mathcal{C}$, we denote by $u|U$ the morphism $\mathcal{F}|U \rightarrow \mathcal{G}|U$ consisting of the $u_V$ for $V \subset U$.

(3.1.6). Suppose now that the category $\mathcal{C}$ admits inductive limits (T, 1.8); then, for any presheaf (and in particular any sheaf) $\mathcal{F}$ on $X$ with values in $\mathcal{C}$ and each $x \in X$, we can define the stalk $\mathcal{F}_x$ as the object of $\mathcal{C}$ defined by the inductive limit of the $\mathcal{F}(U)$ with respect to the filtered set (for $\supseteq$) of the open neighborhoods $U$ of $x$ in $X$, and the morphisms $\rho^U_x : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$ if $u : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of presheaves with values in $\mathcal{C}$, we define for each $x \in X$ the morphism $u_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ as the inductive limit of $u_V : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ with respect to all open neighborhoods of $x$; we thus define $\mathcal{F}_x$ as a covariant functor in $\mathcal{F}$, with values in $\mathcal{C}$, for all $x \in X$.

When $\mathcal{C}$ is further defined by a kind of structure with morphisms $\Sigma$, we call sections over $U$ of a sheaf $\mathcal{F}$ with values in $\mathcal{C}$ the elements of $\mathcal{F}(U)$, and we write $\Gamma(U, \mathcal{F})$ instead of $\mathcal{F}(U)$; for $s \in \Gamma(U, \mathcal{F})$, $V$ an open set contained in $U$, we write $s|V$ instead of $\rho^U_V(s)$; for all $x \in U$, the canonical image of $s$ in $\mathcal{F}_x$ is the germ of $s$ at the point $x$, denoted by $s_x$ (we will never replace the notation $s(x)$ in this sense, this notation being reserved for another notion relating to sheaves which will be considered in this treatise (5.5.1)).

If then $u : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves with values in $\mathcal{C}$, we will write $u(s)$ instead of $u_V(s)$ for all $s \in \Gamma(V, \mathcal{F})$.

If $\mathcal{F}$ is a sheaf of commutative groups, or rings, or modules, we say that the set of $x \in X$ such that $\mathcal{F}_x \neq \{0\}$ is the support of $\mathcal{F}$, denoted $\text{Supp}(\mathcal{F})$; this set is not necessarily closed in $X$.

When $\mathcal{C}$ is defined by a type of structure with morphisms, we systematically refrain from using the point of view of “étalé spaces” in terms of relating to sheaves with values in $\mathcal{C}$; in other words, we will never consider a sheaf as a topological space (nor even as the whole union of its stalks), and we will not consider also a morphism $u : \mathcal{F} \rightarrow \mathcal{G}$ of such sheaves on $X$ as a continuous map of topological spaces.

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This is because in the category $\mathcal{C}$, any morphism that is a bijection (as a map of sets) is an isomorphism. This is no longer true when $\mathcal{C}$ is the category of topological spaces, for example.
3.2. Presheaves on an open basis

(3.2.1). We will restrict to the following categories $\mathcal{C}$ admitting projective limits (generalized, that is, corresponding to not necessarily filtered preordered sets, cf. (T, 1.8)). Let $X$ be a topological space, $\mathcal{B}$ an open basis for the topology of $X$. We will call a presheaf on $\mathcal{B}$, with values in $\mathcal{C}$, a family of objects $\mathcal{F}(U) \in \mathcal{C}$, corresponding to each $U \in \mathcal{B}$, and a family of morphisms $\rho_U^W : \mathcal{F}(V) \to \mathcal{F}(U)$ defined for any pair $(U, V)$ of elements of $\mathcal{B}$ such that $U \subset V$, with the conditions $\rho_U^U =$ identity and $\rho_W^W = \rho_V^V \circ \rho_U^W$ if $U, V, W$ in $\mathcal{B}$ are such that $U \subset V \subset W$. We can associate a presheaf with values in $\mathcal{C}$: $U \mapsto \mathcal{F}(U)$ in the ordinary sense, taking for all open $U$, $\mathcal{F}(U) = \varprojlim \mathcal{F}(V)$, where $V$ runs through the ordered set (for $\subset$, not filtered in general) of $V \in \mathcal{B}$ sets such that $V \subset U$, since the $(V)$ form a projective system for the $\rho_W^V (V \subset W \subset U, V \in \mathcal{B}, W \in \mathcal{B})$. Indeed, if $U, U'$ are two open sets of $X$ such that $U \subset U'$, we define $\rho_{U'}^U$ as the projective limit (for $V \subset U$) of the canonical morphisms $\mathcal{F}(U') \to \mathcal{F}(V)$, for all open $U \subset V \subset U'$, the composition of morphisms $\rho_{U''}^U \circ \rho_{U'}^{U''} = \rho_{U''}^U \circ \rho_U^{U''} \circ \rho_{U'}^U$ for any pair of indices $\alpha, \beta$ and any $V \in \mathcal{B}$ such that $V \subset U_\alpha \cap U_\beta$.

The condition is obviously necessary. To show that it is sufficient, consider first a second basis $\mathcal{B}'$ of the topology of $X$, contained in $\mathcal{B}$, and show that if $\mathcal{F}''$ denotes the presheaf induced by the subfamily $(\mathcal{F}(V))_{V \in \mathcal{B}'}$, $\mathcal{F}''$ is canonically isomorphic to $\mathcal{F}'$. Indeed, first the projective limit (for $V \in \mathcal{B}'$, $V \subset U$) of the canonical morphisms $\mathcal{F}(U') \to \mathcal{F}(V)$ is a morphism $\mathcal{F}(U') \to \mathcal{F}(V)$ for all open $U$. If $U \in \mathcal{B}$, this morphism is an isomorphism, because by hypothesis the canonical morphisms $\mathcal{F}(U) \to \mathcal{F}(V)$ for $V \in \mathcal{B}'$, $V \subset U$, factorize as $\mathcal{F}(U) \to \mathcal{F}(U') \to \mathcal{F}(V)$, and it is immediate to see that the composition of morphisms $\mathcal{F}(U) \to \mathcal{F}(U')$ and $\mathcal{F}(U') \to \mathcal{F}(V)$ thus defined are the identities. This being so, for all open $U$, the morphisms $\mathcal{F}(U) \to \mathcal{F}(W) = \mathcal{F}(W)$ for $W \in \mathcal{B}$ and $W \subset U$ satisfy the conditions characterizing the projective limit of $\mathcal{F}(W)$ ($W \in \mathcal{B}$, $W \subset U$), which proves our assertion given the uniqueness of a projective limit up to isomorphism.

This being so, let $U$ be any open set of $X$, $(U_\alpha)$ a covering of $U$ by the open sets contained in $U$, and $\mathcal{B}'$ the subfamily of $\mathcal{B}$ formed by the sets of $\mathcal{B}$ contained in at least one $U_\alpha$; it is clear that $\mathcal{B}'$ is still a basis of the topology of $U$, so $\mathcal{F}'(U)$ (resp., $\mathcal{F}''(U_\alpha)$) is the projective limit of $\mathcal{F}(V)$ for $V \in \mathcal{B}'$ and $V \subset U$ (resp., $V \subset U_\alpha$), the axiom (F) is then immediately verified by virtue of the definition of the projective limit.

When (F$_0$) is satisfied, we will say by abuse of language that the presheaf $\mathcal{F}$ on the basis $\mathcal{B}$ is a sheaf.

(3.2.2). Let $\mathcal{F}, \mathcal{G}$ be two presheaves on a basis $\mathcal{B}$, with values in $\mathcal{C}$; we define a morphism $u : \mathcal{F} \to \mathcal{G}$ as a family $(u_V)_{V \in \mathcal{B}}$ of morphisms $u_V : \mathcal{F}(V) \to \mathcal{G}(V)$ satisfying the usual compatibility conditions with the restriction morphisms $\rho_V^W$. With the notation of (3.2.1), we have a morphism $u' : \mathcal{F}' \to \mathcal{G}'$ of (ordinary) presheaves by taking for $u'_U$ the projective limit of the $u_V$ for $V \in \mathcal{B}$ and $V \subset U$; the

---

5If $X$ is a Noetherian space, we can still define $\mathcal{F}'(U)$ and show that it is a presheaf (in the ordinary sense) when one supposes only that $\mathcal{C}$ admits projective limits for finite projective systems. Indeed, if $U$ is any open set of $X$, there is a finite covering $(V_i)$ of $U$ consisting of sets of $\mathcal{B}$; for every couple $(i, j)$ of indices, let $(V_{ij})$ be a finite covering of $V_i \cap V_j$ formed by sets of $\mathcal{B}$. Let $I$ be the set of $i$ and triples $(i, j, k)$, ordered only by the relations $i > (i, j, k), j > (i, j, k)$; we then take $\mathcal{F}'(U)$ to be the projective limit of the system of $\mathcal{F}(V_i)$ and $\mathcal{F}(V_{ij})$; it is easy to verify that this does not depend on the coverings $(V_i)$ and $(V_{ij})$ and that $U \mapsto \mathcal{F}'(U)$ is a presheaf.

6It also means that the pair formed by $\mathcal{F}(U)$ and the $\rho_U^W$ is a solution to the universal problem defined in (3.1.1) by the data of $A_\alpha = \mathcal{F}(U_\alpha), A_{\alpha \beta} = \Pi \mathcal{F}(V)$ (for $V \in \mathcal{B}$ such that $V \subset U_\alpha \cap U_\beta$) and $\rho_{U \beta} = (\rho_U^V) : \mathcal{F}(U_\alpha) \to \Pi \mathcal{F}(V)$ defined by the condition that for $V \in \mathcal{B}, V' \in \mathcal{B}, W \in \mathcal{B}, V \cup V' \subset U_\alpha \cap U_\beta, W \subset V \cap V'$, $\rho_W^V \circ \rho_U^W = \rho_W^{V'} \circ \rho_U^{V'}$. 

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verification of the compatibility conditions with the $\rho^{\mu U}_{U}$ follows from the functorial properties of the projective limit.

(3.2.4). If the category $\mathcal{C}$ admits inductive limits, and if $\mathcal{F}$ is a presheaf on the basis $\mathcal{B}$, with values in $\mathcal{C}$, for each $x \in X$ the neighborhoods of $x$ belonging to $\mathcal{B}$ form a cofinal set (for $\subset$) in the set of neighborhoods of $x$, therefore, if $\mathcal{F}'$ is the (ordinary) presheaf corresponding to $\mathcal{F}$, the stalk $\mathcal{F}'_x$ is equal to $\lim_{\mathcal{B}} \mathcal{F}(V)$ over the set of $V \in \mathcal{B}$ containing $x$. If $u : \mathcal{F} \to \mathcal{G}$ is morphism of presheaves on $\mathcal{B}$ with values in $\mathcal{C}$, $u' : \mathcal{F}' \to \mathcal{G}'$ the corresponding morphism of ordinary presheaves, $u'_x$ is likewise the inductive limit of the morphisms $u_V : \mathcal{F}(V) \to \mathcal{G}(V)$ for $V \in \mathcal{B}$, $x \in V$.

(3.2.5). We return to the general conditions of (3.2.1). If $\mathcal{F}$ is an ordinary sheaf with values in $\mathcal{C}$, $\mathcal{F}_1$ the sheaf on $\mathcal{B}$ obtained by the restriction of $\mathcal{F}$ to $\mathcal{B}$, then the ordinary sheaf $\mathcal{F}'_1$ obtained from $\mathcal{F}_1$ by the procedure of (3.2.1) is canonically isomorphic to $\mathcal{F}$, by virtue of the condition (F) and the uniqueness properties of the projective limit. We identify the ordinary sheaf $\mathcal{F}$ with $\mathcal{F}'_1$.

If $\mathcal{G}$ is a second (ordinary) sheaf on $X$ with values in $\mathcal{C}$, and $u : \mathcal{F} \to \mathcal{G}$ a morphism, the preceding remark shows that the data of the morphism $u_V : \mathcal{F}(V) \to \mathcal{G}(V)$ for only the $V \in \mathcal{B}$ completely determines $u$; conversely, it is sufficient, the $u_V$ being given for $V \in \mathcal{B}$, to verify the commutative diagram with the restriction morphisms $\rho^{\mu V}_U$ for $V \in \mathcal{B}$, $W \in \mathcal{B}$, and $V \subset W$, for there to exist a morphism $u'$ and a unique $\mathcal{F}$ in $\mathcal{G}$ such that $u'_V = u_V$ for each $V \in \mathcal{B}$ (3.2.3).

(3.2.6). Suppose that $\mathcal{C}$ admits projective limits. Then the category of sheaves on $X$ with values in $\mathcal{C}$ admits projective limits; if $(\mathcal{H}_\lambda)$ is a projective system of sheaves on $X$ with values in $\mathcal{C}$, the $\mathcal{F}(U) = \lim_{\lambda} \mathcal{F}_\lambda(U)$ indeed define a presheaf with values in $\mathcal{C}$, and the verification of the axiom (F) follows from the transitivity of projective limits; the fact that $\mathcal{F}$ is then the projective limit of the $\mathcal{F}_\lambda$ is immediate.

When $\mathcal{C}$ is the category of sets, for each projective system $(\mathcal{H}_\lambda)$ such that $\mathcal{H}_0$ is a subsheaf of $\mathcal{F}_\lambda$ for each $\lambda$, $\lim_{\lambda} \mathcal{H}_\lambda$ canonically identifies with a subsheaf of $\lim_{\lambda} \mathcal{F}_\lambda$. If $\mathcal{C}$ is the category of abelian groups, the covariant functor $\lim_{\lambda} \mathcal{F}_\lambda$ is additive and left exact.

3.3. Gluing sheaves

(3.3.1). Suppose still that the category $\mathcal{C}$ admits (generalized) projective limits. Let $X$ be a topological space, $\mathcal{U} = (U_\lambda)_{\lambda \in L}$ an open cover of $X$, and for each $\lambda \in L$, let $\mathcal{F}_\lambda$ be a sheaf on $U_\lambda$, with values in $\mathcal{C}$; for each pair of indices $(\lambda, \mu)$, suppose that we are given an isomorphism $\theta_{\lambda\mu} : \mathcal{F}_\mu | (U_\lambda \cap U_\mu) \simeq \mathcal{F} | (U_\lambda \cap U_\mu)$; in addition, suppose that for each triple $(\lambda, \mu, \nu)$, if we denote by $\theta^{\lambda\mu}_{\nu\lambda}, \theta^{\lambda\nu}_{\mu\lambda}, \theta^{\mu\lambda}_{\nu\lambda}$ the restrictions of $\theta_{\lambda\mu}, \theta_{\mu\lambda}, \theta_{\nu\lambda}$ to $U_\lambda \cap U_\mu \cap U_\nu$, then we have $\theta^{\lambda\mu}_{\nu\lambda} = \theta^{\lambda\mu}_{\nu\lambda} \circ \theta^{\lambda\nu}_{\mu\lambda}$ (gluing condition for the $\theta_{\lambda\mu}$).

Then there exists a sheaf $\mathcal{F}$ on $X$ with values in $\mathcal{C}$, and for each $\lambda$ an isomorphism $\eta_\lambda : \mathcal{F} | U_\lambda \simeq \mathcal{F}_\lambda$ such that, for each pair $(\lambda, \mu)$, if we denote by $\eta^{\lambda\mu}_\lambda$ and $\eta^{\mu\lambda}_\mu$ the restrictions of $\eta_\lambda$ and $\eta_\mu$ to $U_\lambda \cap U_\mu$, then we have $\theta_{\lambda\mu} = \eta^{\lambda\mu}_\lambda \circ \eta^{\mu\lambda}_\mu$; in addition, $\mathcal{F}$ and the $\eta_\lambda$ are determined up to unique isomorphism by these conditions. The uniqueness indeed follows immediately from (3.2.5). To establish the existence of $\mathcal{F}$, denote by $\mathcal{B}$ the open basis consisting of the open sets contained in at least one $U_\lambda$, and for each $U \in \mathcal{B}$, choose (by the Hilbert function $\tau$) one of the $\mathcal{F}_\lambda(U)$ for only the $\lambda$ such that $U \subset U_\lambda$; if we denote this object by $\mathcal{F}(U)$, the $\rho^{\mu U}_U$ for $U \subset V, V \in \mathcal{B}$ are defined in an evident way (by means of the $\theta_{\lambda\mu}$), and the transitivity conditions is a consequence of the gluing condition; in addition, the verification of (F) is immediate, so the presheaf on $\mathcal{B}$ thus clearly defines a sheaf, and we deduce by the general procedure (3.2.1) an (ordinary) sheaf still denoted $\mathcal{F}$ and which answers the question. We say that $\mathcal{F}$ is obtained by gluing the $\mathcal{F}_\lambda$ by means of the $\theta_{\lambda\mu}$ and we usually identify the $\mathcal{F}_\lambda$ and $\mathcal{F} | U_\lambda$ by means of the $\eta_\lambda$.

It is clear that each sheaf $\mathcal{F}$ on $X$ with values in $\mathcal{C}$ can be considered as being obtained by the gluing of the sheaves $\mathcal{F}_\lambda = \mathcal{F} | U_\lambda$ (where $(U_\lambda)$ is an arbitrary open cover of $X$), by means of the isomorphisms $\theta_{\lambda\mu}$ reduced to the identity.

(3.3.2). With the same notation, let $\mathcal{G}_\lambda$ be a second sheaf on $U_\lambda$ (for each $\lambda \in L$) with values in $\mathcal{C}$, and for each pair $(\lambda, \mu)$ let us be given an isomorphism $\omega_{\lambda\mu} : \mathcal{G}_\mu | (U_\lambda \cap U_\mu) \simeq \mathcal{G}_\lambda | (U_\lambda \cap U_\mu)$, these isomorphisms satisfying the gluing condition. Finally, suppose that we are given for each $\lambda$ a
morphism \( u_\lambda : \mathcal{F}_\lambda \to \mathcal{G}_\lambda \), and that the diagrams

\[
\begin{align*}
\mathcal{F}|_{U_\lambda} | (U_\lambda \cap U_\mu) & \xrightarrow{u_\mu} \mathcal{G}|_{U_\lambda} | (U_\lambda \cap U_\mu) \\
\mathcal{F}_\lambda | (U_\lambda \cap U_\mu) & \xrightarrow{u_\lambda} \mathcal{G}_\lambda | (U_\lambda \cap U_\mu)
\end{align*}
\]

are commutative. Then, if \( \mathcal{G} \) is obtained by gluing the \( \mathcal{G}_\lambda \) by means of the \( \omega_{\lambda\mu} \), there exists a unique morphism \( u : \mathcal{F} \to \mathcal{G} \) such that the diagrams

\[
\begin{align*}
\mathcal{F}|_{U_\lambda} & \xrightarrow{u|_{U_\lambda}} \mathcal{G}|_{U_\lambda} \\
\mathcal{F}_\lambda & \xrightarrow{u_\lambda} \mathcal{G}_\lambda
\end{align*}
\]

are commutative; this follows immediately from (3.3.3). The correspondence between the family \( (u_\lambda) \) and \( u \) is in a functorial bijection with the subset of \( \Pi_\lambda \text{Hom}(\mathcal{F}_\lambda, \mathcal{G}_\lambda) \) satisfying the conditions (3.3.2.1) on \( \text{Hom}(\mathcal{F}, \mathcal{G}) \).

(3.3.3). With the notation of (3.3.1), let \( V \) be an open set of \( X \); it is immediate that the restrictions to \( V \cap U_\lambda \cap U_\mu \) of the \( \theta_{\lambda\mu} \) satisfy the gluing condition for the induced sheaves \( \mathcal{F}_\lambda | (V \cap U_\lambda) \) and that the sheaves on \( V \) obtained by gluing the latter identifies canonically with \( \mathcal{F}|V \).

### 3.4. Direct images of presheaves

(3.4.1). Let \( X, Y \) be two topological spaces, \( \psi : X \to Y \) a continuous map. Let \( \mathcal{F} \) be a presheaf on \( X \) with values in a category \( \mathcal{C} \); for each open \( U \subset Y \), let \( \mathcal{G}(U) = \mathcal{F}(\psi^{-1}(U)) \), and if \( U, V \) are two open subsets of \( Y \) such that \( U \subset V \), let \( \rho^V_U \) be the morphism \( \mathcal{F}(\psi^{-1}(V)) \to \mathcal{F}(\psi^{-1}(U)) \); it is immediate that the \( \mathcal{G}(U) \) and the \( \rho^V_U \) define a presheaf on \( Y \) with values in \( \mathcal{C} \), that we call the direct image of \( \mathcal{F} \) by \( \psi \) and we denote it by \( \psi_* \mathcal{F} \). If \( \mathcal{F} \) is a sheaf, we immediately verify the axiom (F) for the presheaf \( \mathcal{G} \), so \( \psi_* \mathcal{F} \) is a sheaf.

(3.4.2). Let \( \mathcal{F}_1, \mathcal{F}_2 \) be two presheaves of \( X \) with values in \( \mathcal{C} \), and let \( u : \mathcal{F}_1 \to \mathcal{F}_2 \) be a morphism. When \( U \) varies over the set of open subsets of \( Y \), the family of morphisms \( \mathcal{G}_2(\psi^{-1}(U)) : \mathcal{F}_1(\psi^{-1}(U)) \to \mathcal{F}_2(\psi^{-1}(U)) \) satisfies the compatibility conditions with the restriction morphisms, and as a result defines a morphism \( \psi_* (u) : \psi_* \mathcal{F}_1 \to \psi_* \mathcal{F}_2 \). If \( v : \mathcal{F}_2 \to \mathcal{F}_3 \) is a morphism from \( \mathcal{F}_2 \) to a third presheaf on \( X \) with values in \( \mathcal{C} \), we have \( \psi_* (v \circ u) = \psi_* (v) \circ \psi_* (u) \); in other words, \( \psi_* \mathcal{F} \) is a covariant functor in \( \mathcal{F} \), from the category of presheaves (resp. sheaves) on \( X \) with values in \( \mathcal{C} \), to that of presheaves (resp. sheaves) on \( Y \) with values in \( \mathcal{C} \).

(3.4.3). Let \( Z \) be a third topological space, \( \psi' : Y \to Z \) a continuous map, and let \( \psi'' = \psi' \circ \psi \). It is clear that we have \( \psi''_* (\mathcal{F}) = \psi'_* (\psi_* (\mathcal{F})) \) for each presheaf \( \mathcal{F} \) on \( X \) with values in \( \mathcal{C} \); in addition, for each morphism \( u : \mathcal{F} \to \mathcal{G} \) of such presheaves, we have \( \psi''_* (u) = \psi'_* (\psi_* (u)) \). In other words, \( \psi''_* \) is the composition of the functors \( \psi'_* \) and \( \psi_* \), and this can be written as

\[
(\psi' \circ \psi)_* = \psi'_* \circ \psi_*. 
\]

In addition, for each open set \( U \) of \( Y \), the image under the restriction \( \psi | \psi^{-1}(U) \) of the induced presheaf \( \mathcal{F}| \psi^{-1}(U) \) is none other than the induced presheaf \( \psi_* (\mathcal{F})|U \).

(3.4.4). Suppose that the category \( \mathcal{C} \) admits inductive limits, and let \( \mathcal{F} \) be a presheaf on \( X \) with values in \( \mathcal{C} \); for all \( x \in X \), the morphisms \( \Gamma(\psi^{-1}(U), \mathcal{F}) \to \mathcal{F}_x(U) \) an open neighborhood of \( \psi(x) \) in \( Y \) form an inductive limit, which gives by passing to the limit a morphism \( \psi_\star : (\psi_* (\mathcal{F}))(\psi(x)) \to \mathcal{F}_x \) of the stalks; in general, these morphisms are neither injective or surjective. It is functorial; indeed, if
$u : \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism of presheaves on $X$ with values in $\mathcal{C}$, the diagram

$$
\begin{array}{ccc}
(\psi_*(\mathcal{F}_1))_{\psi(x)} & \xrightarrow{\psi_*} & (\mathcal{F}_1)_x \\
(\psi_*(u))_{\psi(x)} \downarrow \quad & & \downarrow u_x \\
(\psi_*(\mathcal{F}_2))_{\psi(x)} & \xrightarrow{\psi_*} & (\mathcal{F}_2)_x
\end{array}
$$

is commutative. If $Z$ is a third topological space, $\psi : Y \to Z$ a continuous map, and $\psi'' = \psi' \circ \psi$, then we have $\psi''_x = \psi_x \circ \psi'_*(\psi(x))$ for $x \in X$.

(3.4.5). Under the hypotheses of (3.4.4), suppose in addition that $\psi$ is a homeomorphism from $X$ to the subspace $\psi(X)$ of $Y$. Then, for each $x \in X$, $\psi_x$ is an isomorphism. This applies in particular to the canonical injection $j$ of a subset $X$ of $Y$ into $Y$.

(3.4.6). Suppose that $\mathcal{C}$ be the category of groups, or of rings, etc. If $\mathcal{F}$ is a sheaf on $X$ with values in $\mathcal{C}$, of support $S$, and if $y \notin \psi(S)$, then it follows from the definition of $\psi_*(\mathcal{F})$ that $(\psi_*(\mathcal{F}))[y] = \{0\}$, or in other words, that the support of $\psi_*(\mathcal{F})$ is contained in $\psi(S)$; but it is not necessarily contained in $\psi(S)$. Under the same hypotheses, if $j$ is the canonical injection of a subset $X$ of $Y$ into $Y$, the sheaf $j_*(\mathcal{F})$ induces $\mathcal{F}$ on $X$; if moreover $X$ is closed in $Y$, $j_*(\mathcal{F})$ is the sheaf on $Y$ which induces $\mathcal{F}$ on $X$ and $0$ on $Y - X$ (G, II, 2.9.2), but it is in general distinct from the latter when we suppose that $X$ is locally closed but not closed.

3.5. Inverse images of presheaves

(3.5.1). Under the hypotheses of (3.4.1), if $\mathcal{G}$ (resp. $\mathcal{H}$) is a presheaf on $X$ (resp. $Y$) with values in $\mathcal{C}$, then each morphism $u : \mathcal{G} \to \psi_*(\mathcal{F})$ of presheaves on $Y$ is called a $\phi$-morphism from $\mathcal{G}$ to $\mathcal{F}$, and we denote it also by $\mathcal{G} \to \mathcal{F}$. We denote also by $\text{Hom}_X(\mathcal{G}, \mathcal{F})$ the set of $\psi$-morphisms from $\mathcal{G}$ to $\mathcal{F}$. For each pair $(U, V)$, where $U$ is an open set of $X$, $V$ an open set of $Y$ such that $\psi(U) \subset V$, we have a morphism $u_{U,V} : \mathcal{G}(V) \to \mathcal{F}(U)$ by composing the restriction morphism $\mathcal{F}(\psi^{-1}(V)) \to \mathcal{F}(U)$ and the morphism $u_V : \mathcal{G}(V) \to \psi_*(\mathcal{F})(V) = \mathcal{F}(\psi^{-1}(V))$; it is immediate that these morphisms render commutative the diagrams

$$
\begin{array}{ccc}
\mathcal{G}(V) & \xrightarrow{u_{U,V}} & \mathcal{F}(U) \\
\downarrow & & \downarrow \\
\mathcal{G}(V') & \xrightarrow{u_{U',V'}} & \mathcal{F}(U')
\end{array}
$$

for $U' \subset U$, $V' \subset V$, $\psi(U') \subset V'$. Conversely, the data of a family $(u_{U,V})$ of morphisms rendering commutative the diagrams (3.5.1.1) define a $\psi$-morphism $u$, since it suffices to take $u_V = u_{\psi^{-1}(V),V}$.

If the category $\mathcal{C}$ admits (generalized) projective limits, and if $\mathcal{B}$, $\mathcal{B}'$ are bases for the topologies of $X$ and $Y$ respectively, to define a $\psi$-morphism $u$ of sheaves, we can restrict to giving the $u_{U,V}$ for $U \in \mathcal{B}$, $V \in \mathcal{B}'$, and $\psi(U) \subset V$, satisfying the compatibility conditions of (3.5.1.1) for $U$, $U'$ in $\mathcal{B}$ and $V$, $V'$ in $\mathcal{B}'$; it indeed suffices to define $u_W$, for each open $W \subset Y$, as the projective limit of the $u_{U,V}$ for $V \subset W$, $U \in \mathcal{B}$ and $\psi(U) \subset V$.

When the category $\mathcal{C}$ admits inductive limits, we have, for each $x \in X$, a morphism $\mathcal{G}(V) \to \mathcal{F}(\psi^{-1}(V)) \to \mathcal{F}_x$, for each open neighborhood $V$ of $\psi(x)$ in $Y$, and these morphisms form an inductive system which gives by passing to the limit a morphism $\mathcal{G}_x \to \mathcal{F}_x$.

(3.5.2). Under the hypotheses of (3.4.3), let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be presheaves with values in $\mathcal{C}$ on $X$, $Y$, $Z$ respectively, and let $u : \mathcal{G} \to \psi_*(\mathcal{F})$, $v : \mathcal{H} \to \psi'_*(\mathcal{F})$ be a $\psi$-morphism and a $\psi'$-morphism respectively. We obtain a $\psi''$-morphism $w : \mathcal{H} \xrightarrow{v} \psi'_*(\mathcal{F}) \xrightarrow{\psi'_*(u)} \psi'_*(\psi_*(\mathcal{F})) = \psi''_*(\mathcal{F})$, that we call, by definition, the composition of $u$ and $v$. We can therefore consider the pairs $(X, \mathcal{F})$ consisting of a topological space $X$ and a presheaf $\mathcal{F}$ on $X$ (with values in $\mathcal{C}$) as forming a category, the morphisms being the pairs $(\psi, \theta) : (X, \mathcal{F}) \to (Y, \mathcal{G})$ consisting of a continuous map $\psi : X \to Y$ and of a $\psi$-morphism $\theta : \mathcal{G} \to \mathcal{F}$. 
(3.5.3). Let \( \psi : X \to Y \) be a continuous map, \( \mathcal{F} \) a presheaf on \( Y \) with values in \( \mathcal{C} \). We call the inverse image of \( \mathcal{F} \) under \( \psi \) the pair \( (\mathcal{F}', \rho) \), where \( \mathcal{F}' \) is a sheaf on \( X \) with values in \( \mathcal{C} \), and \( \rho : \mathcal{F} \to \mathcal{F}' \) a \( \psi \)-morphism (in other words a homomorphism \( \mathcal{F} \to \psi_*(\mathcal{F}') \)) such that, for each sheaf \( \mathcal{F} \) on \( X \) with values in \( \mathcal{C} \), the map

\[
\text{Hom}_X(\mathcal{F}', \mathcal{F}) \to \text{Hom}_Y(\mathcal{F}, \psi_*(\mathcal{F}'))
\]

sending \( v \) to \( \psi_*(v) \circ \rho \), is a bijection; this map, being functorial in \( \mathcal{F} \), then defines an isomorphism of functors in \( \mathcal{F} \). The pair \((\mathcal{F}', \rho)\) is the solution of a universal problem, and we say it is determined up to unique isomorphism when it exists. We then write \( \mathcal{F}' = \psi^*(\mathcal{F}) \), \( \rho = \rho_{\mathcal{F}} \), and by abuse of language, we say that \( \psi^*(\mathcal{F}) \) is the inverse image sheaf of \( \mathcal{F} \) under \( \psi \), and we agree that \( \psi^*(\mathcal{F}) \) is considered as equipped with a canonical \( \psi \)-morphism \( \rho_{\mathcal{F}} : \mathcal{F} \to \psi^*(\mathcal{F}) \), that is to say the canonical homomorphism of presheaves on \( Y \):

\[
(3.5.4.3) \quad \rho_{\mathcal{F}} : \mathcal{F} \to \psi_*(\psi^*(\mathcal{F})).
\]

For each homomorphism \( v : \psi^*(\mathcal{F}) \to \mathcal{F} \) (where \( \mathcal{F} \) is a sheaf on \( X \) with values in \( \mathcal{C} \)), we put \( v^\flat = \psi_*(v) \circ \rho_{\mathcal{F}} : \mathcal{F} \to \psi_*(\mathcal{F}) \). By definition, each morphism of presheaves \( u : \mathcal{F} \to \psi_*(\mathcal{F}) \) is of the form \( v^\flat \) for a unique \( v \), which we will denote \( u^\flat \). In other words, each morphism \( u : \mathcal{F} \to \psi_*(\mathcal{F}) \) of presheaves factorizes in a unique way as

\[
(3.5.3.3) \quad u : \mathcal{F} \xrightarrow{\rho_{\mathcal{F}}} \psi_*(\psi^*(\mathcal{F})) \xrightarrow{\psi_*(u^\flat)} \psi_*(\mathcal{F}).
\]

(3.5.4). Suppose now that the category \( \mathcal{C} \) be such\(^7\) that each presheaf \( \mathcal{F} \) on \( Y \) with values in \( \mathcal{C} \) admits an inverse image under \( \psi \), and we denote it by \( \psi^*(\mathcal{F}) \).

We will see that we can define \( \psi^*(\mathcal{F}) \) as a covariant functor in \( \mathcal{F} \), from the category of presheaves on \( Y \) with values in \( \mathcal{C} \), to that of sheaves on \( X \) with values in \( \mathcal{C} \), in such a way that the isomorphism \( v \mapsto v^\flat \) is an isomorphism of bifunctors

\[
(3.5.4.1) \quad \text{Hom}_X(\psi^*(\mathcal{F}), \mathcal{F}) \simeq \text{Hom}_Y(\mathcal{F}, \psi_*(\mathcal{F}))
\]
in \( \mathcal{F} \) and \( \mathcal{F} \).

Indeed, for each morphism \( w : \mathcal{F}_1 \to \mathcal{F}_2 \) of presheaves on \( Y \) with values in \( \mathcal{C} \), consider the composite morphism \( \mathcal{F}_1 \xrightarrow{w} \mathcal{F}_2 \xrightarrow{\rho_{\mathcal{F}_2}} \psi_*(\psi^*(\mathcal{F}_2)) \); to it corresponds a morphism \( (\rho_{\mathcal{F}_2} \circ w)^\flat \) : \( \psi^*(\mathcal{F}_1) \to \psi^*(\mathcal{F}_2) \), that we denote by \( \psi^*(w) \). We therefore have, according to (3.5.3.3),

\[
(3.5.4.2) \quad \psi_*(\psi^*(w)) \circ \rho_{\mathcal{F}_1} = \rho_{\mathcal{F}_2} \circ w.
\]

For each morphism \( u : \mathcal{F}_2 \to \psi_*(\mathcal{F}) \), where \( \mathcal{F} \) is a sheaf on \( X \) with values in \( \mathcal{C} \), we have, according to (3.5.3.3), (3.5.4.2), and the definition of \( u^\flat \), that

\[
(u^\flat \circ \psi^*(w))^\flat = \psi_*(u^\flat) \circ \psi_*(\psi^*(w)) \circ \rho_{\mathcal{F}_1} = \psi_*(u^\flat) \circ \rho_{\mathcal{F}_2} \circ w = u \circ w
\]

where again

\[
(3.5.4.3) \quad (u \circ w)^\flat = u^\flat \circ \psi^*(w).
\]

If we take in particular for \( u \) a morphism \( \mathcal{F}_2 \xrightarrow{w} \mathcal{F}_3 \xrightarrow{\rho_{\mathcal{F}_3}} \psi_*(\psi^*(\mathcal{F}_3)) \), it becomes \( \psi^*(w) \circ w = (\rho_{\mathcal{F}_3} \circ w^\flat \circ w)^\flat = (\rho_{\mathcal{F}_3} \circ w^\flat) \circ \psi^*(w) = \psi^*(w^\flat) \circ \psi^*(w) \), hence our assertion.

Finally, for each sheaf \( \mathcal{F} \) on \( X \) with values in \( \mathcal{C} \), let \( i_{\mathcal{F}} \) be the identity morphism of \( \psi_*(\mathcal{F}) \) and denote by

\[
\sigma_{\mathcal{F}} : \psi^*(\psi_*(\mathcal{F})) \to \mathcal{F}
\]

the morphism \( (i_{\mathcal{F}})^\flat \); the formula (3.5.4.3) gives in particular the factorization

\[
(3.5.4.4) \quad u^\flat : \psi^*(\mathcal{F}) \xrightarrow{\psi^*(u)} \psi^*(\psi_*(\mathcal{F})) \xrightarrow{\sigma_{\mathcal{F}}} \mathcal{F}
\]

for each morphism \( u : \mathcal{F} \to \psi_*(\mathcal{F}) \). We say that the morphism \( \sigma_{\mathcal{F}} \) is canonical.

---

\(^7\)In the book mentioned in the introduction, we will give very general conditions on the category \( \mathcal{C} \) ensuring the existence of inverse images of presheaves with values in \( \mathcal{C} \).
(3.5.5). Let $\psi' : Y \to Z$ be a continuous map, and suppose that each presheaf $\mathcal{H}$ on $Z$ with values in $\mathcal{C}$ admits an inverse image $\psi''(\mathcal{H})$ under $\psi'$. Then (with the hypotheses of (3.5.4)) each presheaf $\mathcal{H}$ on $Z$ with values in $\mathcal{C}$ admits an inverse image under $\psi'' = \psi' \circ \psi$ and we have a canonical functorial isomorphism

$$\psi''(\mathcal{H}) \cong \psi^*(\psi'''(\mathcal{H})).$$

This indeed follows immediately from the definitions, taking into account that $\psi'' = \psi' \circ \psi$. In addition, if $u : \mathcal{I} \to \psi'_{*}(\mathcal{I})$ is a $\psi'$-morphism, $v : \mathcal{H} \to \psi'_{*}(\mathcal{I})$ a $\psi'$-morphism, and $w = \psi'_*(u) \circ v$ their composition (3.5.2), then we have immediately that $w^\sharp$ is the composite morphism

$$w^\sharp : \psi^*(\psi'''(\mathcal{H})) \xrightarrow{\psi'^\sharp(w)} \psi^*(\mathcal{I}) \xrightarrow{\psi^\sharp} \mathcal{I}.$$

(3.5.6). We take in particular for $\psi$ the identity map $1_X : X \to X$. Then if the inverse image under $\psi$ of a presheaf $\mathcal{F}$ on $X$ with values in $\mathcal{C}$ exists, we say that this inverse image is the sheaf associated to the presheaf $\mathcal{F}$. Each morphism $u : \mathcal{F} \to \mathcal{F}'$ from $\mathcal{F}$ to a sheaf $\mathcal{F}'$ with values in $\mathcal{C}$ factorizes in a unique way as $\mathcal{F} \xrightarrow{F_u} 1_X(\mathcal{F}) \xrightarrow{\mathcal{I}_u} \mathcal{F}'$.

3.6. Simple and locally simple sheaves

(3.6.1). We say that a presheaf $\mathcal{F}$ on $X$, with values in $\mathcal{C}$, is constant if the canonical morphisms $\mathcal{F}(X) \to \mathcal{F}(U)$ are isomorphisms for each nonempty open $U \subset C$; we note that $\mathcal{F}$ is not necessarily a sheaf. We say that a sheaf is simple if it is the associated sheaf (3.5.6) of a constant presheaf. We say that a sheaf $\mathcal{F}$ is locally simple if each $x \in X$ admits an open neighborhood $U$ such that $\mathcal{F}|U$ is simple.

(3.6.2). Suppose that $X$ is irreducible (2.1.1); then the following properties are equivalent:

(a) $\mathcal{F}$ is a constant presheaf on $X$;
(b) $\mathcal{F}$ is a simple sheaf on $X$;
(c) $\mathcal{F}$ is a locally simple sheaf on $X$.

Indeed, let $\mathcal{F}$ be a constant presheaf on $X$; if $U$, $V$ are two nonempty open sets in $X$, then $U \cap V$ is nonempty, so $\mathcal{F}(X) \to \mathcal{F}(U) \to \mathcal{F}(U \cap V)$ and $\mathcal{F}(X) \to \mathcal{F}(U)$ are isomorphisms, and similarly both $\mathcal{F}(V) \to \mathcal{F}(U \cap V)$ and $\mathcal{F}(V)$ are isomorphisms. We therefore conclude immediately that the axiom (F) of (3.1.2) is clearly satisfied, $\mathcal{F}$ is isomorphic to its associated sheaf, and as a result (a) implies (b).

Now let $(U_a)$ be an open cover of $X$ by nonempty open sets and let $\mathcal{F}$ be a sheaf on $X$ such that $\mathcal{F}|U_a$ is simple for each $a$; as $U_a$ is irreducible, $\mathcal{F}|U_a$ is a constant presheaf according to the above. As $U_a \cap U_b$ is not empty, $\mathcal{F}(U_a) \to \mathcal{F}(U_a \cap U_b)$ and $\mathcal{F}(U_b) \to \mathcal{F}(U_a \cap U_b)$ are isomorphisms, hence we have a canonical isomorphism $\theta_{ab} : \mathcal{F}(U_a) \to \mathcal{F}(U_b)$ for each pair of indices. But then if we apply the condition (F) for $U = X$, we see that for each index $a_0$, $\mathcal{F}(U_{a_0})$ and the $\theta_{a_0}$ are solutions to the universal problem, which (according to the uniqueness) implies that $\mathcal{F}(X) \to \mathcal{F}(U_{a_0})$ is an isomorphism, and hence proves that (c) implies (a).

3.7. Inverse images of presheaves of groups or rings

(3.7.1). We will show that when we take $\mathcal{C}$ to be the category of sets, the inverse image under $\psi$ for each presheaf $\mathcal{G}$ with values in $\mathcal{C}$ always exists (the notation and hypotheses on $X$, $Y$, $\psi$ being that of (3.5.3)). Indeed, for each open $U \subset X$, define $\psi''(U)$ as follows: an element $s'$ of $\mathcal{G}'(U)$ is a family $(s'_x)_{x \in U}$, where $s'_x \in \mathcal{G}_x(\psi(x))$ for each $x \in U$, and where, for each $x \in U$, the following condition is satisfied: there exists an open neighborhood $V$ of $\psi(x)$ in $Y$, a neighborhood $W \subset \psi^{-1}(V) \cap U$ of $x$, and an element $s \in \mathcal{G}(V)$ such that $s'_x = s_{\psi(x)}$ for all $z \in W$. We verify immediately that $U \to \psi''(U)$ clearly satisfies the axioms of a sheaf.

Now let $\mathcal{F}$ be a sheaf of sets on $X$, and let $u : \mathcal{G} \to \psi'_*(\mathcal{F})$, $v : \mathcal{G}' \to \mathcal{F}$ be morphisms. We define $u^\#$ and $v^\#$ in the following manner: if $s'$ is a section of $\mathcal{G}'$ over a neighborhood of $\psi(x)$ in $X$ and if $V$ is an open neighborhood of $\psi(x)$ and $s \in \mathcal{G}(V)$ such that we have $s'_x = s_{\psi(x)}$ for $z$ in a neighborhood of $x$ contained in $\psi^{-1}(V) \cap U$, we take $u^\#(s'_x) = u_{\psi(x)}(s_{\psi(x)})$. Similarly, if $s \in \mathcal{G}(V)$ ($V$ open in $Y$), $v^\#(s)$ is the section of $\mathcal{F}$ over $\psi^{-1}(V)$, the image under $v$ of the section $s'$ of $\mathcal{G}'$ such that $s'_x = s_{\psi(x)}$.
for all \( x \in \psi^{-1}(V) \). In addition, the canonical homomorphism (3.5.3) \( \rho : \mathcal{G} \to \psi_* (\psi^*(\mathcal{G})) \) is defined in the following manner: for each open \( V \subset Y \) and each section \( s \in \Gamma(V, \mathcal{G}) \), \( \rho(s) \) is the section \( (s_{\psi(x)})_{x \in \psi^{-1}(V)} \) of \( \psi^*(\mathcal{G}) \) over \( \psi^{-1}(V) \). The verification of the relations \( (v^2)^2 = u, (v^2)^3 = v \), and \( v^2 = \psi_*(u) \circ \rho \) is immediate, and proves our assertion.

We check that, if \( \psi : \mathcal{G}_1 \to \mathcal{G}_2 \) is a homomorphism of sheaves of sets on \( Y \), \( \psi^*(w) \) is expressed in the following manner: if \( s' = (s'_x)_{x \in U} \) is a section of \( \psi^*(\mathcal{G}_1) \) over an open set \( U \) of \( X \), then \( (\psi^*(w))(s') \) is the family \( (w_{\psi(x)}(s'_x))_{x \in U} \). Finally, it is immediate that for each open set \( V \) of \( Y \), the inverse image of \( \mathcal{G}/V \) under the restriction of \( \psi \) to \( \psi^{-1}(V) \) is identical to the induced sheaf \( \psi^*(\mathcal{G})|_{\psi^{-1}(V)} \).

When \( \psi \) is the identity \( 1_X \), we recover the definition of a sheaf of sets associated to a presheaf (G, II, 1.2). The above considerations apply without change when \( C \) is the category of sheaves of sets (or of groups, or of rings) (not necessarily commutative).

When \( X \) is any subset of a topological space \( Y \), and \( j \) the canonical injection \( X \to Y \), for each sheaf \( \mathcal{G} \) on \( Y \) with values in a category \( C \), we call the induced sheaf of \( \mathcal{G} \) by \( \mathcal{G} \) the inverse image \( j^*(\mathcal{G}) \) (whenever it exists); for the sheaves of sets (or of groups, or of rings) we recover the usual definition (G, II, 1.5).

(3.7.2). Keeping the notation and hypotheses of (3.5.3), suppose that \( \mathcal{G} \) is a sheaf of groups (resp. of rings) on \( Y \). The definition of sections of \( \psi^*(\mathcal{G}) \) (3.7.1) shows (taking into account (3.4.4)) that the homomorphism of stalks \( \psi_* \circ \rho|_x : \mathcal{G}|_x \to (\psi^*(\mathcal{G}))|_x \) is a functorial isomorphism in \( \mathcal{G} \), that identifies the two stalks; with this identification, \( u^2 \) is identical to the homomorphism defined in (3.5.1), and in particular, we have \( \text{Supp}(\psi^*(\mathcal{G})) = \psi^{-1}(\text{Supp}(\mathcal{G})) \).

An immediate consequence of this result is that the functor \( \psi^*(\mathcal{G}) \) is exact in \( \mathcal{G} \) on the abelian category of sheaves of abelian groups.

### 3.8. Sheaves on pseudo-discrete spaces

(3.8.1). Let \( X \) be a topological space whose topology admits a basis \( B \) consisting of open quasi-compact subsets. Let \( \mathcal{F} \) be a sheaf of sets on \( X \); if we equip each of the \( \mathcal{F}(U) \) with the discrete topology, \( U \mapsto \mathcal{F}(U) \) is a presheaf of topological spaces. We will see that there exists a sheaf of topological spaces \( \mathcal{F}' \) associated to \( \mathcal{F} \) (3.5.6) such that \( \Gamma(U, \mathcal{F}') \) is the discrete space \( \mathcal{F}(U) \) for each open quasi-compact subset \( U \). It will suffice to show that the presheaf \( U \mapsto \mathcal{F}(U) \) of discrete topological spaces on \( B \) satisfy the condition \( (F_0) \) of (3.2.2), and more generally that if \( U \) is an open quasi-compact subset and if \( (U_a) \) is a cover of \( U \) by sets of \( B \), then the least fine topology \( \mathcal{T} \) on \( \Gamma(U, \mathcal{F}) \) renders continuous the maps \( \Gamma(U, \mathcal{F}) \to \Gamma(U_a, \mathcal{F}) \) is the discrete topology. There exists a finite number of indices \( a_i \) such that \( U = \bigcup U_{ai} \). Let \( s \in \Gamma(U, \mathcal{F}) \) and let \( s_i \) be its image in \( \Gamma(U_{ai}, \mathcal{F}) \); the intersection of the inverse images of the sets \( \{s_i\} \) is by definition a neighborhood of \( s \) for \( \mathcal{F} \), but since \( \mathcal{F} \) is a sheaf of sets and the \( U_{ai} \) cover \( U \), this intersection is reduced to \( s \), hence our assertion.

We note that if \( U \) is an open non quasi-compact subset of \( X \), the topological space \( \Gamma(U, \mathcal{F}') \) still has \( \Gamma(U, \mathcal{T}) \) as the underlying set, but the topology is not discrete in general: it is the least fine rendering commutative the maps \( \Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F}) \), for \( V \in B \) and \( V \subset U \) (the \( \Gamma(V, \mathcal{F}) \) being discrete).

The above considerations apply without modification to sheaves of groups or of rings (not necessarily commutative), and associate to them sheaves of topological groups or topological rings, respectively. To summarize, we say that the sheaf \( \mathcal{F}' \) is the pseudo-discrete sheaf of spaces (resp. groups, rings) associated to a sheaf of sets (resp. groups, rings) \( \mathcal{F} \).

(3.8.2). Let \( \mathcal{F}, \mathcal{G} \) be two sheaves of sets (resp. groups, rings) on \( X \), \( u : \mathcal{F} \to \mathcal{G} \) a homomorphism. Then \( u \) is thus a continuous homomorphism \( \mathcal{F}' \to \mathcal{G}' \), if we denote by \( \mathcal{F}' \) and \( \mathcal{G}' \) the pseudo-discrete sheaves associated to \( \mathcal{F} \) and \( \mathcal{G} \); this follows in effect from (3.2.5).

(3.8.3). Let \( \mathcal{F} \) be a sheaf of sets, \( \mathcal{H} \) a subsheaf of \( \mathcal{F} \), \( \mathcal{F}' \) and \( \mathcal{H}' \) the pseudo-discrete sheaves associated to \( \mathcal{F} \) and \( \mathcal{H} \) respectively. Then, for each open \( U \subset X \), \( \Gamma(U, \mathcal{H}') \) is closed in \( \Gamma(U, \mathcal{F}') \); indeed, it is the intersection of the inverse images of the \( \Gamma(V, \mathcal{H}) \) (for \( V \in B, V \subset U \)) under the continuous maps \( \Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F}) \), and \( \Gamma(V, \mathcal{H}) \) is closed in the discrete space \( \Gamma(V, \mathcal{F}) \).

### §4. RINGED SPACES

4.1. Ringed spaces, sheaves of \( \mathcal{A} \)-modules, \( \mathcal{A} \)-algebras
A ringed space (resp. topologically ringed space) is a pair \((X, \mathcal{A})\) consisting of a topological space \(X\) and a sheaf of (not necessarily commutative) rings (resp. of a sheaf of topological rings) \(\mathcal{A}\) on \(X\); we say that \(X\) is the underlying topological space of the ringed space \((X, \mathcal{A})\), and \(\mathcal{A}\) the structure sheaf. The latter is denoted \(\mathcal{O}_X\), and its stalk at a point \(x \in X\) is denoted \(\mathcal{O}_{X,x}\) or simply \(\mathcal{O}_x\) when there is no chance of confusion.

We denote by \(1\) or \(e\) the unit section of \(\mathcal{O}_X\) over \(X\) (the unit element of \(\Gamma(X, \mathcal{O}_X)\)).

As in this treatise we will have to consider in particular sheaves of commutative rings, it will be understood, when we speak of a ringed space \((X, \mathcal{A})\) without specification, that \(\mathcal{A}\) is a sheaf of commutative rings.

The ringed spaces with not-necessarily-commutative structure sheaves (resp. the topologically ringed spaces) form a category, where we define a morphism \((X, \mathcal{A}) \rightarrow (Y, \mathcal{B})\) as a couple \((\psi, \theta) = \Psi\) consisting of a continuous map \(\psi: X \rightarrow Y\) and a \(\psi\)-morphism \(\theta: \mathcal{B} \rightarrow \mathcal{A}\) (3.5.1) of sheaves of rings (resp. of sheaves of topological rings); the composition of a second morphism \(\Psi' = (\psi', \theta'): (Y, \mathcal{B}) \rightarrow (Z, \mathcal{C})\) and of \(\Psi\), denoted \(\Psi'' = \Psi' \circ \Psi\), is the morphism \((\psi'', \theta'')\) where \(\psi'' = \psi' \circ \psi\), and \(\theta''\) is the composition of \(\theta\) and \(\theta'\) (equal to \(\psi'_\ast(\theta) \circ \theta'\), cf. (3.5.2)). For ringed spaces, remember that we then have \(\theta'' = \theta' \circ \psi'\ast(\theta')\) (3.5.5); therefore if \(\theta\) and \(\theta'\) are injective (resp. surjective), then the same is true of \(\theta''\), taking into account that \(\psi_x \circ \rho_{\psi(x)}\) is an isomorphism for all \(x \in X\) (3.7.2). We verify immediately, thanks to the above, that when \(\psi\) is an injective continuous map and when \(\theta\) is a surjective homomorphism of sheaves of rings, the morphism \((\psi, \theta)\) is a monomorphism (T, 1.1) in the category of ringed spaces.

By abuse of language, we will often replace \(\psi\) by \(\Psi\) in notation, for example in writing \(\Psi^{-1}(U)\) in place of \(\psi^{-1}(U)\) for a subset \(U\) of \(Y\), when the is no risk of confusion.

For each subset \(M\) of \(X\), the pair \((M, \mathcal{A}|M)\) is evidently a ringed space, said to be induced on \(M\) by the ringed space \((X, \mathcal{A})\) (and is still called the restriction of \((X, \mathcal{A})\) to \(M\)). If \(j\) is the canonical injection \(M \rightarrow X\) and \(\omega\) is the identity map of \(\mathcal{A}|M\), \((j, \omega)\) is a monomorphism \((M, \mathcal{A}|M) \rightarrow (X, \mathcal{A})\) of ringed spaces, called the canonical injection. The composition of a morphism \(\Psi: (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})\) and this injection is called the restriction of \(\Psi\) to \(M\).

We will not revisit the definitions of \(\mathcal{A}\)-modules or algebraic sheaves on a ringed space \((X, \mathcal{A})\) (G, II, 2.2); when \(\mathcal{A}\) is a sheaf of not necessarily commutative rings, by \(\mathcal{A}\)-module we will always mean “left \(\mathcal{A}\)-module” unless expressly stated otherwise. The \(\mathcal{A}\)-submodules of \(\mathcal{A}\) will be called sheaves of ideals (left, right, or two-sided) in \(\mathcal{A}\) or \(\mathcal{A}\)-ideals.

When \(\mathcal{A}\) is a sheaf of commutative rings, and in the definition of \(\mathcal{A}\)-modules, we replace everywhere the module structure by that of an algebra, we obtain the definition of an \(\mathcal{A}\)-algebra on \(X\). It is the same to say that an \(\mathcal{A}\)-algebra (not necessarily commutative) is a \(\mathcal{A}\)-module \(\mathcal{C}\), given with a homomorphism of \(\mathcal{A}\)-modules \(\phi: \mathcal{C} \otimes \mathcal{A} \mathcal{C} \rightarrow \mathcal{C}\) and a section \(e\) over \(X\), such that: 1st the diagram

\[
\begin{array}{ccc}
\mathcal{C} \otimes \mathcal{A} \mathcal{C} & \xrightarrow{\phi} & \mathcal{C} \\
\downarrow{1 \otimes \phi} & & \downarrow{\phi} \\
\mathcal{C} & \xrightarrow{\phi} & \mathcal{C}
\end{array}
\]

is commutative; 2nd for each open \(U \subset X\) and each section \(s \in \Gamma(U, \mathcal{C})\), we have \(\phi((e|U) \otimes s) = \phi(s \otimes (e|U)) = s\). We say that \(\mathcal{C}\) is a commutative \(\mathcal{A}\)-algebra if the diagram

\[
\begin{array}{ccc}
\mathcal{C} \otimes \mathcal{A} \mathcal{C} & \xrightarrow{\phi} & \mathcal{C} \\
\downarrow{\phi} & & \downarrow{\phi} \\
\mathcal{C} & \xrightarrow{\phi} & \mathcal{C}
\end{array}
\]

is commutative, \(\phi\) denoting the canonical symmetry (twist) map of the tensor product \(\mathcal{C} \otimes \mathcal{A} \mathcal{C}\).

The homomorphisms of \(\mathcal{A}\)-algebras are also defined as the homomorphisms of \(\mathcal{A}\)-modules in (G, II, 2.2), but naturally no longer form an abelian group.

If \(\mathcal{M}\) is a \(\mathcal{A}\)-submodule of an \(\mathcal{A}\)-algebra \(\mathcal{C}\), the \(\mathcal{A}\)-subalgebra of \(\mathcal{C}\) generated by \(\mathcal{M}\) is the sum of the images of the homomorphisms \(\mathcal{C}^n \mathcal{M} \rightarrow \mathcal{C}\) (for each \(n \geq 0\)). This is also the sheaf associated
to the presheaf $U \mapsto \mathcal{B}(U)$ of algebras, $\mathcal{B}(U)$ being the subalgebra of $\Gamma(U, \mathcal{C})$ generated by the submodule $\Gamma(U, \mathcal{M})$.

**4.1.4.** We say that a sheaf of rings $\mathcal{A}$ on a topological space $X$ is *reduced at a point $x$* in $X$ if the stalk $\mathcal{A}_x$ is a reduced ring (1.1.1); we say that $\mathcal{A}$ is reduced if it is reduced at all points of $X$. Recall that a ring $A$ is called *regular* if each of the local rings $A_p$ (where $p$ varies over the set of prime ideals of $A$) is a regular local ring; we will say that a sheaf of rings $\mathcal{A}$ on $X$ is regular at a point $x$ (resp. regular) if the stalk $\mathcal{A}_x$ is a regular ring (resp. if $\mathcal{A}$ is regular at each point). Finally, we will say that a sheaf of rings $\mathcal{A}$ on $X$ is *normal at a point $x$* (resp. normal) if the stalk $\mathcal{A}_x$ is an integral and integrally closed ring (resp. if $\mathcal{A}$ is normal at each point). We will say that a ringed space $(X, \mathcal{A})$ has any of these preceeding properties if the sheaf of rings $\mathcal{A}$ has that property.

A *graded* sheaf of rings $\mathcal{A}$ is by definition a sheaf of rings that is the direct sum (G, II, 2.7) of a family $(\mathcal{A}_n)_{n \in \mathbb{Z}}$ of sheaves of abelian groups with the conditions $\mathcal{A}_n \mathcal{A}_m \subset \mathcal{A}_{n+m}$; a graded $\mathcal{A}$-module is an $\mathcal{A}$-module $\mathcal{F}$ that is the direct sum of a family $(\mathcal{F}_n)_{n \in \mathbb{Z}}$ of sheaves of abelian groups, satisfying the conditions $\mathcal{A}_n \mathcal{F}_m \subset \mathcal{F}_{n+m}$. It is equivalent to say that $(\mathcal{A}_m)\mathcal{F}(\mathcal{A}_m)\mathcal{F}(\mathcal{A}_m)\mathcal{F}(\mathcal{A}_m)$ is integral and integrally closed.

**4.1.5.** Given a ringed space $(X, \mathcal{A})$ (not necessarily commutative), we will not recall here the definitions of the bifunctors $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$, $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$, and $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ (G, II, 2.8 and 2.2) in the categories of left or right (depending on the case) $\mathcal{A}$-modules, with values in the category of sheaves of abelian groups (or more generally of $\mathcal{C}$-modules, if $\mathcal{C}$ is the center of $\mathcal{A}$). The stalk $(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G})_x$ for each point $x \in X$ canonically identifies with $\mathcal{F}_x \otimes_{\mathcal{A}_x} \mathcal{G}_x$ and we define a canonical and functorial homomorphism $(\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}))_x \rightarrow \mathcal{H}om_{\mathcal{A}_x}(\mathcal{F}_x, \mathcal{G}_x)$ which is in general neither injective nor surjective. The bifunctors considered above are additive and in particular, commute with finite direct limits; $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ is right exact in $\mathcal{F}$ and in $\mathcal{A}$, commutes with inductive limits, and $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{G}$ (resp. $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$) canonically identifies with $\mathcal{A}$ (resp. $\mathcal{F}$). The functors $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ and $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ are left exact in $\mathcal{F}$ and $\mathcal{G}$; more precisely, if we have an exact sequence of the form $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{G}'$, the sequence

$$0 \rightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{F}', \mathcal{G}) \rightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) \rightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}')$$

is exact, and if we have an exact sequence of the form $\mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{G}' \rightarrow 0$, the sequence

$$0 \rightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{F}', \mathcal{G}) \rightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) \rightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{F}', \mathcal{G}')$$

is exact, with the analogous properties for the functor $\mathcal{H}om$. In addition, $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ canonically identifies with $\mathcal{G}$; finally, for each open $U \subset X$, we have

$$\Gamma(U, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})) = \mathcal{H}om_{\mathcal{A}_U}(\mathcal{F}|U, \mathcal{G}|U).$$

For each left (resp. right) $\mathcal{A}$-module, we define the dual of $\mathcal{F}$ and denote it by $\mathcal{F}^\vee$ the right (resp. left) $\mathcal{A}$-module $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{A})$.

Finally, if $\mathcal{A}$ is a sheaf of commutative rings, $\mathcal{F}$ an $\mathcal{A}$-module, $U \mapsto \wedge^p \Gamma(U, \mathcal{F})$ is a presheaf whose associated sheaf is an $\mathcal{A}$-module denoted $\wedge^p \mathcal{F}$ and is called the $p$-th exterior power of $\mathcal{F}$; we verify easily that the canonical map of the presheaf $U \mapsto \wedge^p \Gamma(U, \mathcal{F})$ to the associated sheaf $\wedge^p \mathcal{F}$ is injective, and for each $x \in X$, $(\wedge^p \mathcal{F})_x = \wedge^p \mathcal{F}_x$. It is clear that $\wedge^p \mathcal{F}$ is a covariant functor in $\mathcal{F}$.

**4.1.6.** Suppose that $\mathcal{A}$ is a sheaf of not-necessarily-commutative rings, $\mathcal{J}$ a left sheaf of ideals of $\mathcal{A}$, $\mathcal{F}$ an $\mathcal{A}$-module; we then denote by $\mathcal{J} \mathcal{F}$ the $\mathcal{A}$-submodule of $\mathcal{F}$, the image of $\mathcal{J} \otimes_{\mathcal{Z}} \mathcal{F}$ (where $\mathcal{Z}$ is the sheaf associated to the constant presheaf $U \mapsto \mathcal{Z}$) under the canonical map $\mathcal{J} \otimes_{\mathcal{A}} \mathcal{F} \rightarrow \mathcal{F}$; it is clear that for each $x \in X$, we have $(\mathcal{J} \mathcal{F})_x = \mathcal{J}_x \mathcal{F}_x$. When $\mathcal{A}$ is commutative, $\mathcal{J} \mathcal{F}$ is also the canonical image of $\mathcal{J} \otimes_{\mathcal{A}} \mathcal{F} \rightarrow \mathcal{F}$. It is immediate that $\mathcal{J} \mathcal{F}$ is also the $\mathcal{A}$-module associated to the presheaf $U \mapsto \Gamma(U, \mathcal{J})\Gamma(U, \mathcal{F})$. If $\mathcal{J}_1$, $\mathcal{J}_2$ are two left sheaves of ideals of $\mathcal{A}$, we have $\mathcal{J}_1(\mathcal{J}_2 \mathcal{F}) = (\mathcal{J}_1 \mathcal{J}_2) \mathcal{F}$.

**4.1.7.** Let $(X, \mathcal{A}_\lambda)_{\lambda \in I}$ be a family of ringed spaces; for each couple $(\lambda, \mu)$, suppose we are given an open subset $V_{\lambda\mu}$ of $X_{\lambda}$, and an isomorphism of ringed spaces $\phi_{\lambda\mu} : (V_{\lambda\mu}, \mathcal{A}_\mu|V_{\lambda\mu}) \simeq (V_{\lambda\mu}, \mathcal{A}_\lambda|V_{\lambda\mu})$, with $V_{\lambda\lambda} = X_\lambda$, $\phi_{\lambda\lambda}$ being the identity. Furthermore, suppose that, for each triple $(\lambda, \mu, \nu)$, if we denote by $\phi_{\lambda\mu}$ the restriction of $\phi_{\lambda\mu}$ to $V_{\lambda\mu} \cap V_{\lambda\nu}$, $\phi_{\lambda\mu}$ is an isomorphism from $(V_{\lambda\mu} \cap V_{\lambda\nu}, \mathcal{A}_\lambda|V_{\lambda\mu} \cap V_{\lambda\nu})$ to $(V_{\lambda\nu} \cap V_{\lambda\nu}, \mathcal{A}_\mu|V_{\lambda\nu} \cap V_{\lambda\nu})$ and that we have $\phi_{\lambda\nu} = \phi_{\lambda\mu} \circ \phi_{\mu\nu}$ (gluing condition for the $\phi_{\lambda\mu}$).
We can first consider the topological space obtained by gluing (by means of the $\phi_{\lambda\mu}$) of the $X_\lambda$ along the $V_{\lambda\mu}$; if we identify $X_\lambda$ with the corresponding open subset $X'_\lambda$ in $X$, the hypotheses imply that the three sets $V_{\lambda\mu} \cap V_{\lambda\nu}, V_{\mu\lambda} \cap V_{\lambda\nu}, V_{\lambda\nu} \cap V_{\nu\mu}$ identify with $X'_\lambda \cap X'_\mu \cap X'_\nu$. We can also transport to $X'_\lambda$ the ringed space structure of $X_\lambda$, and if $\mathcal{A}'_{\lambda}$ are the transported sheaves of rings corresponding to the $\mathcal{A}_\lambda$, the $\mathcal{A}'_{\lambda}$ satisfy the gluing condition (3.3.1) and therefore define a sheaf of rings $\mathcal{A}$ on $X$; we say that $(X, \mathcal{A})$ is the ringed space obtained by gluing the $(X_\lambda, \mathcal{A}_\lambda)$ along the $V_{\lambda\mu}$, by means of the $\phi_{\lambda\mu}$.

### 4.2. Direct image of an $\mathcal{A}$-module

(4.2.1). Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be two ringed spaces, $\Psi = (\psi, \theta)$ a morphism $(X, \mathcal{A}) \to (Y, \mathcal{B}); \psi_*(\mathcal{A})$ is then a sheaf of rings on $Y$, and $\theta$ a homomorphism $\mathcal{B} \to \psi_*(\mathcal{A})$ of sheaves of rings. Then let $\mathcal{F}$ be an $\mathcal{A}$-module; the direct image $\psi_*(\mathcal{F})$ is a sheaf of abelian groups on $Y$. In addition, for each open $U \subset Y$,

$$\Gamma(U, \psi_*(\mathcal{F})) = \Gamma(\psi^{-1}(U), \mathcal{F})$$

is equipped with the structure of a module over the ring $\Gamma(U, \psi_*(\mathcal{A})) = \Gamma(\psi^{-1}(U), \mathcal{B})$; the bilinear maps which define these structures are compatible with the restriction operations, defining on $\psi_*(\mathcal{F})$ the structure of a $\psi_*(\mathcal{A})$-module. The homomorphism $\theta : \mathcal{B} \to \psi_*(\mathcal{A})$ then defines also on $\psi_*(\mathcal{F})$ a $B$-module structure; we say that this $B$-module is the direct image of $\mathcal{F}$ under the morphism $\Psi$, and we denote it by $\Psi_*(\mathcal{F})$. If $\mathcal{F}_1, \mathcal{F}_2$ are two $\mathcal{A}$-modules over $X$ and $u$ an $\mathcal{A}$-homomorphism $\mathcal{F}_1 \to \mathcal{F}_2$, it is immediate (by considering the sections over the open subsets of $Y$) that $\psi_*(u)$ is a $\psi_*(\mathcal{A})$-homomorphism $\psi_*(\mathcal{F}_1) \to \psi_*(\mathcal{F}_2)$, and a fortiori a $B$-homomorphism $\Psi_*(\mathcal{F}_1) \to \Psi_*(\mathcal{F}_2)$; as a $B$-homomorphism, we denote it by $\Psi_*(u)$. So we see that $\Psi_*$ is a covariant functor from the category of $\mathcal{A}$-modules to that of $B$-modules. In addition, it is immediate that this functor is left exact (G, II, 2.12).

On $\psi_*(\mathcal{A})$, the structure of a $B$-module and the structure of a sheaf of rings define a $B$-algebra structure; we denote by $\Psi_*(\mathcal{A})$ this $B$-algebra.

(4.2.2). Let $\mathcal{M}, \mathcal{N}$ be two $\mathcal{A}$-modules. For each open set $U$ of $Y$, we have a canonical map

$$\Gamma(U, \psi_*(\mathcal{M})) \times \Gamma(U, \psi_*(\mathcal{N})) \to \Gamma(U, \psi_*(\mathcal{M} \otimes_\mathcal{A} \mathcal{N}))$$

which is bilinear over the ring $\Gamma(\psi^{-1}(U), \mathcal{A}) = \Gamma(U, \psi_*(\mathcal{A}))$, and a fortiori over $\Gamma(U, \mathcal{B})$; it therefore defines a homomorphism

$$\Gamma(U, \psi_*(\mathcal{M})) \otimes_{\Gamma(U, \mathcal{B})} \Gamma(U, \psi_*(\mathcal{N})) \to \Gamma(U, \psi_*(\mathcal{M} \otimes_\mathcal{A} \mathcal{N}))$$

and as we check immediately that these homomorphisms are compatible with the restriction operations, they give a canonical functorial homomorphism of $B$-modules

(4.2.2.1) $\Psi_*(\mathcal{M}) \otimes_{\Psi_*(\mathcal{A})} \Psi_*(\mathcal{N}) \to \Psi_*(\mathcal{M} \otimes_\mathcal{A} \mathcal{N})$

which is in general neither injective nor surjective. If $\mathcal{P}$ is a third $\mathcal{A}$-module, we check immediately that the diagram

(4.2.2.2) $\Psi_*(\mathcal{M}) \otimes_{\Psi_*(\mathcal{A})} \Psi_*(\mathcal{N}) \otimes_{\Psi_*(\mathcal{A})} \Psi_*(\mathcal{P}) \to \Psi_*(\mathcal{M} \otimes_\mathcal{A} \mathcal{N}) \otimes_{\Psi_*(\mathcal{A})} \Psi_*(\mathcal{P})$

is commutative.

(4.2.3). Let $\mathcal{M}, \mathcal{N}$ be two $\mathcal{A}$-modules. For each open $U \subset Y$, we have by definition that $\Gamma(\psi^{-1}(U), \mathcal{H}om_\mathcal{A}(\mathcal{M}, \mathcal{N})) = \mathcal{H}om_{\mathcal{A}|\mathcal{V}}(\mathcal{M}|\mathcal{V}, \mathcal{N}|\mathcal{V})$, where we put $\mathcal{V} = \psi^{-1}(U)$; the map $u \mapsto \psi_*(u)$ is a homomorphism

$$\mathcal{H}om_{\mathcal{A}|\mathcal{V}}(\mathcal{M}|\mathcal{V}, \mathcal{N}|\mathcal{V}) \to \mathcal{H}om_{\mathcal{B}|\mathcal{U}}(\Psi_*(\mathcal{M})|\mathcal{U}, \Psi_*(\mathcal{N})|\mathcal{U})$$

on the $\Gamma(U, \mathcal{B})$-module structures; these homomorphisms are compatible with the restriction operations, hence they define a canonical functorial homomorphism of $B$-modules

(4.2.3.1) $\Psi_*(\mathcal{H}om_\mathcal{A}(\mathcal{M}, \mathcal{N})) \to \mathcal{H}om_{\mathcal{B}}(\Psi_*(\mathcal{M}), \Psi_*(\mathcal{N}))$. 
(4.2.4). If \( C \) is an \( \mathcal{A} \)-algebra, the composite homomorphism
\[
\Psi_*(C) \otimes_{\mathcal{A}} \Psi_*(C) \rightarrow \Psi_*(C \otimes_{\mathcal{A}} C) \rightarrow \Psi_*(C)
\]
defines on \( \Psi_*(C) \) the structure of a \( \mathcal{B} \)-algebra, as a result of (4.2.2.2). We see similarly that if \( \mathcal{M} \) is a \( C \)-module, \( \Psi_*(\mathcal{M}) \) is canonically equipped with the structure of a \( \Psi_*(C) \)-module.

(4.2.5). Consider in particular the case where \( X \) is a closed subspace of \( Y \) and where \( \psi \) is the canonical injection \( j : X \rightarrow Y \). If \( \mathcal{R} = \mathcal{R}|X = j^*(\mathcal{B}) \) is the restriction of the sheaf of rings \( \mathcal{B} \) to \( X \), an \( \mathcal{A} \)-module \( \mathcal{M} \) can be considered as an \( \mathcal{R} \)-module by means of the homomorphism \( \theta^\#: \mathcal{R} \rightarrow \mathcal{A} \); then \( \Psi_*(\mathcal{M}) \) is the \( \mathcal{A} \)-module which induces \( \mathcal{M} \) on \( X \) and 0 elsewhere. If \( N \) is a second \( \mathcal{A} \)-module, \( \Psi_*(\mathcal{M}) \otimes_{\mathcal{R}} \Psi_*(N) \) canonically identifies with \( \Psi_*(\mathcal{M} \otimes_{\mathcal{R}} N) \) and \( \mathcal{M} \otimes_{\mathcal{R}} N \) with \( \Psi_*(\mathcal{M} \otimes_{\mathcal{R}} N) \).

(4.2.6). Let \((Z, C)\) be a third ringed space, \( \Psi' = (\psi', \theta') \) a morphism \((Y, \mathcal{B}) \rightarrow (Z, C)\); if \( \Psi'' \) is the composite morphism \( \Psi' \circ \Psi \), it is clear that we have \( \Psi'' = \Psi'_* \circ \Psi_* \).

4.3. Inverse image of an \( \mathcal{A} \)-module

(4.3.1). The hypotheses and notation being the same as (4.2.1), let \( \mathcal{I} \) be a \( \mathcal{B} \)-module and \( \psi^*(\mathcal{I}) \) the inverse image (3.7.1) which is therefore a sheaf of abelian groups on \( X \). The definition of sections of \( \psi^*(\mathcal{I}) \) and of \( \psi^*(\mathcal{B}) \) (3.7.1) shows that \( \psi^*(\mathcal{I}) \) is canonically equipped with a \( \psi^*(\mathcal{B}) \)-module structure. On the other hand, the homomorphism \( \phi^\#: \psi^*(\mathcal{B}) \rightarrow \mathcal{A} \) endows \( \phi^\# \) with the a \( \psi^*(\mathcal{B}) \)-module structure, which we denote by \( \phi^!(\mathcal{I}) \) when necessary to avoid confusion; the tensor product \( \psi^*(\mathcal{I}) \otimes_{\psi^*(\mathcal{B})} \phi^!(\mathcal{I}) \) is then equipped with an \( \mathcal{A} \)-module structure. We say that this \( \mathcal{A} \)-module is the inverse image of \( \phi^\# \) under the morphism \( \psi^\# \) and denote it by \( \psi^*(\mathcal{I}) \). If \( \mathcal{I}_1, \mathcal{I}_2 \) are two \( \mathcal{B} \)-modules over \( Y, \psi \) a \( \mathcal{B} \)-homomorphism \( \mathcal{I}_1 \rightarrow \mathcal{I}_2 \), then \( \psi^*(\mathcal{I}_1) \rightarrow \psi^*(\mathcal{I}_2) \), as we check immediately, is a \( \psi^*(\mathcal{B}) \)-homomorphism from \( \psi^*(\mathcal{I}_1) \) to \( \psi^*(\mathcal{I}_2) \); as a result \( \psi^*(\mathcal{I}_1) \otimes_{\psi^*(\mathcal{B})} \psi^*(\mathcal{I}_2) \) is an \( \mathcal{A} \)-homomorphism \( \psi^*(\mathcal{I}_1) \rightarrow \psi^*(\mathcal{I}_2) \), which we denote by \( \psi^*(\mathcal{I}_1) \rightarrow \psi^*(\mathcal{I}_2) \). So we define \( \psi^* \) as a covariant functor from the category of \( \mathcal{B} \)-modules to that of \( \mathcal{A} \)-modules. Here, this functor (contrary to \( \psi^\# \)) is no longer exact in general, but only right exact, the tensorization by \( \mathcal{A} \) being a right exact functor to the category of \( \psi^*(\mathcal{B}) \)-modules.

For each \( x \in X \), we have \((\psi^*(\mathcal{I}))_x = \mathcal{I}_x \otimes_{\mathcal{B}} \mathcal{A}, \mathcal{A}_x \), according to (3.7.2). The support of \( \psi^*(\mathcal{I}) \) is thus contained in \( \psi^{-1}(\text{Supp}(\mathcal{I})) \).

(4.3.2). Let \( (\mathcal{I}_i) \) be an inductive system of \( \mathcal{B} \)-modules, and let \( \mathcal{I} = \lim \mathcal{I}_i \) be its inductive limit. The canonical homomorphisms \( \mathcal{I}_i \rightarrow \mathcal{I} \) define the \( \psi^*(\mathcal{B}) \)-homomorphisms \( \psi^*(\mathcal{I}) \rightarrow \psi^*(\mathcal{I}_i) \), which give a canonical homomorphism \( \lim \psi^*(\mathcal{I}_i) \rightarrow \psi^*(\mathcal{I}) \). As the stalk at a point of an inductive limit of sheaves is the inductive limit of the stalks at the same point (G, II, 1.11), the preceding canonical homomorphism is bijective (3.7.2). In addition, the tensor product commutes with inductive limits of sheaves, and we thus have a canonical functorial isomorphism \( \lim \psi^*(\mathcal{I}_i) \simeq \psi^*(\lim \mathcal{I}_i) \) of \( \mathcal{A} \)-modules.

On the other hand, for a finite direct sum \( \bigoplus \mathcal{I}_i \) of \( \mathcal{B} \)-modules, it is clear that \( \psi^*(\bigoplus \mathcal{I}_i) = \bigoplus \psi^*(\mathcal{I}_i) \), therefore, by tensoring with \( \mathcal{A} \),
\[
\psi^*(\bigoplus \mathcal{I}_i) = \bigoplus \psi^*(\mathcal{I}_i).
\]

By passing to the inductive limit, we deduce, in light of the above, that the above equality is still true for any direct sum.

(4.3.3). Let \( \mathcal{I}_1, \mathcal{I}_2 \) be two \( \mathcal{B} \)-modules; from the definition of the inverse images of sheaves of abelian groups (3.7.1), we obtain immediately a canonical homomorphism \( \psi^*(\mathcal{I}_1) \otimes_{\psi^*(\mathcal{B})} \psi^*(\mathcal{I}_2) \rightarrow \psi^*(\mathcal{I}_1 \otimes_{\mathcal{B}} \mathcal{I}_2) \) of \( \psi^*(\mathcal{B}) \)-modules, and the stalk at a point of a tensor product of sheaves being the tensor product of the stalks at this point (G, II, 2.8), we deduce from (3.7.2) that the above homomorphism is in fact a isomorphism. By tensoring with \( \mathcal{A} \), we obtain a canonical functorial isomorphism
\[
\psi^*(\mathcal{I}_1) \otimes_{\mathcal{A}} \psi^*(\mathcal{I}_2) \simeq \psi^*(\mathcal{I}_1 \otimes_{\mathcal{B}} \mathcal{I}_2).
\]

(4.3.4). Let \( \mathcal{C} \) be a \( \mathcal{B} \)-algebra; the data of the algebra structure on \( \mathcal{C} \) is the same as the data of a \( \mathcal{B} \)-homomorphism \( \mathcal{C} \otimes_{\mathcal{B}} \mathcal{C} \rightarrow \mathcal{C} \) satisfying the associativity and commutativity conditions (conditions which are checked stalk-wise); the above isomorphism allows us to consider this homomorphism
as a homomorphism of $\mathcal{A}$-modules $\Psi^*(\mathcal{C}) \otimes_{\mathcal{A}} \Psi^*(\mathcal{C}) \to \Psi^*(\mathcal{C})$ satisfying the same conditions, so $\Psi^*(\mathcal{C})$ is thus equipped with an $\mathcal{A}$-algebra structure. In particular, it follows immediately from the definitions that the $\mathcal{A}$-algebra $\Psi^*(\mathcal{B})$ is equal to $\mathcal{A}$ (up to a canonical isomorphism).

Similarly, if $\mathcal{M}$ is a $\mathcal{C}$-module, the data of this module structure is the same as that of a $\mathcal{B}$-homomorphism $\mathcal{C} \otimes_{\mathcal{B}} \mathcal{M} \to \mathcal{M}$ satisfying the associativity condition; hence we give a $\Psi^*(\mathcal{C})$-module structure on $\Psi^*(\mathcal{M})$.

**4.3.5.** Let $\mathcal{J}$ be a sheaf of ideals of $\mathcal{B}$; as the functor $\psi^*$ is exact, the $\psi^*(\mathcal{B})$-module $\psi^*(\mathcal{J})$ canonically identifies with a sheaf of ideals of $\psi^*(\mathcal{B})$; the canonical injection $\psi^*(\mathcal{J}) \to \psi^*(\mathcal{B})$ then gives a homomorphism of $\mathcal{A}$-modules $\Psi^*(\mathcal{J}) = \psi^*(\mathcal{J}) \otimes_{\psi^*(\mathcal{B})} \mathcal{A}[\mathcal{J}] \to \mathcal{A}$; we denote by $\Psi^*(\mathcal{J})[\mathcal{J}]$, or $\mathcal{J} \mathcal{A}$ if there is no fear of confusion, the image of $\Psi^*(\mathcal{J})$ under this homomorphism. So we have by definition $\mathcal{J} \mathcal{A} = \mathcal{J}^1((\psi^*(\mathcal{J}))[\mathcal{J}]$ and in particular, for each $x \in X$, $(\mathcal{J} \mathcal{A})_x = \mathcal{J}^1((\psi^*(\mathcal{J}))[\mathcal{J}]$ taking into account the canonical identification between the stalks of $\psi^*(\mathcal{J})$ and those of $\mathcal{J}$ (3.7.2). If $\mathcal{J}_1, \mathcal{J}_2$ are two sheaves of ideals of $\mathcal{B}$, then we have $(\mathcal{J}_1 \mathcal{J}_2) \mathcal{A} = \mathcal{J}_1((\mathcal{J}_2 \mathcal{A}) = (\mathcal{J}_1 \mathcal{A})((\mathcal{J}_2 \mathcal{A})$. If $\mathcal{F}$ is an $\mathcal{A}$-module, we set $\mathcal{J} \mathcal{F} = (\mathcal{J} \mathcal{A})[\mathcal{J}]$.

**4.3.6.** Let $(Z, \mathcal{C})$ be a third ringed space, $\Psi' = (\psi', \theta')$ a morphism $(Y, \mathcal{B}) \to (Z, \mathcal{C})$; if $\Psi''$ is the composite morphism $\Psi' \circ \Psi$, it follows from the definition (4.3.1) and from (4.3.3.1) that we have $\Psi'' = \Psi' \circ \Psi^*$.

**4.4. Relation between direct and inverse images**

**4.4.1.** The hypotheses and notation being the same as in (4.2.1), let $\mathcal{F}$ be a $\mathcal{B}$-module. By definition, a homomorphism $u : \mathcal{F} \to \Psi_*(\mathcal{F})$ of $\mathcal{B}$-modules is still called a $\Psi$-morphisms from $\mathcal{F}$ to $\mathcal{F}$, or simply a homomorphism from $\mathcal{F}$ to $\mathcal{F}$ and we write it as $u : \mathcal{F} \to \mathcal{F}$ when no confusion will occur. To give such a homomorphism is the same as giving, for each pair $(U, V)$ where $U$ is an open set of $X$, $V$ an open set of $Y$ such that $\psi(U) \subset V$, a homomorphism $u_{U,V} : \Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F})$ of $\Gamma(U, \mathcal{B})$-modules, $\Gamma(U, \mathcal{F})$ being considered as a $\Gamma(U, \mathcal{B})$-module by means of the ring homomorphism $\theta_{U,V} : \Gamma(V, \mathcal{B}) \to \Gamma(U, \mathcal{B})$; the $u_{U,V}$ must in addition render commutative the diagrams (3.5.1.1). It suffices, however, to define $u$ by the data of the $u_{U,V}$ when $U$ (resp. $V$) varies over a basis $\mathcal{B}$ (resp. $\mathcal{B}'$) for the topology of $X$ (resp. $Y$) and to check the commutativity of (3.5.1.1) for these restrictions.

**4.4.2.** Under the hypotheses of (4.2.1) and (4.2.6), let $\mathcal{H}$ be a $\mathcal{C}$-module, $v : \mathcal{H} \to \Psi'_*(\mathcal{F})$ a $\Psi'$-morphism; then $w : \mathcal{H} \to \Psi'_*(\mathcal{F}) \to \Psi'_*(\Psi_*(\mathcal{F}))$ is a $\Psi''$-morphism which we call the composition of $u$ and $v$.

**4.4.3.** We will now show that we can define a canonical isomorphism of bifunctors in $\mathcal{F}$ and $\mathcal{G}$.

**Homorphism** (4.4.3.1) $\operatorname{Hom}_{\mathcal{A}}(\Psi^*(\Psi_*(\mathcal{F})), \mathcal{F}) \cong \operatorname{Hom}_{\mathcal{A}}((\Psi_*(\mathcal{F})), \Psi_*(\mathcal{F}))$

which we denote by $v \mapsto v^\theta_\Psi$ (or simply $v \mapsto v^\theta$ if there is no chance of confusion); we denote by $u^\theta_\Psi$ and $u^\theta_{\Psi_1}$, the inverse isomorphism. This definition is the following: by composing $v : \Psi^*(\mathcal{F}) \to \mathcal{F}$ with the canonical map $\psi^*(\mathcal{F}) \to \Psi^*(\mathcal{F})$, we obtain a homomorphism of sheaves of groups $v^\theta : \psi^*(\mathcal{F}) \to \mathcal{F}$, which is also a homomorphism of $\Psi^*(\mathcal{B})$-modules. We obtain (3.7.1) a homomorphism $v^\theta : \mathcal{F} \to \psi^*(\mathcal{F}) = \Psi_*(\mathcal{F})$, which is also a homomorphism of $\mathcal{B}$-modules as we check easily; we take $v^\theta_\Psi = v^\theta$. Similarly, for $u : \mathcal{F} \to \Psi_*(\mathcal{F})$, which is a homomorphism of $\mathcal{B}$-modules, we obtain (3.7.1) a homomorphism $u^\theta : \psi^*(\mathcal{F}) \to \mathcal{F}$ of $\psi^*(\mathcal{B})$-modules, hence by tensoring with $\mathcal{A}$ we have a homomorphism of $\mathcal{A}$-modules $\Psi^*(\mathcal{F}) \to \mathcal{F}$, which we denote by $u^\theta_\Psi$. It is immediate to check that $(u^\theta_\Psi)^\theta = u$ and $((v^\theta_\Psi)^\theta = v$, so we have established the functorial nature in $\mathcal{F}$ of the isomorphism $v \mapsto v^\theta_\Psi$. The functorial nature in $\mathcal{F}$ of $u \mapsto u^\theta_\Psi$ is then formally shown as in (3.5.4) (reasoning that would also prove the functorial nature of $\Psi^*$ established in (4.3.1) directly).

If we take for $v$ the identity homomorphism of $\Psi^*(\mathcal{B})$, $v^\theta_\Psi$ is a homomorphism

**4.4.3.2** $\rho_\Psi : \mathcal{F} \to \Psi_*(\Psi^*(\mathcal{F}));$

if we take for $u$ the identity homomorphism of $\Psi_*(\mathcal{F})$, $u^\theta_\Psi$ is a homomorphism

**4.4.3.3** $\sigma_\Psi : \Psi^*(\Psi_*(\mathcal{F})) \to \mathcal{F};$
We denote by $u$ (4.4.7).

Let $is a homomorphism of $\mathcal{A}$ and these canonical morphisms are functorial in $v$.

Form an inductive system of homomorphisms (4.2.2.1).

The homomorphism $u_x$ is obtained also by passing to the inductive limit relative to the homomorphisms $\Gamma(V, \mathcal{G}) \to \Gamma(\psi^{-1}(V), \mathcal{F}) \to \mathcal{F}_x$, where $V$ varies over the neighborhoods of $\psi(x)$.

(4.4.4). Let $\mathcal{F}_1, \mathcal{F}_2$ be $\mathcal{A}$-modules, $\mathcal{F}_1, \mathcal{F}_2$ be $\mathcal{B}$-modules, $u_i (i = 1, 2)$ a homomorphism from $\mathcal{F}_i$ to $\mathcal{F}_i$. We denote by $u_1 \otimes u_2$ the homomorphism $u : \mathcal{F}_1 \otimes \mathcal{F}_2 \to \mathcal{F}_1 \otimes \mathcal{F}_2$ such that $u^i = (u_1)^i \otimes (u_2)^i$ (taking into account (4.3.3.1)); we check that $u$ is also the composition $\mathcal{F}_1 \otimes \mathcal{F}_2 \to \Psi_s(\mathcal{F}_1) \otimes B \Psi_s(\mathcal{F}_2)$, where the first arrow is the ordinary tensor product $u_1 \otimes u_2$ and the second is the canonical homomorphism (4.2.2.1).

(4.4.5). Let $(\mathcal{F}_\lambda)_{\lambda \in L}$ be an inductive system of $\mathcal{B}$-modules, and, for each $\lambda \in L$, let $u_\lambda$ be a homomorphism $\mathcal{F}_\lambda \to \Psi_s(\mathcal{F})$, form an inductive limit; we put $\mathcal{F} = \lim \mathcal{F}_\lambda$ and $u = \lim u_\lambda$; then the $(u_\lambda)^i$ form an inductive system of homomorphisms $\Psi^s(\mathcal{F}_\lambda) \to \mathcal{F}$, and the inductive limit of this system is none other than $u^i$.

(4.4.6). Let $\mathcal{M}, \mathcal{N}$ be two $\mathcal{B}$-modules, $V$ an open set of $Y, U = \psi^{-1}(V)$; the map $v \to \Psi^s(v)$ is a homomorphism

$$\text{Hom}_{\mathcal{B}|V}(\mathcal{M}|V, \mathcal{N}|V) \to \text{Hom}_{\mathcal{B}|U}(\Psi^s(\mathcal{M}|U, \Psi^s(\mathcal{N}|U|U)$$

for the $\Gamma(V, \mathcal{B})$-module structures $(\text{Hom}_{\mathcal{B}|U}(\Psi^s(\mathcal{M}|U, \Psi^s(\mathcal{N}|U))$ is normaly equipped with the a $\Gamma(U, \Psi^s(\mathcal{B}))$-module structure, and thanks to the canonical homomorphism (3.7.2) $\Gamma(V, \mathcal{B}) \to \Gamma(U, \Psi^s(\mathcal{B}))$, it is also a $\Gamma(V, \mathcal{B})$-module. We see immediately that these homomorphisms are compatible with the restriction morphisms, and as a result define a canonical functorial homomorphism

$$\gamma : \text{Hom}_{\mathcal{B}}(\mathcal{M}, \mathcal{N}) \to \Psi_s(\text{Hom}_{\mathcal{B}}(\Psi^s(\mathcal{M}), \Psi^s(\mathcal{N}))$$

it also corresponds to this homomorphism the homomorphism

$$\gamma^2 : \Psi^s(\text{Hom}_{\mathcal{B}}(\mathcal{M}, \mathcal{N})) \to \text{Hom}_{\mathcal{B}}(\Psi^s(\mathcal{M}), \Psi^s(\mathcal{N}))$$

and these canonical morphisms are functorial in $\mathcal{M}$ and $\mathcal{N}$.

(4.4.7). Suppose that $\mathcal{F}$ (resp. $\mathcal{G}$) is an $\mathcal{A}$-algebra (resp. a $\mathcal{B}$-algebra). If $u : \mathcal{G} \to \Psi_s(\mathcal{F})$ is a homomorphism of $\mathcal{B}$-algebras, $u^i$ is a homomorphism of $\mathcal{B}$-algebras, this follows from the commutativity of the diagram

$\begin{array}{ccc}
\mathcal{G} \otimes \mathcal{B} \mathcal{G} & \longrightarrow & \mathcal{G} \\
\downarrow & & \downarrow u \\
\Psi_s(\mathcal{F} \otimes \mathcal{B} \mathcal{F}) & \longrightarrow & \Psi_s(\mathcal{F})
\end{array}$

and from (4.4.4). Similarly, if $v : \Psi^s(\mathcal{G}) \to \mathcal{F}$ is a homomorphism of $\mathcal{A}$-algebras, $v^i : \mathcal{G} \to \Psi_s(\mathcal{F})$ is a homomorphism of $\mathcal{B}$-algebras.

(4.4.8). Let $(Z, \mathcal{C})$ be a third ringed space, $\mathcal{F}' = (\psi', \theta')$ a morphism $(Y, \mathcal{G}) \to (Z, \mathcal{C})$, and $\Psi'' : (X, \mathcal{F}) \to (Z, \mathcal{C})$ the composite morphism $\Psi' \circ \Psi$. Let $\mathcal{H}$ be a $\mathcal{C}$-module, $\psi'$ a homomorphism from $\mathcal{H}$ to $\mathcal{F}$; the composition $v'' = v \circ \psi'$ is by definition the homomorphism from $\mathcal{H}$ to $\mathcal{F}$ defined by

$$\begin{array}{ccc}
\mathcal{H} & \xrightarrow{v'} & \Psi''(\mathcal{G}) \\
\downarrow & & \downarrow \\
\Psi_s((\mathcal{H})) & \longrightarrow & \Psi_s(\Psi_s(\mathcal{F}));
\end{array}$$

we check that $v''$ is the homomorphism

$$\begin{array}{c}
\Psi^s(\Psi^s(\mathcal{H})) \xrightarrow{\Psi^s(v')} \Psi^s(\mathcal{G}) \xrightarrow{v''} \mathcal{F}.
\end{array}$$
§5. QUASI-COHERENT AND COHERENT SHEAVES

5.1. Quasi-coherent sheaves

(5.1.1). Let \( (X, \mathcal{O}_X) \) be a ringed space, \( \mathcal{F} \) an \( \mathcal{O}_X \)-module. The data of a homomorphism \( u : \mathcal{O}_X \to \mathcal{F} \) of \( \mathcal{O}_X \)-modules is equivalent to that of the section \( s = u(1) \in \Gamma(X, \mathcal{F}) \). Indeed, when \( s \) is given, for each section \( t \in \Gamma(U, \mathcal{O}_X) \), we necessarily have \( u(t) = t \cdot (s|U) \); we say that \( u \) is defined by the section \( s \). If now \( I \) is any set of indices, consider the direct sum sheaf \( \mathcal{O}_X^{(I)} \), and for each \( i \in I \), let \( h_i \) be the canonical injection of the \( i \)-th factor into \( \mathcal{O}_X^{(I)} \); we know that \( u \mapsto (u \circ h_i) \) is an isomorphism from \( \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{(I)}, \mathcal{F}) \) to the product \( (\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}))^I \). So there is a canonical one-to-one correspondence between the homomorphisms \( u : \mathcal{O}_X^{(I)} \to \mathcal{F} \) and the families of sections \( (s_i)_{i \in I} \) of \( \mathcal{F} \) over \( X \). The homomorphism \( u \) corresponding to \( (s_i) \) sends an element \( (a_i) \in (\Gamma(U, \mathcal{O}_X))^{(I)} \) to \( \sum_{i \in I} a_i \cdot (s_i|U) \).

We say that \( \mathcal{F} \) is generated by the family \( (s_i) \) if the homomorphism \( \mathcal{O}_X^{(I)} \to \mathcal{F} \) defined for each family is surjective (in other words, if, for each \( x \in X \), \( \mathcal{F}_x \) is an \( \mathcal{O}_x \)-module generated by the \( (s_i)_x \)). We say that \( \mathcal{F} \) is generated by its sections over \( X \) if it is generated by the family of all these sections (or by a subfamily), in other words, if there exists a surjective homomorphism \( \mathcal{O}_X^{(I)} \to \mathcal{F} \) for a suitable \( I \).

We note that an \( \mathcal{O}_X \)-module \( \mathcal{F} \) can be such that there exists a point \( x_0 \in X \) for which \( \mathcal{F}|U \) is not generated by its sections over \( U \), regardless of the choice of neighborhood \( U \) of \( x_0 \): it suffices to take \( X = \mathbb{R} \), for \( \mathcal{O}_X \) the simple sheaf \( \mathbb{Z} \), for \( \mathcal{F} \) the algebraic subsheaf of \( \mathcal{O}_X \) such that \( \mathcal{F}_0 = \{0\} \), \( \mathcal{F}_x = \mathbb{Z} \) for \( x \neq 0 \), and finally \( x_0 = 0 \): the only section of \( \mathcal{F}|U \) over \( U \) is 0 for a neighborhood \( U \) of 0.

(5.1.2). Let \( f : X \to Y \) be a morphism of ringed spaces. If \( \mathcal{F} \) is a \( \mathcal{O}_X \)-module generated by its sections over \( X \), then the canonical homomorphism \( f^* (f_*(\mathcal{F})) \to \mathcal{F} \) (4.4.3.3) is surjective; indeed, with the notation of (5.1.1), \( s_i \otimes 1 \) is a section of \( f^* (f_*(\mathcal{F})) \) over \( X \), and its image in \( \mathcal{F} \) is \( s_i \). The example in (5.1.1) where \( f \) is the identity shows that the inverse of this proposition is false in general.

If \( \mathcal{G} \) is an \( \mathcal{O}_Y \)-module generated by its sections over \( Y \), then \( f^* (\mathcal{G}) \) is generated by its sections over \( X \), since \( f^* \) is a right exact functor.

(5.1.3). We say that an \( \mathcal{O}_X \)-module \( \mathcal{F} \) is quasi-coherent if for each \( x \in X \) there is an open neighborhood \( U \) of \( x \) such that \( \mathcal{F}|U \) is isomorphic to the cokernel of a homomorphism of the form \( \mathcal{O}_X^{(I)}|U \to \mathcal{O}_X^{(J)}|U \), where \( I \) and \( J \) are sets of arbitrary indices. It is clear that \( \mathcal{O}_X \) is itself a quasi-coherent \( \mathcal{O}_X \)-module, and that any direct sum of quasi-coherent \( \mathcal{O}_X \)-modules is again a quasi-coherent \( \mathcal{O}_X \)-module. We say that an \( \mathcal{O}_X \)-algebra \( \mathcal{A} \) is quasi-coherent if it is quasi-coherent as an \( \mathcal{O}_X \)-module.

(5.1.4). Let \( f : X \to Y \) be a morphism of ringed spaces. If \( \mathcal{G} \) is a quasi-coherent \( \mathcal{O}_Y \)-module, then \( f^* (\mathcal{G}) \) is a quasi-coherent \( \mathcal{O}_X \)-module. Indeed, for each \( x \in X \), there is an open neighborhood \( V \) of \( f(x) \) in \( Y \) such that \( \mathcal{G}|V \) is the cokernel of a homomorphism \( \mathcal{O}_Y^{(I)}|V \to \mathcal{O}_Y^{(J)}|V \). If \( U = f^{-1}(V) \), and if \( f_U \) is the restriction of \( f \) to \( U \), then we have \( f^* (\mathcal{G})|U = f_U^*(\mathcal{G}|V) \); as \( f_U^* \) is right exact and commutes with direct sums, \( f_U^*(\mathcal{G}|V) \) is the cokernel of a homomorphism \( \mathcal{O}_X^{(I)}|U \to \mathcal{O}_X^{(J)}|U \).

5.2. Sheaves of finite type

(5.2.1). We say that an \( \mathcal{O}_X \)-module \( \mathcal{F} \) is of finite type if for each \( x \in X \) there exists an open neighborhood \( U \) of \( x \) such that \( \mathcal{F}|U \) is generated by a finite family of sections over \( U \), or if it is isomorphic to a sheaf quotient of a sheaf of the form \( (\mathcal{O}_X|U)^p \) where \( p \) is finite. Each sheaf quotient of a sheaf of finite type is again a sheaf of finite type, as well as each finite direct sum and each finite tensor product of sheaves of finite type. An \( \mathcal{O}_X \)-module of finite type is not necessarily quasi-coherent, as we can see for the \( \mathcal{O}_X \)-module \( \mathcal{O}_X/\mathcal{F} \), where \( \mathcal{F} \) is the example in (5.1.1). If \( \mathcal{F} \) is of finite type, then \( \mathcal{F}_x \) is a \( \mathcal{O}_x \)-module of finite type for each \( x \in X \), but the example in (5.1.1) shows that this condition is necessary but not sufficient in general.

(5.2.2). Let \( \mathcal{F} \) be an \( \mathcal{O}_X \)-module of finite type. If \( s_i \) \((1 \leq i \leq n)\) are the sections of \( \mathcal{F} \) over an open neighborhood \( U \) of a point \( x \in X \) and the \( (s_i)_x \) generate \( \mathcal{F}_x \), then there exists an open neighborhood \( V \subset U \) of \( x \) such that the \( (s_i)_y \) generate \( \mathcal{F}_y \) for all \( y \in Y \) (FAC, I, 2, 12, prop. 1). In particular, we conclude that the support of \( \mathcal{F} \) is closed.
Similarly, if \( u : \mathcal{F} \to \mathcal{G} \) is a homomorphism such that \( u_x = 0 \), then there exists a neighborhood \( U \) of \( x \) such that \( u_y = 0 \) for all \( y \in U \).

**5.2.3.** Suppose that \( X \) is quasi-compact, and let \( \mathcal{F} \) and \( \mathcal{G} \) be two \( O_X \)-modules such that \( \mathcal{G} \) is of finite type, \( u : \mathcal{F} \to \mathcal{G} \) a surjective homomorphism. In addition, suppose that \( \mathcal{F} \) is the inductive limit of an inductive system \((\mathcal{F}_\lambda)\). Then there exists an index \( \mu \) such that the homomorphism \( \mathcal{F}_\mu \to \mathcal{G} \) is surjective. Indeed, for each \( x \in X \), there exists a finite system of sections \( s_i \) of \( \mathcal{G} \) over an open neighborhood \( U(x) \) of \( x \) such that the \( (s_i) \), generate \( \mathcal{G} \) for all \( y \in U(x) \); there is then an open neighborhood \( V(x) \subset U(x) \) of \( x \) and \( n \) sections \( t_i \) of \( \mathcal{F} \) over \( V(x) \) such that \( s_i | V(s) = u(t_i) \) for all \( i \); we can also suppose that the \( t_i \) are the canonical images of sections of a similar sheaf \( \mathcal{F}_\lambda(x) \) over \( V(x) \). We then cover \( X \) with a finite number of neighborhoods \( V(x_k) \), and let \( \mu \) be the maximal index of the \( \lambda(x_k) \); it is clear that this index gives the answer.

Suppose still that \( X \) is quasi-compact, and let \( \mathcal{F} \) be an \( O_X \)-module of finite type generated by its sections over \( X \) (5.1.1); then \( \mathcal{F} \) is generated by a finite subfamily of these sections: indeed, it suffices to cover \( X \) by a finite number of open neighborhoods \( U_k \) such that, for each \( k \), there is a finite number of sections \( s_{ik} \) of \( \mathcal{F} \) over \( X \) whose restrictions to \( U_k \) generate \( \mathcal{F}|U_k \); it is clear that the \( s_{ik} \) then generate \( \mathcal{F} \).

(5.2.4). Let \( f : X \to Y \) be a morphism of ringed spaces. If \( \mathcal{G} \) is an \( \mathcal{O}_Y \)-module of finite type, then \( f^*(\mathcal{G}) \) is an \( \mathcal{O}_X \)-module of finite type. Indeed, for each \( x \in X \), there is an open neighborhood \( V \) of \( f(x) \) in \( Y \) and a surjective homomorphism \( v : \mathcal{O}_Y|V \to \mathcal{G}|V \). If \( U = f^{-1}(V) \) and if \( f|U \) is the restriction of \( f \) to \( U \), then we have \( f^*(\mathcal{G})|U = f^*_U(\mathcal{G}|V) \); since \( f^*_U \) is right exact (4.3.1) and commutes with direct sums (4.3.2), \( f^*_U(v) \) is a surjective homomorphism \( \mathcal{O}_X|U \to f^*(\mathcal{G})|U \).

(5.2.5). We say that an \( \mathcal{O}_X \)-module \( \mathcal{F} \) admits a finite presentation if for each \( x \in X \) there exists an open neighborhood \( U \) of \( x \) such that \( \mathcal{F}|U \) is isomorphic to a cokernel of a \( (\mathcal{O}_X|U) \)-homomorphism \( \mathcal{O}_X^p|U \to \mathcal{O}_X^q|U \), \( p \) and \( q \) being two integers \( > 0 \). Such an \( \mathcal{O}_X \)-module is therefore of finite type and quasi-coherent. If \( f : X \to Y \) is a morphism of ringed spaces, and if \( \mathcal{G} \) is an \( \mathcal{O}_Y \)-module admitting a finite presentation, then \( f^*(\mathcal{G}) \) admits a finite presentation, as shown in the argument of (5.1.4).

(5.2.6). Let \( \mathcal{F} \) be an \( \mathcal{O}_X \)-module admitting a finite presentation (5.2.5); then, for each \( \mathcal{O}_X \)-module \( \mathcal{H} \), the canonical functorial homomorphism

\[
(\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}))_x \longrightarrow \text{Hom}_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{H}_x)
\]

is bijective (T, 4.1.1).

(5.2.7). Let \( \mathcal{F} \) and \( \mathcal{G} \) be two \( \mathcal{O}_X \)-modules admitting a finite presentation. If for some \( x \in X \), \( \mathcal{F}_x \) and \( \mathcal{G}_x \) are isomorphic as \( \mathcal{O}_x \)-modules, then there exists an open neighborhood \( U \) of \( x \) such that \( \mathcal{F}|U \) and \( \mathcal{G}|U \) are isomorphic. Indeed, if \( \phi : \mathcal{F}_x \to \mathcal{G}_x \) and \( \psi : \mathcal{G}_x \to \mathcal{F}_x \) are an isomorphism and its inverse isomorphism, then there exists, according to (5.2.6), an open neighborhood \( V \) of \( x \) and a section \( u \) (resp. \( v \)) of \( \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \) (resp. \( \text{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}) \)) over \( V \) such that \( u_x = \phi \) (resp. \( v_x = \psi \)). As \( (u \circ v)_x \) and \( (v \circ u)_x \) are the identity automorphisms, there exists an open neighborhood \( U \subset V \) of \( x \) such that \( (u \circ v)|U \) and \( (v \circ u)|U \) are the identity automorphisms, hence the proposition.

### 5.3. Coherent sheaves

(5.3.1). We say that an \( \mathcal{O}_X \)-module \( \mathcal{F} \) is coherent if it satisfies the two following conditions:

(a) \( \mathcal{F} \) is of finite type.

(b) for each open \( U \subset X \), integer \( n > 0 \), and homomorphism \( u : \mathcal{O}_X^n|U \to \mathcal{F}|U \), the kernel of \( u \) is of finite type.

We note that these two conditions are of a local nature.

For most of the proofs of the properties of coherent sheaves in what follows, cf. (FAC, I, 2).

(5.3.2). Each coherent \( \mathcal{O}_X \)-module admits a finite presentation (5.2.5); the inverse is not necessarily true, since \( \mathcal{O}_X \) itself is not necessarily a coherent \( \mathcal{O}_X \)-module.

Each \( \mathcal{O}_X \)-submodule of finite type of a coherent \( \mathcal{O}_X \)-module is coherent; each finite direct sum of coherent \( \mathcal{O}_X \)-modules is a coherent \( \mathcal{O}_X \)-module.

(5.3.3). If \( 0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \) is an exact sequence of \( \mathcal{O}_X \)-modules and if two of these \( \mathcal{O}_X \)-modules are coherent, then so is the third.
(5.3.4). If $\mathcal{F}$ and $\mathcal{G}$ are two coherent $\mathcal{O}_X$-modules, $u : \mathcal{F} \to \mathcal{G}$ a homomorphism, then $\text{Im}(u)$, $\text{Ker}(u)$, and $\text{Coker}(u)$ are coherent $\mathcal{O}_X$-modules. In particular, if $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-submodules of a coherent $\mathcal{O}_X$-module, then $\mathcal{F} + \mathcal{G}$ and $\mathcal{F} \cap \mathcal{G}$ are coherent.

If $\mathcal{A} \to \mathcal{B} \to \mathcal{C} \to \mathcal{D} \to \mathcal{E}$ is an exact sequence of $\mathcal{O}_X$-modules, and if $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$, $\mathcal{E}$ are coherent, then $\mathcal{C}$ is coherent.

(5.3.5). If $\mathcal{F}$ and $\mathcal{G}$ are two coherent $\mathcal{O}_X$-modules, then so are $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ are $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

(5.3.6). Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, $\mathcal{J}$ a coherent sheaf of ideals of $\mathcal{O}_X$. Then the $\mathcal{O}_X$-module $\mathcal{J} \mathcal{F}$ is coherent, as the image of $\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F}$ under the canonical homomorphism $\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$ ((5.3.4) and (5.3.5)).

(5.3.7). We say that an $\mathcal{O}_X$-algebra $\mathcal{A}$ is coherent if it is coherent as an $\mathcal{O}_X$-module. In particular, $\mathcal{O}_X$ is a coherent sheaf of rings if and only if for each open $U \subset X$ and each homomorphism of the form $u : \mathcal{O}_X^p|U \to \mathcal{O}_X^q|U$, the kernel of $u$ is an $\mathcal{O}_X|U$-module of finite type.

If $\mathcal{O}_X$ is a coherent sheaf of rings, then each $\mathcal{O}_X$-module $\mathcal{F}$ admitting a finite presentation (5.2.5) is coherent, according to (5.3.4).

The annihilator of an $\mathcal{O}_X$-module $\mathcal{F}$ is the kernel $\mathcal{J}$ of the canonical homomorphism $\mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ which sends each section $s \in \Gamma(U, \mathcal{O}_X)$ to the multiplication by $s$ map in $\text{Hom}(\mathcal{F}|U, \mathcal{F}|U)$; if $\mathcal{O}_X$ is coherent and if $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $\mathcal{J}$ is coherent ((5.3.4) and (5.3.5)) and for each $x \in X$, $\mathcal{J}_x$ is the annihilator of $\mathcal{F}_x$ (5.2.6).

(5.3.8). Suppose that $\mathcal{O}_X$ is coherent; let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, $x$ a point of $X$, $M$ a submodule of finite type of $\mathcal{F}_x$; then there exists an open neighborhood $U$ of $x$ and a coherent $\mathcal{O}_X|U$-submodule $\mathcal{G}$ of $\mathcal{F}|U$ such that $\mathcal{G}_x = M$ (T, 4.1, Lemma 1).

This result, along with the properties of the $\mathcal{O}_X$-submodules of a coherent $\mathcal{O}_X$-module, impose the necessary conditions on the rings $\mathcal{O}_x$ such that $\mathcal{O}_X$ is coherent. For example (5.3.4), the intersection of two ideals of finite type of $\mathcal{O}_x$ must still be an ideal of finite type.

(5.3.9). Suppose that $\mathcal{O}_X$ is coherent, and let $M$ be an $\mathcal{O}_x$-module admitting a finite presentation, therefore isomorphic to a cokernel of a homomorphism $\phi : \mathcal{O}_x^p \to \mathcal{O}_x^q$; then there exists an open neighborhood $U$ of $x$ and a coherent $(\mathcal{O}_X|U)$-module $\mathcal{F}$ such that $\mathcal{F}_x$ is isomorphic to $M$. Indeed, according to (5.2.6), there exists a section $u$ of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_x^q, \mathcal{O}_x^p)$ over an open neighborhood $U$ of $x$ such that $u_x = \phi$; the cokernel $\mathcal{F}$ of the homomorphism $u : \mathcal{O}_X^p|U \to \mathcal{O}_X^q|U$ gives the answer (5.3.4).

(5.3.10). Suppose that $\mathcal{O}_X$ is coherent, and let $\mathcal{J}$ be a coherent sheaf of ideals of $\mathcal{O}_X$. For a $(\mathcal{O}_X/\mathcal{J})$-module $\mathcal{F}$ to be coherent, it is necessary and sufficient for it to be coherent as a $\mathcal{O}_X$-module. In particular, $\mathcal{O}_X/\mathcal{J}$ is a coherent sheaf of rings.

(5.3.11). Let $f : X \to Y$ be a morphism of ringed spaces, and suppose that $\mathcal{O}_X$ is coherent; then, for each coherent $\mathcal{O}_Y$-module $\mathcal{G}$, $f^*(\mathcal{G})$ is a coherent $\mathcal{O}_X$-module. Indeed, with the notation of (5.2.4), we can assume that $\mathcal{G}|V$ is the cokernel of a homomorphism $v : \mathcal{O}_Y^p|V \to \mathcal{O}_Y^q|V$; as $f^*_V$ is right exact, $f^*(\mathcal{G})|U = f^*_U(\mathcal{G}|V)$ is the cokernel of the homomorphism $f^*_U(v) : \mathcal{O}_X^p|U \to \mathcal{O}_X^q|U$, hence our assertion.

(5.3.12). Let $Y$ be a closed subset of $X$, $j : Y \to X$ the canonical injection, $\mathcal{O}_Y$ a sheaf of rings on $Y$, and set $\mathcal{O}_X = j_*(\mathcal{O}_Y)$. For a $\mathcal{O}_Y$-module $\mathcal{G}$ to be of finite type (resp. quasi-coherent, coherent), it is necessary and sufficient for $j_*(\mathcal{G})$ to be an $\mathcal{O}_X$-module of finite type (resp. quasi-coherent, coherent).
5.4. Locally free sheaves

(5.4.1). Let $X$ be a ringed space. We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is locally free if for each $x \in X$ there exists an open neighborhood $U$ of $x$ such that $\mathcal{F}|_U$ is isomorphic to a $(\mathcal{O}_X|_U)$-module of the form $\mathcal{O}_X^{(1)}|_U$, where $I$ can depend on $U$. If for each $U$, $I$ is finite, then we say that $\mathcal{F}$ is of finite rank; if for each $U$, $I$ has the same finite number of elements $n$, we say that $\mathcal{F}$ is of rank $n$. A locally free $\mathcal{O}_X$-module of rank 1 is called invertible (cf. (5.4.3)). If $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module of finite rank, then for each $x \in X$, $\mathcal{F}_x$ is a free $\mathcal{O}_x$-module of finite rank $n(x)$, and there exists a neighborhood $U$ of $x$ such that $\mathcal{F}|_U$ is of rank $n(x)$; if $X$ is connected, then $n(x)$ is constant.

It is clear that each locally free sheaf is quasi-coherent, and if $\mathcal{O}_X$ is a coherent sheaf of rings, then each locally free $\mathcal{O}_X$-module of finite rank is coherent.

If $\mathcal{L}$ is locally free, then $\mathcal{L} \otimes \mathcal{O}_X \mathcal{F}$ is an exact functor in $\mathcal{F}$ to the category of $\mathcal{O}_X$-modules.

We will mostly consider locally free $\mathcal{O}_X$-modules of finite rank, and when we speak of locally free sheaves without specifying, it will be understood that they are of finite rank.

Suppose that $\mathcal{O}_X$ is coherent, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then, if at a point $x \in X$, $\mathcal{F}_x$ is an $\mathcal{O}_x$-module of rank $n$, there exists a neighborhood $U$ of $x$ such that $\mathcal{F}|_U$ is locally free of rank $n$; in fact, $\mathcal{F}_x$ is then isomorphic to $\mathcal{O}_x^n$, and the proposition follows from (5.2.7).

(5.4.2). If $\mathcal{L}$, $\mathcal{F}$ are two $\mathcal{O}_X$-modules, we have a canonical functorial homomorphism

$$\mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} = \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F} \rightarrow \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{F})$$

defined in the following way: for each open set $U$, send any pair $(u, t)$, where $u \in \Gamma(U, \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)) = \text{Hom}(\mathcal{L}|_U, \mathcal{O}_X|_U)$ and $t \in \Gamma(U, \mathcal{F})$, to the element of $\text{Hom}(\mathcal{L}|_U, \mathcal{F}|_U)$ which, for each $x \in U$, sends $s_x \in \mathcal{L}_x$ to the element $u_x(s_x)t_x$ of $\mathcal{F}_x$. If $\mathcal{L}$ is locally free of finite rank, then this homomorphism is bijective; the property being local, we can in fact reduce to the case where $\mathcal{L} = \mathcal{O}_X^n$; as for each $\mathcal{O}_X$-module $\mathcal{F}$, $\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathcal{F})$ is canonically isomorphic to $\mathcal{F}^n$, we have reduced to the case $\mathcal{L} = \mathcal{O}_X$, which is immediate.

(5.4.3). If $\mathcal{L}$ is invertible, then so is its dual $\mathcal{L}^\vee = \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$, since we can immediately reduce (as the question is local) to the case $\mathcal{L} = \mathcal{O}_X$. In addition, we have a canonical isomorphism

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L} \simeq \mathcal{O}_X$$

as, according to (5.3.2), it suffices to define a canonical isomorphism $\text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \simeq \mathcal{O}_X$. For each $\mathcal{O}_X$-module $\mathcal{F}$, we have a canonical homomorphism $\mathcal{O}_X \simeq \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{F})$ (5.3.7). It remains to prove that if $\mathcal{F} = \mathcal{L}$ is invertible, then this homomorphism is bijective, and as the question is local, it reduces to the case $\mathcal{L} = \mathcal{O}_X$, which is immediate.

Due to the above, we put $\mathcal{L}^{-1} = \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$, and we say that $\mathcal{L}^{-1}$ is the inverse of $\mathcal{L}$. The terminology “invertible sheaf” can be justified in the following way when $X$ is a point and $\mathcal{O}_X$ is a local ring $A$ with maximal ideal $m$; if $M$ and $M'$ are two $A$-modules ($M$ being of finite type) such that $M \otimes_A M'$ is isomorphic to $A$, then as $(A/m) \otimes_A (M \otimes_A M')$ identifies with $(M/mM) \otimes_{A/m} (M'/mM')$, this latter tensor product of vector spaces over the field $A/m$ is isomorphic to $A/m$, which requires $M/mM$ and $M'/mM'$ to be of dimension 1. For each element $z \in M$ not in $mM$, we have $M = Az + mM$, which implies that $M = Az$ according to Nakayama’s Lemma, $M$ being of finite type. Moreover, as the annihilator of $z$ kills $M \otimes_A M'$, which is isomorphic to $A$, this annihilator is $\{0\}$, and as a result $M$ is isomorphic to $A$. In the general case, this shows that $\mathcal{L}$ is an $\mathcal{O}_X$-module of finite type, such that there exists an $\mathcal{O}_X$-module $\mathcal{F}$ for which $\mathcal{L} \otimes \mathcal{O}_X \mathcal{F}$ is isomorphic to $\mathcal{O}_X$, and if in addition the rings $\mathcal{O}_x$ are local rings, then $\mathcal{L}_x$ is an $\mathcal{O}_x$-module isomorphic to $\mathcal{O}_x$ for each $x \in X$. If $\mathcal{O}_X$ and $\mathcal{L}$ are assumed to be coherent, then we conclude that $\mathcal{L}$ is invertible according to (5.2.7).

(5.4.4). If $\mathcal{L}$ and $\mathcal{L}'$ are two invertible $\mathcal{O}_X$-modules, then so is $\mathcal{L} \otimes \mathcal{O}_X \mathcal{L}'$, since the question is local, we can assume that $\mathcal{L} = \mathcal{O}_X$, and the result is then trivial. For each integer $n \geq 1$, we denote by $\mathcal{L}^\otimes n$ the tensor product of $n$ copies of the sheaf $\mathcal{L}$; we set by convention $\mathcal{L}^\otimes 0 = \mathcal{O}_X$, and for $n \geq 1, \mathcal{L}^\otimes (-n) = (\mathcal{L}^{-1})^\otimes n$. With these notation, there is then a canonical functorial isomorphism

$$\mathcal{L}^\otimes m \otimes \mathcal{O}_X \mathcal{L}^\otimes n \simeq \mathcal{L}^\otimes (m+n)$$
for any rational integers \( m \) and \( n \): indeed, by definition, we immediately reduce to the case where \( m = -1, n = 1 \), and the isomorphism in question is then that defined in (5.4.3).

(5.4.5). Let \( f : Y \to X \) be a morphism of ringed spaces. If \( \mathcal{L} \) is a locally free (resp. invertible) \( \mathcal{O}_X \)-module, then \( f^*(\mathcal{L}) \) is a locally free (resp. invertible) \( \mathcal{O}_Y \)-module: this follows immediately from the fact that the inverse images of two locally isomorphic \( \mathcal{O}_X \)-modules are locally isomorphic, that \( f^* \) commutes with finite direct sums, and that \( f^*(\mathcal{O}_X) = \mathcal{O}_Y \) (4.3.4). In addition, we know that we have a canonical functorial homomorphism \( f^*(\mathcal{L}^n) \to (f^*(\mathcal{L}))^\otimes n \) (4.4.6), and when \( \mathcal{L} \) is locally free, this homomorphism is bijective; indeed, we again reduce to the case where \( \mathcal{L} = \mathcal{O}_X \) which is trivial. We conclude that if \( \mathcal{L} \) is invertible, then \( f^*(\mathcal{L}^\otimes n) \) canonically identifies with \( (f^*(\mathcal{L}))^\otimes n \) for each rational integer \( n \).

(5.4.6). Let \( \mathcal{L} \) be an invertible \( \mathcal{O}_X \)-module; we denote by \( \Gamma_*(X, \mathcal{L}) \) or simply \( \Gamma_*(\mathcal{L}) \) the abelian group direct sum \( \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{L}^\otimes n) \); we equip it with the structure of a graded module where \( \Gamma_*(\mathcal{O}_X) \) is the structure of a graded ring. We conclude that there exists a functorial homomorphism of graded rings \( \Gamma_*(\mathcal{L}) \to \Gamma_*(\mathcal{L})^\otimes \) (5.4.5). One can show that there exists a functorial homomorphism of graded rings \( \Gamma_*(\mathcal{L}) \to \Gamma_*(\mathcal{L})^\otimes \) (5.4.7). Let \( \mathcal{F} \) be any \( \mathcal{O}_X \)-module, then we set

\[
\Gamma_*(\mathcal{L}, \mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F} \otimes \mathcal{O}_X \mathcal{L}^\otimes n).
\]

We equip this abelian group with the structure of a graded module over the graded ring \( \Gamma_*(\mathcal{L}) \) in the following way: to a pair \((s_n, u_m)\), where \( s_n \in \Gamma(X, \mathcal{L}^\otimes n) \) and \( u_m \in \Gamma(X, \mathcal{F} \otimes \mathcal{O}_X \mathcal{L}^\otimes m) \), we associate the section of \( \mathcal{F} \otimes \mathcal{O}_X \mathcal{L}^\otimes (n+m) \) which canonically corresponds (5.4.4) to \( s_n \otimes u_m \); the verification of the module axioms is immediate. For \( X \) and \( \mathcal{L} \) fixed, \( \Gamma_*(\mathcal{L}, \mathcal{F}) \) is a covariant functor in \( \mathcal{F} \) with values in the category of graded \( \Gamma_*(\mathcal{L}) \)-modules; for \( X \) and \( \mathcal{F} \) fixed, it is a covariant functor in \( \mathcal{L} \) with values in the category of abelian groups.

If \( f : Y \to X \) is a morphism of ringed spaces, the canonical homomorphism (4.4.3.2) \( \rho : \mathcal{L}^\otimes n \to f_*(f^*(\mathcal{L}^\otimes n)) \) defines a homomorphism of abelian groups \( \Gamma(X, \mathcal{L}^\otimes n) \to \Gamma(Y, f^*(\mathcal{L}^\otimes n)) \), and as \( f^*(\mathcal{L}^\otimes n) = (f^*(\mathcal{L}))^\otimes n \), it follows from the definitions of the canonical homomorphisms (4.4.3.2) and (5.4.4.1) that the above homomorphisms define a functorial homomorphism of graded rings \( \Gamma_*(\mathcal{L}) \to \Gamma_*(\mathcal{L})^\otimes \). The same canonical homomorphism (4.4.3) similarly defines a homomorphism of abelian groups \( \Gamma(X, \mathcal{F} \otimes \mathcal{O}_X \mathcal{L}^\otimes n) \to \Gamma(Y, f^*(\mathcal{F} \otimes \mathcal{O}_X \mathcal{L}^\otimes n)) \), and as

\[
f^*(\mathcal{F} \otimes \mathcal{O}_X \mathcal{L}^\otimes n) = f^*(\mathcal{F}) \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y \mathcal{L})^\otimes n \quad \text{(4.3.3.1),}
\]

these homomorphism (for \( n \) variable) define a di-homomorphism of graded modules \( \Gamma_*(\mathcal{L}, \mathcal{F}) \to 0_1 \) \( 51 \Gamma, (f^*(\mathcal{L}), f^*(\mathcal{F})). \n
(5.4.7). One can show that there exists a set \( \mathfrak{M} \) (also denoted \( \mathfrak{M}(X) \)) of invertible \( \mathcal{O}_X \)-modules such that each invertible \( \mathcal{O}_X \)-module is isomorphic to a unique element of \( \mathfrak{M} \); we define on \( \mathfrak{M} \) a composition law by sending two elements \( \mathcal{L} \) and \( \mathcal{L}' \) of \( \mathfrak{M} \) to the unique element of \( \mathfrak{M} \) isomorphic to \( \mathcal{L} \otimes \mathcal{O}_X \mathcal{L}' \). With this composition law, \( \mathfrak{M} \) is a group isomorphic to the cohomology group \( H^1(X, \mathcal{O}_X^\times) \), where \( \mathcal{O}_X^\times \) is the subsheaf of \( \mathcal{O}_X \) such that \( \Gamma(U, \mathcal{O}_X^\times) \) is the group of invertible elements of the ring \( \Gamma(U, \mathcal{O}_X) \) for each open \( U \subset X \), \( \mathcal{O}_X^\times \) is therefore a sheaf of multiplicative abelian groups.

We will note that for all open \( U \subset X \), the group of sections \( \Gamma(U, \mathcal{O}_X^\times) \) canonically identifies with the automorphism group of the \( \mathcal{O}_X|U \)-module \( \mathcal{O}_X|U \), the identification sending a section \( \theta \) of \( \mathcal{O}_X|U \) over \( U \) to the automorphism \( u_{\theta} \) of \( \mathcal{O}_X|U \) such that \( u_{\theta}(s_x) = \theta(s_x) \) for all \( x \in X \) and all \( s_x \in \mathcal{O}_X \). Then let \( U = (U_x) \) be an open cover of \( X \); the data, for each pair of indices \((\lambda, \mu)\), of an automorphism \( \theta_{\lambda\mu} \) of \( \mathcal{O}_X|U_{\lambda} \cap U_{\mu} \) is the same as giving a 1-cochain of the cover \( U \), with values in \( \mathcal{O}_X^\times \), and say that the \( \theta_{\lambda\mu} \) satisfy the gluing condition (3.3.1), meaning that the corresponding cochain is a cocycle. Similarly, the data, for each \( \lambda \), of an automorphism \( \omega_{\lambda} \) of \( \mathcal{O}_X|U_{\lambda} \) is the same as the data of a 0-cocycle of the cover \( U \), with values in \( \mathcal{O}_X^\times \), and its coboundary corresponds to the family of automorphisms \( (\omega_{\lambda}|U_{\lambda} \cap U_{\mu}) \circ (\omega_{\mu}|U_{\lambda} \cap U_{\mu})^{-1} \). We can send each 1-cocycle of \( U \) with values in \( \mathcal{O}_X^\times \) to the element of \( \mathfrak{M} \) isomorphic to an invertible \( \mathcal{O}_X \)-module obtained by gluing with respect to the family of

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\(^8\) See the book in preparation cited in the introduction.
automorphisms $(\theta_{\lambda\mu})$ corresponding to this cocycle, and to two cohomologous cocycles correspond two equal elements of $\mathcal{M}$ (3.3.2); in other words, we thus define a map $\phi_{\mu} : H^1(U, \mathcal{O}_X^*) \to \mathcal{M}$. In addition, if $\mathcal{B}$ is a second open cover of $X$, finer than $U$, then the diagram

$$
\begin{array}{ccc}
H^1(U, \mathcal{O}_X^*) & \xrightarrow{\phi_{\mu}} & \mathcal{M} \\
\downarrow{\phi_{\beta}} & & \downarrow{0_1} \\
H^1(\mathcal{B}, \mathcal{O}_X^*) & & 
\end{array}
$$

where the vertical arrow is the canonical homomorphism (G, II, 5.7), is commutative, as a result of (3.3.3). By passing to the inductive limit, we therefore obtain a map $H^1(X, \mathcal{O}_X^*) \to \mathcal{M}$, the Čech cohomology group $\check{H}^1(X, \mathcal{O}_X^*)$ identifying as we know with the first cohomology group $H^1(X, \mathcal{O}_X^*)$ (G, II, 5.9, Cor. of Thm. 5.9.1). This map is surjective: indeed, by definition, for each invertible $\mathcal{O}_X$-module $\mathcal{L}$, there is an open cover $(U_\lambda)$ of $X$ such that $\mathcal{L}$ is obtained by gluing the sheaves $\mathcal{O}_X|U_\lambda$ (3.3.1). It is also injective, since it suffices to prove for the maps $H^1(U, \mathcal{O}_X^*) \to \mathcal{M}$, and this follows from (3.3.2). It remains to show that the bijection thus defined is a group homomorphism.

Given two invertible $\mathcal{O}_X$-modules $\mathcal{L}$ and $\mathcal{L}'$, there is an open cover $(U_\lambda)$ such that $\mathcal{L}|U_\lambda$ and $\mathcal{L}'|U_\lambda$ are isomorphic to $\mathcal{O}_X|U_\lambda$ for each $\lambda$, so there is for each index $\lambda$ an element $a_\lambda$ (resp. $a'_\lambda$) of $\Gamma(U_\lambda, \mathcal{L})$ (resp. $\Gamma(U_\lambda, \mathcal{L}')$) such that the elements of $\Gamma(U_\lambda, \mathcal{L})$ (resp. $\Gamma(U_\lambda, \mathcal{L}')$) are the $s_\lambda \cdot a_\lambda$ (resp. $s_\lambda \cdot a'_\lambda$), where $s_\lambda$ varies over $\Gamma(U_\lambda, \mathcal{O}_X)$. The corresponding cocycles $(\epsilon_{\lambda\mu}, (\epsilon'_{\lambda\mu})$ are such that $s_\lambda \cdot a_\lambda = s_\mu \cdot a_\mu$ (resp. $s_\lambda \cdot a'_\lambda = s_\mu \cdot a'_\mu$) over $U_\lambda \cap U_\mu$, is equivalent to $s_\lambda = \epsilon_{\lambda\mu} s_\mu$ (resp. $s_\lambda = \epsilon'_{\lambda\mu} s_\mu$) over $U_\lambda \cap U_\mu$. As the sections of $\mathcal{L} \otimes \mathcal{O}_X \mathcal{L}'$ over $U_\lambda$ are the finite sums of the $s_\lambda \cdot a_\lambda \cdot a'_\mu$ where $s_\lambda$ and $s'_\mu$ vary over $\Gamma(U_\lambda, \mathcal{O}_X)$, it is clear that the cocycle $(\epsilon_{\lambda\mu}, \epsilon'_{\lambda\mu})$ corresponds to $\mathcal{L} \otimes \mathcal{O}_X \mathcal{L}'$, which finishes the proof.  

(5.4.8) Let $f = (\psi, \omega)$ be a morphism $Y \to X$ of ringed spaces. The functor $f^*(\mathcal{L})$ to the category of free $\mathcal{O}_Y$-modules defines a map (which we still denote $f^*$) by abuse of language) from the set $\mathcal{M}(X)$ to the set $\mathcal{M}(Y)$. Second, we have a canonical homomorphism (T, 3.2.2)

$$
(5.4.8.1) \quad H^1(X, \mathcal{O}_X^*) \longrightarrow H^1(Y, \mathcal{O}_Y^*).
$$

When we canonically identify (2.5.7) $\mathcal{M}(X)$ and $H^1(X, \mathcal{O}_X^*)$ (resp. $\mathcal{M}(Y)$ and $H^1(Y, \mathcal{O}_Y^*)$), the homomorphism (5.4.8.1) identifies with the map $f^*$. Indeed, if $\mathcal{L}$ comes from a cocycle $(\epsilon_{\lambda\mu})$ corresponding to an open cover $(U_\lambda)$ of $X$, then it suffices to show that $f^*(\mathcal{L})$ comes from a cocycle whose cohomology class is the image under (5.4.8.1) of $(\epsilon_{\lambda\mu})$. If $\theta_{\lambda\mu}$ is the automorphism of $\mathcal{O}_X|U_\lambda \cap U_\mu$ which corresponds to $\epsilon_{\lambda\mu}$, then it is clear that $f^*(\mathcal{L})$ is obtained by gluing the $\mathcal{O}_Y|\mathcal{F}^{-1}(U_\lambda)$ by means of the automorphisms $f^*(\theta_{\lambda\mu})$, and it then suffices to check that these latter automorphisms corresponds to the cocycle $(\omega^*(\epsilon_{\lambda\mu}))$, which follows immediately from the definitions (we can identify $\epsilon_{\lambda\mu}$ with its canonical image under $\rho (3.7.2)$, a section of $\omega^*(\mathcal{O}_X^*)$ over $\mathcal{F}^{-1}(U_\lambda \cap U_\mu)$).

(5.4.9) Let $\mathcal{E}$ and $\mathcal{F}$ be two $\mathcal{O}_X$-modules, $\mathcal{F}$ assumed to be locally free, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module extension of $\mathcal{F}$ by $\mathcal{E}$, in other words there exists an exact sequence $0 \to \mathcal{E} \xrightarrow{f} \mathcal{F} \xrightarrow{p} \mathcal{F} \to 0$. Then, for each $x \in X$, there exists an open neighborhood $U$ of $x$ such that $\mathcal{F}|U$ is isomorphic to the direct sum $\mathcal{E}|U \oplus \mathcal{F}|U$. We can reduce to the case where $\mathcal{F} = \mathcal{O}_X^*$; let $c_i (1 \leq i \leq n)$ be the canonical sections (5.5.5) of $\mathcal{O}_X^*$; there then exists an open neighborhood $U$ of $x$ and $n$ sections $s_i$ of $\mathcal{F}$ over $U$ such that $p(s_i|U) = c_i|U$ for $1 \leq i \leq n$. That being so, let $f$ be the homomorphism $\mathcal{F}|V \to \mathcal{F}|U$ defined by the sections $s_i|U$ (5.1.1). It is immediate that for each open $V \subset U$, and each section $s \in \Gamma(V, \mathcal{F})$ we have $s - f(p(s)) \in \Gamma(V, \mathcal{E})$, hence our assertion.

(5.4.10) Let $f : X \to Y$ be a morphism of ringed spaces, $\mathcal{F}$ an $\mathcal{O}_X$-module, and $\mathcal{L}$ a locally free $\mathcal{O}_Y$-module of finite rank. Then there exists a canonical isomorphism

$$
(5.4.10.1) \quad f_* (\mathcal{F}) \otimes \mathcal{O}_Y \mathcal{L} \cong f_* (\mathcal{F} \otimes f^*(\mathcal{L})).
$$

---

9For a general form of this result, see the book cited in the note on p. 51.
Indeed, for each \( \mathcal{O}_Y \)-module \( \mathcal{L} \), we have a canonical homomorphism

\[
\rho \quad \begin{array}{c}
\mathfrak{f}_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{L} \xrightarrow{1 \otimes \rho} \mathfrak{f}_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathfrak{f}_*(\mathcal{L}) \\
\xrightarrow{\alpha} \mathfrak{f}_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathfrak{f}^*(\mathcal{L}))
\end{array}
\]

\( \rho \) the homomorphism (4.43.2) and \( \alpha \) the homomorphism (4.2.2.1). To show that when \( \mathcal{L} \) is locally free, this homomorphism is bijective, it suffices, since the questions is local, to consider the case where \( \mathcal{L} = \mathcal{O}_X^n \); in addition, \( \mathfrak{f}_* \) and \( \mathfrak{f}^* \) commute with finite direct sums, so we can assume \( n = 1 \), and in this case the proposition follows immediately from the definitions and from the relation \( \mathfrak{f}^*(\mathcal{O}_Y) = \mathcal{O}_X \).

5. Sheaves on a locally ringed space

(5.5.1). We say that a ringed space \( (X, \mathcal{O}_X) \) is a locally ringed space if for each \( x \in X \), \( \mathcal{O}_x \) is a local ring; these ringed spaces will be by far the most frequent ringed spaces that we will consider in this work. We then denote by \( m_x \) the maximal ideal of \( \mathcal{O}_x \), by \( k(x) \) the residue field \( \mathcal{O}_x/m_x \); for each \( \mathcal{O}_X \)-module \( \mathcal{F} \), each open set \( U \) of \( X \), each point \( x \in U \), and each section \( f \in \Gamma(U, \mathcal{F}) \), we denote by \( f(x) \) the class of the germ \( f_x \in \mathcal{F}_x \) mod. \( m_x \mathcal{F}_x \), and we say that this is the value of \( f \) at the point \( x \). The relation \( f(x) = 0 \) then means that \( f_x \in m_x \mathcal{F}_x \); when this is so, we say (by abuse of language) that \( f \) is zero at \( x \). We will take care not to confuse this relation with \( f_x = 0 \).

(5.5.2). Let \( X \) be a locally ringed space, \( \mathcal{L} \) an invertible \( \mathcal{O}_X \)-module, and \( f \) a section of \( \mathcal{L} \) over \( X \). There is then an equivalence between the three following properties for a point \( x \in X \):

(a) \( f_x \) is a generator of \( \mathcal{L}_x \);
(b) \( f_x \notin m_x \mathcal{L}_x \) (in other words, \( f(x) \neq 0 \));
(c) there exists a section \( g \) of \( \mathcal{L}^{-1} \) over an open neighborhood \( V \) of \( x \) such that the canonical image of \( f \otimes g \) in \( \Gamma(V, \mathcal{O}_X) \) (5.4.3) is the unit section.

Indeed, since the questions is local, we can reduce to the case where \( \mathcal{L} = \mathcal{O}_X \); the equivalence of (a) and (b) are then evident, and it is clear that (c) implies (b). Conversely, if \( f_x \notin m_x \), then \( f_x \) is invertible in \( \mathcal{O}_x \), say \( f_x g_x = 1_x \). By definition of germs of sections, this means that there exists a neighborhood \( V \) of \( x \) and a section \( g \) of \( \mathcal{O}_X \) over \( V \) such that \( f g = 1 \) in \( V \), hence (c).

It follows immediately from the condition (c) that the set \( X_f \) of \( x \) satisfying the equivalent conditions (a), (b), (c) is open in \( X \); following the terminology introduced in (5.5.1), this is the set of the \( x \) for which \( f \) does not vanish.

(5.5.3). Under the hypotheses of (5.5.2), let \( \mathcal{L}' \) be a second invertible \( \mathcal{O}_X \)-module; then, if \( f \in \Gamma(X, \mathcal{L}) \), \( g \in \Gamma(X, \mathcal{L}') \), we have

\[
X_f \cap X_g = X_{fg}.
\]

We can in fact reduce immediately to the case where \( \mathcal{L} = \mathcal{L}' = \mathcal{O}_X \) (since the questions is local); as \( f \otimes g \) then canonically identifies with the product \( fg \), the proposition is evident.

(5.5.4). Let \( \mathcal{F} \) be a locally free \( \mathcal{O}_X \) of rank \( n \); it is immediate that \( \wedge^p \mathcal{F} \) is a locally free \( \mathcal{O}_X \)-module of rank \( \binom{n}{p} \) if \( p \leq n \) and \( 0 \) if \( p > n \), since the question is local and we can reduce to the case where \( \mathcal{F} = \mathcal{O}_X^m \); in addition, for each \( x \in X \), \( \wedge^p \mathcal{F}_x / m_x (\wedge^p \mathcal{F}_x) \) is a vector space of dimension \( \binom{n}{p} \) over \( k(x) \), which canonically identifies with \( \wedge^p (\mathcal{F}_x / m_x \mathcal{F}_x) \). Let \( s_1, \ldots, s_p \) be the sections of \( \mathcal{F} \) over an open subset \( U \) of \( X \), and let \( s = s_1 \wedge \cdots \wedge s_p \), which is a section of \( \wedge^p \mathcal{F} \) over \( U \) (4.1.5); we have \( s(x) = s_1(x) \wedge \cdots \wedge s_p(x) \), and as a result, we say that the \( s_1(x), \ldots, s_p(x) \) are linearly dependent means \( s(x) = 0 \). We conclude that the set of the \( x \in X \) such that \( s_1(x), \ldots, s_p(x) \) are linearly dependent is open in \( X \); it suffices in fact, by reducing to the case where \( \mathcal{F} = \mathcal{O}_X^m \), to apply (5.5.2) to the section image of \( s \) under one of the projections of \( \wedge^p \mathcal{F} = \mathcal{O}_X^{\binom{n}{p}} \) to the \( \binom{n}{p} \) factors.

In particular, if \( s_1, \ldots, s_n \) are \( n \) sections of \( \mathcal{F} \) over \( U \) such that \( s_1(x), \ldots, s_n(x) \) are linearly independent for each point \( x \in U \), then the homomorphism \( u : \mathcal{O}_X^n|U \to \mathcal{F}|U \) defined by the \( s_i \) (5.1.1) is an isomorphism; indeed, we can restrict to the case where \( \mathcal{F} = \mathcal{O}_X^m \) and where we canonically identify \( \wedge^n \mathcal{F} \) and \( \mathcal{O}_X^m ; s = s_1 \wedge \cdots \wedge s_n \) is then an invertible section of \( \mathcal{O}_X \) over \( U \), and we define an inverse homomorphism for \( u \) by means of the Cramer formulas.
Let \( \mathcal{E} \) and \( \mathcal{F} \) be two locally free \( \mathcal{O}_X \)-modules (of finite rank), and let \( u : \mathcal{E} \to \mathcal{F} \) be a homomorphism. For there to exist a neighborhood \( U \) of \( x \in X \) such that \( u|U \) is injective and that \( \mathcal{F}|U \) is the direct sum of the \( u(\mathcal{E})|U \) and of a locally free \( (\mathcal{O}_X|U) \)-submodule \( \mathcal{G} \), it is necessary and sufficient that \( u_x : \mathcal{E}_x \to \mathcal{F}_x \) gives, by passing to quotients, an injective homomorphism of vector spaces \( \mathcal{E}_x/m_x \mathcal{E}_x \to \mathcal{F}_x/m_x \mathcal{F}_x \). The condition is indeed necessary, since \( \mathcal{F}_x \) is then the direct sum of the free \( \mathcal{O}_x \)-modules \( u_x(\mathcal{E}_x) \) and \( \mathcal{G}_x \), so \( \mathcal{F}_x/m_x \mathcal{F}_x \) is the direct sum of \( u_x(\mathcal{E}_x)/m_x u_x(\mathcal{E}_x) \) and of \( \mathcal{G}_x/m_x \mathcal{G}_x \). The condition is sufficient, since we can reduce to the case where \( \mathcal{E} = \mathcal{O}_X^m \); let \( s_1, \ldots, s_m \) be the images under \( u \) of \( \mathcal{O}_X^m \) such that \((e_i)_y \) is equal to the \( i \)-th element of the canonical basis of \( \mathcal{O}_Y^m \) for each \( y \in Y \) (canonical sections of \( \mathcal{O}_X^m \)). By hypothesis, the \( s_1(x), \ldots, s_m(x) \) are linearly independent, so if \( \mathcal{F} \) is of rank \( n \), then there exist \( n - m \) sections \( s_{m+1}, \ldots, s_n \) of \( \mathcal{F} \) over a neighborhood \( V \) of \( x \) such that the \( s_i(x) \) \((1 \leq i \leq n)\) form a basis for \( \mathcal{F}_x/m_x \mathcal{F}_x \). There then exists \((\ref{5.5.4})\) a neighborhood \( U \subset V \) of \( x \) such that the \( s_i(y) \) \((1 \leq i \leq n)\) form a basis for \( \mathcal{F}_y/m_y \mathcal{F}_y \) for each \( y \in V \), and we conclude \((\ref{5.5.4})\) that there is an isomorphism from \( \mathcal{F}|U \) to \( \mathcal{O}_X^m|U \), sending the \( s_i|U \) \((1 \leq i \leq m)\) to the \( e_i|U \), which finishes the proof.

\section{6. FLATNESS}

The notion of flatness is due to J.-P. Serre \cite{Ser56}; in the following, we omit the proofs of the results which are presented in the Algèbre commutative of N. Bourbaki, to which we refer the reader. We assume that all rings are commutative.\footnote{See the exposé cited of N. Bourbaki for the generalization from most of the results to the noncommutative case.}

If \( M, N \) are two \( A \)-modules, \( M' \) (resp. \( N' \)) a submodule of \( M \) (resp. \( N \)), we denote by \( \text{Im}(M' \otimes_A N) \to \mathcal{O}_X^m \bigcap \mathcal{O}_X^n \) the submodule of \( M \otimes_A N \), the image under the canonical map \( M' \otimes_A N' \to M \otimes_A N \).

\subsection{6.1. Flat modules}

The following conditions are equivalent:

(a) The functor \( \text{Tor}_i^A(M, N) \) is exact in \( N \) on the category of \( A \)-modules;

(b) \( \text{Tor}_i^A(M, N) = 0 \) for each \( i > 0 \) and for each \( A \)-module \( N \);

(c) \( \text{Tor}_i^A(M, N) = 0 \) for each \( A \)-module \( N \).

When \( M \) satisfies these conditions, we say that \( M \) is a flat \( A \)-module. It is clear that each free \( A \)-module is flat.

For \( M \) to be a flat \( A \)-module, it suffices that for each ideal \( \mathfrak{J} \) of \( A \), of finite type, the canonical map \( M \otimes_A \mathfrak{J} \to M \otimes_A J = M \) is injective.

Each inductive limit of flat \( A \)-modules is a flat \( A \)-module. For a direct sum \( \bigoplus_{\lambda \in \Lambda} M_\lambda \) of \( A \)-modules to be a flat \( A \)-modules, it is necessary and sufficient that each of the \( A \)-modules \( M_\lambda \) is flat. In particular, every projective \( A \)-module is flat.

Let \( 0 \to M' \to M \to M'' \to 0 \) be an exact sequence of \( A \)-modules, such that \( M'' \) is flat. Then, for each \( A \)-module \( N \), the sequence

\[ 0 \to M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0 \]

is exact. In addition, for \( M \) to be flat, it is necessary and sufficient that \( M'' \) is (but it can be that \( M \) and \( M' \) are flat without \( M'' = M/M' \) being so).

Let \( M \) be a flat \( A \)-module, \( N \) any \( A \)-module; for two submodules \( N' \), \( N'' \) of \( N \), we then have

\[ \text{Im}(M \otimes_A (N' + N'')) = \text{Im}(M \otimes_A N') + \text{Im}(M \otimes_A N''), \]

\[ \text{Im}(M \otimes_A (N' \cap N'')) = \text{Im}(M \otimes_A N') \cap \text{Im}(M \otimes_A N'') \]

(images taken in \( M \otimes_A N \)).

Let \( M \) and \( N \) be two \( A \)-modules, \( M' \) (resp. \( N' \)) a submodule of \( M \) (resp. \( N \)), and suppose that one of the modules \( M/M' \), \( N/N' \) is flat. Then we have \( \text{Im}(M' \otimes_A N) = \text{Im}(M' \otimes_A N) \cap (M \otimes_A N') \) (images in \( M \otimes_A N \)). In particular, if \( \mathfrak{J} \) is an ideal of \( A \) and if \( M/M' \) is flat, then we have \( \mathfrak{J}M' = M' \cap \mathfrak{J}M \).
6.2. Change of ring  When an additive group $M$ is equipped with multiple modules structures relative to the rings $A, B, ...$, we say that $M$ is flat as an $A$-module, $B$-module, ..., we sometimes also say that $M$ is $A$-flat, $B$-flat, ... .

(6.2.1). Let $A$ and $B$ be two rings, $M$ an $A$-module, $N$ an $(A, B)$-bimodule. If $M$ is flat and if $N$ is $B$-flat, then $M \otimes_A N$ is $B$-flat. In particular, if $M$ and $N$ are two flat $A$-modules, then $M \otimes_A N$ is a flat $A$-module. If $B$ is an $A$-algebra and if $M$ is a flat $A$-module, then the $B$-module $M_\otimes B = M \otimes_A B$ is flat. Finally, if $B$ is an $A$-algebra which is flat as an $A$-module, and if $N$ is a flat $B$-module, then $N$ is also $A$-flat.

(6.2.2). Let $A$ be a ring, $B$ an $A$-algebra which is flat as an $A$-module. Let $M, N$ be two $A$-modules, such that $M$ admits a finite presentation; then the canonical homomorphism

$$\Hom_A(M, N) \otimes_A B \to \Hom_B(M \otimes_A B, N \otimes_A B)$$

(sending $u \otimes b$ to the homomorphism $m \otimes b' \mapsto u(m) \otimes b'b$) is an isomorphism.

(6.2.3). Let $(A, \varphi_\ell)$ be a filtered inductive system of rings; let $A = \varprojlim \ A_\ell$. On the other hand, for each $\ell$, let $M_\ell$ be an $A_\ell$-module, and for $\ell \leq \mu$ let $\varphi_\mu : M_\mu \to M_\ell$ be a $\varphi_\ell$-homomorphism, such that $(M_\ell, \varphi_\ell)$ is an inductive system; $M = \varprojlim \ M_\ell$ is then an $A$-module. This being so, if for each $\ell, M_\ell$ is a flat $A_\ell$-module, then $M$ is a flat $A$-module. Indeed, let $J$ be an ideal of finite type of $A$; by definition of the inductive limit, there exists an index $\ell$ and an ideal $J_\ell$ of $A_\ell$ such that $J_\ell = J A_\ell$. If we put $J_\ell = J_\mu A_\mu$ for $\mu \geq \ell$, we also have $J = \varprojlim J_\mu$ (where $\mu$ varies over the indices $\geq \ell$), hence (the functor $\varprojlim$ being exact and commuting with tensor products)

$$M \otimes_A J = \varprojlim (M_\mu \otimes_{A_\mu} J_\mu) = \varprojlim J_\mu M_\mu = J M.$$

6.3. Local nature of flatness

(6.3.1). If $A$ is a ring, $S$ a multiplicative subset of $A$, $S^{-1}A$ is a flat $A$-module. Indeed, for each $A$-module $N$, $N \otimes_A S^{-1}A$ identifies with $S^{-1}N$ (1.2.5) and we know (1.3.2) that $S^{-1}N$ is an exact functor in $N$.

If now $M$ is a flat $A$-module, $S^{-1}M = M \otimes_A S^{-1}A$ is a flat $S^{-1}A$-module (6.2.1), so it is also $A$-flat according to the above and from (6.2.1). In particular, if $P$ is an $S^{-1}A$-module, we can consider it as an $A$-module isomorphic to $S^{-1}P$; for $P$ to be $A$-flat, it is necessary and sufficient that it is $S^{-1}A$-flat.

(6.3.2). Let $A$ be a ring, $B$ an $A$-algebra, and $T$ a multiplicative subset of $B$. If $P$ is a $B$-module which is $A$-flat, $T^{-1}P$ is $A$-flat. Indeed, for each $A$-module $N$, we have $(T^{-1}P) \otimes_A N = (T^{-1}B \otimes_B P) \otimes_A N = T^{-1}B \otimes_B (P \otimes_A N) = T^{-1}(P \otimes_A N); T^{-1}(P \otimes_A N)$ is an exact functor in $N$, being the composition of the two exact functors $P \otimes_A N$ (in $N$) and $T^{-1}Q$ (in $Q$). If $S$ is a multiplicative subset of $A$ such that its image in $B$ is contained in $T$, then $T^{-1}P$ is equal to $S^{-1}(T^{-1}P)$, so it is also $S^{-1}A$-flat according to (6.3.1).

(6.3.3). Let $\phi : A \to B$ be a ring homomorphism, $M$ a $B$-module. The following properties are equivalent:

(a) $M$ is a flat $A$-module.

(b) For each maximal ideal $n$ of $B$, $M_n$ is a flat $A$-module.

(c) For each maximal ideal $n$ of $B$, by setting $m = \phi^{-1}(n)$, $M_n$ is a flat $A_m$-module.

Indeed, as $M_n = (M_n)_m$, the equivalence of (b) and (c) follows from (6.3.1), and the fact that (a) implies (b) is a particular case of (6.3.2). It remains to see that (b) implies (a), that is to say, that for each injective homomorphism $u : N' \to N$ of $A$-modules, the homomorphism

$$v = 1 \otimes u : M \otimes_A N' \to M \otimes_A N$$

is injective. We have that $v$ is also a homomorphism of $B$-modules, and we know that for it to be injective, it suffices that for each maximal ideal $n$ of $B$, $v_n : (M \otimes_A N')_n \to (M \otimes_A N)_n$ is injective. But as

$$(M \otimes_A N)_n = B_n \otimes_B (M \otimes_A N) = M_n \otimes_A N,$$

$v_n$ is none other than the homomorphism $1 \otimes u : M_n \otimes_A N' \to M_n \otimes_A N$, which is injective since $M_n$ is $A$-flat.
In particular (by taking $B = A$), for an $A$-module $M$ to be flat, it is necessary and sufficient that $M_m$ is $A_m$-flat for each maximal ideal $m$ of $A$.

(6.3.4). Let $M$ be an $A$-module; if $M$ is flat, and if $f \in A$ does not divide 0 in $A$, $f$ does not kill any element $\neq 0$ in $M$, since the homomorphism $m \mapsto f \cdot m$ is expressed as $1 \otimes u$, where $u$ is the multiplication $a \mapsto f \cdot a$ on $A$ and $M$ is identified with $M \otimes_A A$; if $u$ is injective, it is the same for $1 \otimes u$ since $M$ is flat. In particular, if $A$ is integral, $M$ is torsion-free.

Conversely, suppose that $A$ is integral, $M$ is torsion-free, and suppose that for each maximal ideal $m$ of $A$, $A_m$ is a discrete valuation ring; then $M$ is $A$-flat. Indeed, it suffices (6.3.3) to prove that $M_m$ is $A_m$-flat, and we can therefore suppose that $A$ is already a discrete valuation ring. But as $M$ is the inductive limit of its submodules of finite type, and these latter submodules are torsion-free, we can in addition reduce to the case where $M$ is of finite type (6.1.2). The proposition follows in this case from that $M$ is a free $A$-module.

In particular, if $A$ is an integral ring, $\phi : A \to B$ a ring homomorphism making $B$ a flat $A$-module and $\neq \{0\}$, $\phi$ is necessarily injective. Conversely, if $B$ is integral, $A$ a subring of $B$, and if for each maximal ideal $m$ of $A$, $A_m$ is a discrete valuation ring, then $B$ is $A$-flat.

6.4. Faithfully flat modules

(6.4.1). For an $A$-module $M$, the following four properties are equivalent:

(a) For a sequence $N' \to N \to N''$ of $A$-modules to be exact, it is necessary and sufficient that the sequence $M \otimes_A N' \to M \otimes_A N \to M \otimes_A N''$ is exact;

(b) $M$ is flat for each $A$-module $N$, the relation $M \otimes_A N = 0$ implies $N = 0$;

(c) $M$ is flat for each homomorphism $\nu : N \to N'$ of $A$-modules, the relation $1_M \otimes \nu = 0, 1_M$ being the identity automorphism of $M$;

(d) $M$ is flat for each maximal ideal $m$ of $A$, $mM \neq M$.

When $M$ satisfies these conditions, we say that $M$ is a faithfully flat $A$-module; $M$ is then necessarily a faithful module. In addition, if $u : N \to N'$ is a homomorphism of $A$-modules, then for $u$ to be injective (resp. surjective, bijective), it is necessary and sufficient that $1 \otimes u : M \otimes_A N \to M \otimes_A N'$ is so.

(6.4.2). A free module $\neq \{0\}$ is faithfully flat; it is the same for the direct sum of a flat module and a faithfully flat module. If $S$ is a multiplicative subset of $A$, then $S^{-1}A$ is a faithfully flat $A$-module if $S$ consists of invertible elements (so $S^{-1}A = A$).

(6.4.3). Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of $A$-modules; if $M'$ and $M''$ are flat, and if one of the two is faithfully flat, then $M$ is also faithfully flat.

(6.4.4). Let $A$ and $B$ be two rings, $M$ an $A$-module, $N$ an $(A, B)$-bimodule. If $M$ is faithfully flat and if $N$ is a faithfully flat $B$-module, then $M \otimes_A N$ is a faithfully flat $B$-module. In particular, if $M$ and $N$ are two faithfully flat $A$-modules, then so is $M \otimes_A N$. If $B$ is an $A$-algebra and if $M$ is a faithfully flat $A$-module, the $B$-module $M_B$ is faithfully flat.

(6.4.5). If $M$ is a faithfully flat $A$-modules and if $S$ is a multiplicative subset of $A$, $S^{-1}M$ is a faithfully flat $S^{-1}A$-module, since $S^{-1}M = M \otimes_A (S^{-1}A)$ (6.4.4). Conversely, if for each maximal ideal $m$ of $A$, $M_m$ is a faithfully flat $A_m$-module, then $M$ is a faithfully flat $A$-module, since $M$ is $A$-flat (6.3.3), and we have

$$M_m/mM_m = (M \otimes_A A_m) \otimes_{A_m} (A_m/mA_m) = M \otimes_A (A/m) = M/mM,$$

so the hypotheses imply that $M/mM \neq 0$ for each maximal ideal $m$ of $A$, which proves our assertion (6.4.1).
6.5. Restriction of scalars

(6.5.1). Let \( A \) be a ring, \( \phi : A \to B \) a ring homomorphism making \( B \) an \( A \)-algebra. Suppose that there exists a \( B \)-module \( N \) which is a faithfully flat \( A \)-module. Then, for each \( A \)-module \( M \), the homomorphism \( x \mapsto 1 \otimes x \) from \( M \) to \( B \otimes_A M = M_B \) is injective. In particular, \( \phi \) is injective; for each ideal \( a \) of \( A \), we have \( \phi^{-1}(aB) = a \); for each maximal (resp. prime) ideal \( m \) of \( A \), there exists a maximal (resp. prime) ideal \( n \) of \( B \) such that \( \phi^{-1}(n) = m \).

(6.5.2). When the conditions of (6.5.1) are satisfied, we identify \( A \) with the subring of \( B \) by \( \phi \) and more generally, for each \( A \)-module \( M \), we identify \( M \) with an \( A \)-submodule of \( M_B \). We note that if \( B \) is also Noetherian, then so is \( A \), since the map \( a \mapsto aB \) is an increasing injection from the set of ideals of \( A \) to the set of ideals of \( B \); the existence of an infinite strictly increasing sequence of ideals of \( A \) thus implies the existence of an analogous sequence of ideals of \( B \).

6.6. Faithfully flat rings

(6.6.1). Let \( \phi : A \to B \) be a ring homomorphism making \( B \) an \( A \)-algebra. The following five properties are equivalent:

(a) \( B \) is a faithfully flat \( A \)-module (in other words, \( M_B \) is an exact and faithful functor in \( M \)).
(b) The homomorphism \( \phi \) is injective and the \( A \)-module \( B/\phi(A) \) is flat.
(c) The \( A \)-module \( B \) is flat (in other words, the functor \( M_B \) is exact), and for each \( A \)-module \( M \), the homomorphism \( x \mapsto 1 \otimes x \) from \( M \) to \( M_B \) is injective.
(d) The \( A \)-module \( B \) is flat and for each ideal \( a \) of \( A \), we have \( \phi^{-1}(aB) = a \).
(e) The \( A \)-module \( B \) is flat and for each maximal ideal \( m \) of \( A \), there exists a maximal ideal \( n \) of \( B \) such that \( \phi^{-1}(n) = m \).

When these conditions are satisfied, we identify \( A \) with a subring of \( B \).

(6.6.2). Let \( A \) be a local ring, \( m \) its maximal ideal, and \( B \) an \( A \)-algebra such that \( mB \neq B \) (which is so when for example \( B \) is a local ring and \( A \to B \) is a local homomorphism). If \( B \) is a flat \( A \)-module, \( B \) is a faithfully flat \( A \)-module. Indeed, this follows from (6.4.1, (d)). Under the indicated conditions, we thus see that if \( B \) is Noetherian, then so too is \( A \) (6.5.2).

(6.6.3). Let \( B \) be an \( A \)-algebra which is a faithfully flat \( A \)-module. For each \( A \)-module \( M \) and each \( A \)-submodule \( M' \) of \( M \), we have (by identifying \( M \) with an \( A \)-submodule of \( M_B \)) \( M' = M \cap M_B \).

(6.6.4). Let \( B \) be an \( A \)-algebra, \( N \) a faithfully flat \( B \)-module. For \( B \) to be a flat (resp. faithfully flat) \( A \)-module, it is necessary and sufficient that \( M_B \) is a flat (resp. faithfully flat) \( B \)-module.

In particular, let \( C \) be a \( B \)-algebra; if the ring \( C \) is faithfully flat over \( B \) and \( B \) is faithfully flat over \( A \), then \( C \) is faithfully flat over \( A \); if \( C \) is faithfully flat over \( B \) and over \( A \), then \( B \) is faithfully flat over \( A \).

6.7. Flat morphisms of ringed spaces

(6.7.1). Let \( f : X \to Y \) be a morphism of ringed spaces, and let \( \mathcal{F} \) be a \( \mathcal{O}_X \)-module. We say that \( \mathcal{F} \) is \( f \)-flat (or \( Y \)-flat when there is no chance of confusion with \( f \) at a point \( x \in X \) if \( \mathcal{F}_x \) is a flat \( \mathcal{O}_{f(x)} \)-module; we say that \( \mathcal{F} \) is \( f \)-flat over \( y \in Y \) if \( \mathcal{F} \) is \( f \)-flat for all the points \( x \in f^{-1}(y) \); we say that \( \mathcal{F} \) is \( f \)-flat if \( \mathcal{F} \) is \( f \)-flat at all the points of \( X \). We say that the morphism \( f \) is \( f \)-flat at \( x \in X \) (resp. flat over \( y \in Y \), resp. flat) if \( \mathcal{O}_X \) is \( f \)-flat at \( x \) (resp. \( f \)-flat over \( y \), resp. \( f \)-flat). If \( f \) is a flat morphism, we then say that \( X \) is \( f \)-flat over \( Y \), or \( Y \)-flat.

(6.7.2). With the notation of (6.7.1), if \( \mathcal{F} \) is \( f \)-flat at \( x \), for each open neighborhood \( U \) of \( y = f(x) \), the functor \( (\mathcal{F}^* \otimes_{\mathcal{O}_Y} \mathcal{F})_x \) is exact on the category of \( (\mathcal{O}_Y|U) \)-modules; indeed, this stalk canonically identifies with \( \mathcal{F}_y \otimes_{\mathcal{O}_y} \mathcal{F}_x \), and our assertion follows from the definition. In particular, if \( f \) is a flat morphism, the functor \( f^* \) is exact on the category of \( \mathcal{O}_Y \)-modules.

(6.7.3). Conversely, suppose the sheaf of rings \( \mathcal{O}_Y \) is coherent, and suppose that for each open neighborhood \( U \) of \( y \), the functor \( (f^* \mathcal{F})_x \) is exact in \( \mathcal{F} \) on the category of coherent \( (\mathcal{O}_Y|U) \)-modules.
Then $\mathcal{F}$ is $f$-flat at $x$. In fact, it suffices to prove that for each ideal of finite type $\mathfrak{J}$ of $\mathcal{O}_Y$, the canonical homomorphism $\mathfrak{J} \otimes_{\mathcal{O}_Y} \mathcal{F}_x \to \mathcal{F}_x$ is injective (6.1.1). We know (5.3.8) that there then exists an open neighborhood $U$ of $y$ and a coherent sheaf of ideals $\mathcal{F}$ of $\mathcal{O}_Y|U$ such that $\mathcal{F}_y = \mathfrak{J}$, hence the conclusion.

(6.7.4). The results of (6.1) for flat modules are immediately translated into propositions for sheaves with are $f$-flat at a point:

If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of $\mathcal{O}_X$-modules and if $\mathcal{F}''$ is $f$-flat at a point $x \in X$, then, for each open neighborhood $U$ of $y = f(x)$ and each $(\mathcal{O}_Y|U)$-module $\mathcal{I}$, the sequence

$$0 \to (f^*(\mathcal{I}) \otimes_{\mathcal{O}_X} \mathcal{F}')_x \to (f^*(\mathcal{I}) \otimes_{\mathcal{O}_X} \mathcal{F})_x \to (f^*(\mathcal{I}) \otimes_{\mathcal{O}_X} \mathcal{F}'')_x \to 0$$

is exact. For $\mathcal{F}$ to be $f$-flat at $x$, it is necessary and sufficient that $\mathcal{F}'$ is. We have similar statements for the corresponding notions of a $f$-flat $\mathcal{O}_X$-modules over $y \in Y$, or of a $f$-flat $\mathcal{O}_Y$-module.

(6.7.5). Let $f : X \to Y$, $g : Y \to Z$ be two morphisms of ringed spaces; let $x \in X$, $y = f(x)$, and $\mathcal{F}$ be an $\mathcal{O}_X$-module. If $\mathcal{F}$ is $f$-flat at the point $x$ and if the morphism $g$ is flat at the point $y$, then $\mathcal{F}$ is $(g \circ f)$-flat at $x$ (6.2.1). In particular, if $f$ and $g$ are flat morphisms, then $g \circ f$ is flat.

(6.7.6). Let $X$, $Y$ be two ringed spaces, $f : X \to Y$ a flat morphism. Then the canonical homomorphism of bifunctors (4.4.6)

$$f^*(\mathcal{H}\text{om}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G})) \to \mathcal{H}\text{om}_{\mathcal{O}_X}(f^*(\mathcal{F}), f^*(\mathcal{G}))$$

is an isomorphism when $\mathcal{F}$ admits a finite presentation (5.2.5).

Indeed, since the questions is local, we can assume that there exists an exact sequence $\mathcal{O}_Y^m \to \mathcal{O}_Y^n \to \mathcal{F} \to 0$. The two sides of (6.7.6.1) are right exact functors in $\mathcal{O}_Y$ according to the hypotheses on $f$; we then have reduced to proving the proposition in the case where $\mathcal{F} = \mathcal{O}_Y$, in which the result is trivial.

(6.7.8). We say that a morphism $f : X \to Y$ of ringed spaces is faithfully flat if $f$ is surjective and if, for each $x \in X$, $\mathcal{O}_x$ is a faithfully flat $\mathcal{O}_{f(x)}$-module. When $X$ and $Y$ are locally ringed spaces (5.5.1), it is equivalent to say that the morphism $f$ is surjective and flat (6.6.2). When $f$ is faithfully flat, $f^*$ is an exact and faithful functor on the category of $\mathcal{O}_X$-modules (6.6.1, a), and for an $\mathcal{O}_Y$-module $\mathcal{G}$ to be $Y$-flat, it is necessary and sufficient that $f^*(\mathcal{G})$ is (6.6.3).

§7. ADIC RINGS

7.1. Admissible rings

(7.1.1). Recall that in a topological ring $A$ (not necessarily separated), we say that an element $x$ is topologically nilpotent if $0$ is a limit of the sequence $(x^n)_{n \geq 0}$. We say that a topological ring $A$ is linearly topologized if there exists a fundamental system of neighborhoods of $0$ in $A$ of (necessarily open) ideals.

Definition (7.1.2). — In a linearly topologized ring $A$, we say that an ideal $\mathfrak{J}$ is an ideal of definition if $\mathfrak{J}$ is open and if, for each neighborhood $V$ of $0$, there exists an integer $n > 0$ such that $\mathfrak{J}^n \subset V$ (which we express, by abuse of language, by saying that the sequence $(\mathfrak{J}^n)$ tends to $0$). We say that a linearly topologized ring $A$ is preadmissible if there exists in $A$ an ideal of definition; we say that $A$ is admissible if it is preadmissible and in addition it is separated and complete.

It is clear that if $\mathfrak{J}$ is an ideal of definition, $\mathfrak{L}$ an open ideal of $A$, then $\mathfrak{J} \cap \mathfrak{L}$ is also an ideal of definition; the ideals of definition of a preadmissible ring $A$ thus form a fundamental system of neighborhoods of $0$.

Lemma (7.1.3). — Let $A$ be a linearly topologized ring:

(i) For $x \in A$ to be topologically nilpotent, it is necessary and sufficient that for each open ideal $\mathfrak{J}$ of $A$, the canonical image of $x$ in $A/\mathfrak{J}$ is nilpotent. The set $\mathfrak{L}$ of topologically nilpotent elements of $A$ is an ideal.

(ii) Suppose that in addition $A$ is preadmissible, and let $\mathfrak{J}$ be an ideal of definition for $A$. For $x \in A$ to be topologically nilpotent, it is necessary and sufficient that its canonical image in $A/\mathfrak{J}$ is nilpotent; the ideal $\mathfrak{L}$ is the inverse image of the nilradical of $A/\mathfrak{J}$ and is thus open.
Proof. (i) follows immediately from the definitions. To prove (ii), it suffices to note that for each neighborhood \( V \) of 0 in \( A \), there exists an \( n > 0 \) such that \( \mathfrak{J}^n \subset V \); if \( x \in A \) is such that \( x^m \in \mathfrak{J} \), we have \( x^{nq} \in V \) for \( q \geq n \), so \( x \) is topologically nilpotent. \( \square \)

**Proposition (7.1.4).** — Let \( A \) be a preadmissible ring, \( \mathfrak{J} \) an ideal of definition for \( A \).

(i) For an ideal \( \mathfrak{J}' \) of \( A \) to be contained in an ideal of definition, it is necessary and sufficient that there exists an integer \( n > 0 \) such that \( \mathfrak{J}'^n \subset \mathfrak{J} \).

(ii) For an \( x \in A \) to be contained in an ideal of definition, it is necessary and sufficient that it is topologically nilpotent.

**Proof.**

(i) If \( \mathfrak{J}'^n \subset \mathfrak{J} \), then for each open neighborhood \( V \) of 0 in \( A \), there exists an \( m \) such that \( \mathfrak{J}^m \subset V \), thus \( \mathfrak{J}'^{nm} \subset V \).

(ii) The condition is evidently necessary; it is sufficient, since if it satisfied, then there exists an \( n \) such that \( x^n \in \mathfrak{J} \), so \( \mathfrak{J}' = \mathfrak{J} + Ax \) is an ideal of definition, because it is open, and \( \mathfrak{J}'^n \subset \mathfrak{J} \).

**Corollary (7.1.5).** — In a preadmissible ring \( A \), an open prime ideal contains all the ideals of definition.

**Corollary (7.1.6).** — The notation and hypotheses being that of (7.1.4), the following properties of an ideal \( \mathfrak{J}_0 \) of \( A \) are equivalent:

(a) \( \mathfrak{J}_0 \) is the largest ideal of definition of \( A \);

(b) \( \mathfrak{J}_0 \) is a maximal ideal of definition;

(c) \( \mathfrak{J}_0 \) is an ideal of definition such that the ring \( A/\mathfrak{J}_0 \) is reduced.

For there to exist an ideal \( \mathfrak{J}_0 \) to have these properties, it is necessary and sufficient that the nilradical of \( A/\mathfrak{J} \) to be nilpotent; \( \mathfrak{J}_0 \) is then equal to the ideal \( \mathfrak{I} \) of topologically nilpotent elements of \( A \).

**Proof.** It is clear that (a) implies (b), and (b) implies (c) according to (7.1.4, ii); for the same reason, (c) implies (a). The latter assertion follows from (7.1.4, i) and (7.1.3, ii). \( \square \)

When \( \mathfrak{I}/\mathfrak{J}_0 \), the nilradical of \( A/\mathfrak{J}_0 \), is nilpotent, and we denote by \( A_{\text{red}} \) the (reduced) quotient ring \( A/\mathfrak{I} \).

**Corollary (7.1.7).** — A preadmissible Noetherian ring admits a largest ideal of definition.

**Corollary (7.1.8).** — If a preadmissible ring \( A \) is such that, for an ideal of definition \( \mathfrak{J} \), the powers \( \mathfrak{J}^n (n > 0) \) form a fundamental system of neighborhoods of 0, it is the same for the powers \( \mathfrak{J}^m \) for each ideal of definition \( \mathfrak{J}' \) of \( A \).

**Definition (7.1.9).** — We say that a preadmissible ring \( A \) is preadic if there exists an ideal of definition \( \mathfrak{J} \) for \( A \) such that the \( \mathfrak{J}^n \) form a fundamental system of neighborhoods of 0 in \( A \) (or equivalently, such that the \( \mathfrak{J}^n \) are open). We call a ring adic if it is a separated and complete preadic ring.

If \( \mathfrak{J} \) is an ideal of definition for a preadic (resp. adic) ring \( A \), we say that \( A \) is a \( \mathfrak{J} \)-preadic (resp. \( \mathfrak{J} \)-adic) ring, and that its topology is the \( \mathfrak{J} \)-preadic (resp. \( \mathfrak{J} \)-adic) topology. More generally, if \( M \) is an \( A \)-module, the topology on \( M \) having for a fundamental system of neighborhoods of 0 the submodules \( \mathfrak{J}^nM \) is called the \( \mathfrak{J} \)-preadic (resp. \( \mathfrak{J} \)-adic) topology. According to (7.1.8), these topologies are independent of the choice of the ideal of definition \( \mathfrak{J} \).

**Proposition (7.1.10).** — Let \( A \) be an admissible ring, \( \mathfrak{J} \) an ideal of definition for \( A \). Then \( \mathfrak{J} \) is contained in the radical of \( A \).

This statement is equivalent to any of the following corollaries:

**Corollary (7.1.11).** — For each \( x \in \mathfrak{J} \), \( 1 + x \) is invertible in \( A \).

**Corollary (7.1.12).** — For \( f \in A \) to be invertible in \( A \), it is necessary and sufficient that its canonical image in \( A/\mathfrak{J} \) is invertible in \( A/\mathfrak{J} \).

**Corollary (7.1.13).** — For each \( A \)-module \( M \) of finite type, the relation \( M = \mathfrak{J}M \) (equivalent to \( M \otimes_A (A/\mathfrak{J}) = 0 \)) implies that \( M = 0 \).
Corollary (7.1.14). — Let \( u : M \to N \) be a homomorphism of \( A \)-modules, \( N \) being of finite type; for \( u \) to be surjective, it is necessary and sufficient that \( u \otimes 1 : M \otimes_A (A/\mathfrak{N}) \to N \otimes_A (A/\mathfrak{N}) \) is.

PROOF. The equivalence of (7.1.10) and (7.1.11) follows from Bourbaki, Alg., chap. VIII, §6, no. 3, th. 1, and the equivalence of (7.1.10) and (7.1.13) follows from loc. cit., th. 2; the fact that (7.1.10) implies (7.1.14) follows from loc. cit., cor. 4 of the prop. 6; on the other hand, (7.1.14) implies (7.1.13) by applying the zero homomorphism. Finally, (7.1.10) implies that if \( f \) is invertible in \( A/\mathfrak{N} \), then \( f \) is not contained in any maximal ideal of \( A \), thus \( f \) is invertible in \( A \), in other words, (7.1.10) implies (7.1.12); conversely, (7.1.12) implies (7.1.11).

It therefore remains to prove (7.1.11). Now as \( A \) is separated and complete, and the sequence \( (\mathfrak{N}^n) \) tends to 0, it is immediate that the series \( \sum_{n=0}^{\infty}(-1)^n x^n \) is convergent in \( A \), and that if \( y \) is its sum, then we have \( y(1 + x) = 1 \).

\[ \Box \]

7.2. Adic rings and projective limits

(7.2.1). Each projective limit of discrete rings is evidently a linearly topologized ring, separated and compact. Conversely, let \( A \) be a linearly topologized ring, and let \( (\mathfrak{N}_\lambda) \) be a fundamental system of open neighborhoods of 0 in \( A \) consisting of ideals. The canonical maps \( \phi_\lambda : A \to A/\mathfrak{N}_\lambda \) form a projective system of continuous representations and therefore define a continuous representation \( \phi : A \to \varinjlim A/\mathfrak{N}_\lambda \); if \( A \) is separated, then \( \phi \) is a topological isomorphism from \( A \) to an everywhere-dense subring of \( \varinjlim A/\mathfrak{N}_\lambda \); in addition \( A \) is complete, then \( \phi \) is a topological isomorphism from \( A \) to \( \varinjlim A/\mathfrak{N}_\lambda \).

Lemma (7.2.2). — For a linearly topologized ring to be admissible, it is necessary and sufficient that it is isomorphic to a projective limit \( A = \varinjlim A_\lambda \), where \( (A_\lambda, \mu_{\lambda\mu}) \) is a projective limit of discrete rings having for the set of indices a filtered ordered \((\text{by} \leq)\) \( L \) which admits a smallest element denoted 0 and satisfies the following conditions: 1st. the \( u_\lambda : A \to A_\lambda \) are surjective; 2nd. the kernel \( \mathfrak{N}_\lambda \) of \( u_{0\lambda} : A_\lambda \to A_0 \) is nilpotent.

When this is so, the kernel \( \mathfrak{N} \) of \( u_0 : A \to A_0 \) is equal to \( \varinjlim \mathfrak{N}_\lambda \).

PROOF. The necessity of the condition follows from (7.2.1), by choosing \( (\mathfrak{N}_\lambda) \) to be a fundamental system of neighborhoods of 0 consisting of ideals of definitions contained in an ideal of definition \( \mathfrak{N}_0 \) and by applying (7.1.4, i). The converse follows from the definition of the projective limit and from (7.2.1), and the latter assertion is immediate. \[ \Box \]

(7.2.3). Let \( A \) be an admissible topological ring, \( \mathfrak{N} \) an ideal of \( A \) contained in an ideal of definition (in other words (7.1.4) such that \( (\mathfrak{N}^n) \) tends to 0); we can consider on \( A \) the ring topology having for a fundamental system of neighborhoods of 0 the powers \( \mathfrak{N}^n \) \((n > 0)\); we call again this the \( \mathfrak{N} \)-preadic topology. The hypothesis that \( A \) is admissible implies that \( \bigcup_n \mathfrak{N}^n = 0 \), therefore the \( \mathfrak{N} \)-preadic topology on \( A \) is separated; let \( \hat{A} = \varprojlim A/\mathfrak{N}^n \) be the completion of \( A \) for this topology (where the \( A/\mathfrak{N}^n \) are equipped with the discrete topology), and denote by \( u \) (the (not necessarily continuous) ring homomorphism \( A \to \hat{A} \), the projective limit of the sequence of homomorphisms \( u_0 : A \to A/\mathfrak{N}^n \). On the other hand, the \( \mathfrak{N} \)-preadic topology on \( A \) is finer than the given topology \( \mathcal{T} \) on \( A \); as \( A \) is separated and complete for \( \mathcal{T} \), we can extend by continuity the identity map of \( \hat{A} \) (equipped with the \( \mathfrak{N} \)-preadic topology) to \( A \) equipped with \( \mathcal{T} \); this gives a continuous representation \( v : \hat{A} \to A \).

Proposition (7.2.4). — If \( A \) is an admissible ring and \( \mathfrak{N} \) is contained in an ideal of definition of \( A \), then \( A \) is separated and complete for the \( \mathfrak{N} \)-preadic topology.

PROOF. With the notation of (7.2.3), it is immediate that \( v \circ u \) is the identity map of \( A \). On the other hand, \( u_0 \circ v : \hat{A} \to A/\mathfrak{N}^n \) is the extension by continuity (for the \( \mathfrak{N} \)-preadic topology on \( A \) and the discrete topology on \( A/\mathfrak{N}^n \)) of the canonical map \( u_0 \); in other words, it is the canonical map from \( \hat{A} = \varprojlim A/\mathfrak{N}^n \) to \( A/\mathfrak{N}^n \); \( u \circ v \) is therefore the projective limit of this sequence of maps, which is by definition the identity map of \( \hat{A} \); this proves the proposition. \[ \Box \]

Corollary (7.2.5). — Under the hypotheses of (7.2.3), the following conditions are equivalent:

(a) the homomorphism \( u \) is continuous;
(b) the homomorphism \( v \) is bicontinuous;
(c) \( A \) is a \( \mathfrak{N} \)-adic ring.
Corollary (7.2.6). — Let $A$ be an admissible ring, $\mathfrak{J}$ an ideal of definition for $A$. For $A$ to be Noetherian, it is necessary and sufficient for $A/\mathfrak{J}$ to be Noetherian and for $\mathfrak{J}/\mathfrak{J}^2$ to be an $A/\mathfrak{J}$-module of finite type.

These conditions are evidently necessary. Conversely, suppose the conditions are satisfied; as according to (7.2.4) $A$ is complete for the $\mathfrak{J}$-adic topology, for it to be Noetherian, it is necessary and sufficient that the associated graded ring $\text{grad} A$ (for the filtration on the $\mathfrak{J}^n$) is ([(CC, p. 18–07, th. 4)]). Now, let $a_1, \ldots, a_n$ be the elements of $\mathfrak{J}$ whose classes mod. $\mathfrak{J}^2$ are the generators of $\mathfrak{J}/\mathfrak{J}^2$ as a $A/\mathfrak{J}$-module. It is immediate by induction that the classes mod. $\mathfrak{J}^n$ of the monomials of total degree $m$ in the $a_i$ ($1 \leq i \leq n$) form a system of generators for the $A/\mathfrak{J}$-module $\mathfrak{J}^n/\mathfrak{J}^{n+1}$. We conclude that $\text{grad}(A)$ is a ring isomorphic to a quotient of $(A/\mathfrak{J})[T_1, \ldots, T_n]$ ($T_i$ indeterminates), which finishes the proof.

Proposition (7.2.7). — Let $(A_i, u_{ij})$ be a projective system $(i \in \mathbb{N})$ of discrete rings, and for each integer $i$, let $\mathfrak{J}_i$ be the kernel in $A_i$ of the homomorphism $u_{0i} : A_i \to A_0$. We suppose that:

(a) For $i \leq j$, $u_{ij}$ is surjective and its kernel is $\mathfrak{J}_{ij}^{i+1}$ (therefore $A_i$ is isomorphic to $A_j/\mathfrak{J}_{ij}^{i+1}$).

(b) $\mathfrak{J}_1/\mathfrak{J}_1^2 \cong \mathfrak{J}_1$ is a module of finite type over $A_0 = A_1/\mathfrak{J}_1$.

Let $A = \lim_{\leftarrow i} A_i$, and for each integer $n \geq 0$, let $u_n$ be the canonical homomorphism $A \to A_n$, and let $\mathfrak{J}^{(n+1)}$ be its kernel. Then we have these conditions:

(i) $A$ is an adic ring, having $\mathfrak{J} = \mathfrak{J}^{(1)}$ for an ideal of definition.

(ii) We have $\mathfrak{J}^{(n)} = \mathfrak{J}^n$ for each $n \geq 1$.

(iii) $\mathfrak{J}/\mathfrak{J}^2$ is isomorphic to $\mathfrak{J}_1 = \mathfrak{J}_1/\mathfrak{J}_1^2$, and as a result is a module of finite type over $A_0 = A/\mathfrak{J}$.

Proof. The hypothesis of surjectivity on the $u_{ij}$ implies that $u_n$ is surjective; in addition, the hypothesis (a) implies that $\mathfrak{J}_{ij}^{i+1} = 0$, therefore $A$ is an admissible ring (7.2.2); by definition, the $\mathfrak{J}^{(n)}$ form a fundamental system of neighborhoods of 0 in $A$, so (ii) implies (i). In addition, we have $\mathfrak{J} = \lim_{\leftarrow i} \mathfrak{J}_i$ and the maps $\mathfrak{J} \to \mathfrak{J}_i$ are surjective, so (ii) implies (iii), and it remains to prove (ii). By definition, $\mathfrak{J}^{(n)}$ consists of the elements $(x_k)_{k \geq 0}$ of $A$ such that $x_k = 0$ for $k < n$, therefore $\mathfrak{J}^{(n)} \mathfrak{J}^{(m)} < \mathfrak{J}^{(n+m)}$, in other words the $\mathfrak{J}^{(n)}$ form a filtration of $A$. On the other hand, $\mathfrak{J}^{(n)}/\mathfrak{J}^{(n+1)}$ is isomorphic to the projection from $\mathfrak{J}^{(n)}$ to $A_n$; as $\mathfrak{J}^{(n)} = \lim_{\leftarrow i \geq n} \mathfrak{J}^{(i)}$, this projection is none other than $\mathfrak{J}_n$, which is a module over $A_0 = A_n/\mathfrak{J}_n$. Now let $a_j = (a_{jk})_{k \geq 0}$ be $r$ elements of $\mathfrak{J} = \mathfrak{J}^{(1)}$ such that $a_{11}, \ldots, a_{1r}$ form a system of generators for $\mathfrak{J}_1$ over $A_0$; we will see that the set $S_n$ of monomials of total degree $n$ and the $a_j$ generate the ideal $\mathfrak{J}^{(n)}$ of $A$. As $\mathfrak{J}_{ij}^{i+1} = 0$, it is clear first of all that $S_n \subset \mathfrak{J}^{(n)}$; since $A$ is complete for the filtration $(\mathfrak{J}^{(m)})$, it suffices to prove that the set $\mathfrak{S}_m$ of classes mod. $\mathfrak{J}^{(n+1)}$ of elements of $S_n$ generate the graded module $\mathfrak{J}^{(n)}$ over the graded ring $\text{grad}(A)$ for the above filtration ([(CC, p. 18–06, lemme]); according to the definition of the multiplication on $\text{grad}(A)$, it suffices to prove that for each $m$, $\mathfrak{S}_m$ is a system of generators for the $A_0$-module $\mathfrak{J}^{(m)}/\mathfrak{J}^{(m+1)}$, or that $\mathfrak{J}^{(m)}$ is generated by the monomials of degree $m$ in the $a_{jm}$ ($1 \leq j \leq r$). For this, it remains to show that $\mathfrak{J}_m$ is generated (as an $A_m$-module) by the monomials of degree $\leq m$ relative to $a_{jm}$; the proposition being evident by definition for $m = 1$, we argue by induction on $m$, and let $\mathfrak{J}_m$ be the $A_m$-submodule of $\mathfrak{J}_m$ generated by these monomials. The relation $\mathfrak{J}_{m-1} = \mathfrak{J}_m/\mathfrak{J}_m^m$ and the induction hypothesis prove that $\mathfrak{J}_m = \mathfrak{J}_m^m + \mathfrak{J}_m^m$, hence, since $\mathfrak{J}_m^{m+1} = 0$, we have $\mathfrak{J}_m^m = \mathfrak{J}_m^m$, and finally $\mathfrak{J}_m = \mathfrak{J}_m^m$.

Corollary (7.2.8). — Under the conditions of Proposition (7.2.7), for $A$ to be Noetherian, it is necessary and sufficient that $A_0$ is.

Proof. This follows immediately from Corollary (7.2.6).

Proposition (7.2.9). — Suppose the hypotheses of Proposition (7.2.7); for each integer $i$, let $M_i$ be an $A_i$-module, and for $i \leq j$, let $v_{ij} : M_j \to M_i$ be a $u_{ij}$-homomorphism, such that $(M_i, v_{ij})$ is a projective system. In addition, suppose that $M_0$ is an $A_0$-module of finite type and that the $v_{ij}$ are surjective with kernel $\mathfrak{J}_{ij}^{i+1} M_i$. Then $M = \lim_{\leftarrow i} M_i$ is an $A$-module of finite type, and the kernel of the surjective $u_n$-homomorphism $v_n : M \to M_n$ is $\mathfrak{J}^{n+1} M$ (such that $M_n$ identifies with $M/\mathfrak{J}^{n+1} M = M \otimes_A (A/\mathfrak{J}^{n+1})$).

Proof. Let $z_h = (z_{hk})_{k \geq 0}$ be a system of $s$ elements of $M$ such that the $z_{hi}$ $(1 \leq h \leq s)$ forms a system of generators for $M_0$; we will show that the $z_h$ generate the $A$-module $M$. The $A$-module $M$
is separated and complete for the filtration by the $M^{(n)}$, where $M^{(n)}$ is the set of $y = (y_k)_{k \geq 0}$ in $M$ such that $y_k = 0$ for $k < n$; it is clear that we have $\mathfrak{g}^{(n)} M \subset M^{(n)}$ and that $M^{(n)}/M^{(n+1)} = \mathfrak{g}^{(n)} M_n$.

We therefore have reduced to showing that the classes of the $\mathfrak{g}^{(n)}$ modulo $M^{(0)}$ generate the graded module $\mathrm{gr}(M)$ (by the above filtration) over the graded ring $\mathrm{gr}(A)$ [CC, p. 18–06, lemme]; for this, we observe that it suffices to prove that the $z_{hn} (1 \leq h \leq s)$ generate the $A_n$-module $M_n$.

We argue by induction on $n$, the proposition being evident by definition for $n = 0$; the relation $M_{n-1} = M_n/\mathfrak{g}^{(n)} M_n$ and the induction hypothesis show that if $M'_n$ is the submodule of $M_n$ generated by the $z_{hn}$, we have that $M_n = M'_n + \mathfrak{g}^{n} M_n$, and as $\mathfrak{g}_n$ is nilpotent, this implies that $M_n = M'_n$.

Similarly, passing to the associated graded modules shows that the canonical map from $\mathfrak{g}^{(n)}$ to $M^{(n)}$ is surjective (thus bijection), in other words that $\mathfrak{g}^{(n)} M = \mathfrak{g}^{n} M$ is the kernel of $M \to M_{n-1}$. □

**Corollary (7.2.10).** — Let $(N_i, w_{ij})$ be a second projective system of $A_i$-modules satisfying the conditions of Proposition (7.2.9), and let $N = \lim N_i$. There is a bijective correspondence between the projective systems $(h_i)$ of $A_i$-homomorphisms $h_i : \tilde{M}_i \to N_i$ and the homomorphisms of $A$-modules $h : M \to N$ (which is necessarily continuous for the $\mathfrak{g}$-adic topologies).

**Proof.** It is clear that if $h : M \to N$ is an $A$-homomorphism, then we have $h(\mathfrak{g}^n M) \subset \mathfrak{g}^n N$, hence the continuity of $h$; by passing to quotients, there corresponds to $h$ a projective system of $A_i$-homomorphisms $h_i : M_i \to N_i$, whose projective limit is $h$, hence the corollary. □

**Remark (7.2.11).** — Let $A$ be an adic ring with an ideal of definition $\mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{g}^2$ is an $A/\mathfrak{g}$-module of finite type; it is clear that the $A_1$-modules $M_n = \mathfrak{g}^n M_n$, and the induction hypothesis show that if $M_n$ is such that the first $k$ components of $\tilde{u}(z')$ coincide with the $z$; in other words, the image of $\tilde{M}'$ under $\tilde{u}$ is dense in the kernel of $\tilde{v}$.

If we suppose that $A$ is Noetherian, then so is $\tilde{A}$, according to (7.2.12), $\mathfrak{g}/\mathfrak{g}^2$ is then an $A$-module of finite type.

**Theorem (7.3.2).** — (Krull’s Theorem). Let $A$ be a Noetherian ring, $\mathfrak{g}$ an ideal of $A$, $M$ an $A$-module of finite type, and $M'$ a submodule of $M$; then the induced topology on $M'$ by the $\mathfrak{g}$-predic topology of $A$ is identical to the $\mathfrak{g}$-predic topology of $M'$.

This follows immediately from
Lemma (7.3.2.1). — (Artin–Rees Lemma). Under the hypotheses of (7.3.2), there exists an integer \( p \) such that, for \( n \geq p \), we have

\[
M' \cap \mathfrak{J}^n M = \mathfrak{J}^{n-p}(M' \cap \mathfrak{J}^p M).
\]

For the proof, see [CC, p. 2–04].

Corollary (7.3.3). — Under the hypotheses of (7.3.2), the canonical map \( M \otimes_A \hat{A} \to \hat{M} \) is bijective, and the functor \( M \otimes_A \hat{A} \) is exact in \( M \) on the category of \( A \)-modules of finite type; as a result, the separated \( \hat{3} \)-adic completion \( \hat{A} \) is a flat \( A \)-module (6.1.1).

Proof. We first note that \( \hat{M} \) is an exact functor in \( M \) on the category of \( A \)-modules of finite type. Indeed, let \( 0 \to M' \to M \to M'' \to 0 \) be an exact sequence; we have seen that \( v : \hat{M} \to \hat{M}'' \) is surjective (7.3.1); on the other hand, if \( i \) is the canonical homomorphism \( M \to \hat{M} \), it follows from Krull’s Theorem (7.3.2) that the closure of \( i(\hat{M}') \) in \( \hat{M} \) identifies with the separated completion of \( M' \) for the \( \hat{3} \)-adic topology; thus \( \hat{v} \) is injective, and according to (7.3.1), the image of \( \hat{v} \) is equal to the kernel of \( \hat{v} \).

This being so, the canonical map \( M \otimes_A \hat{A} \to \hat{M} \) is obtained by passing to the projective limit of the maps \( M \otimes_A \hat{A} \to M \otimes_A (A/\mathfrak{J}^n) = M/\mathfrak{J}^n M \). It is clear that this map is bijective when \( M \) is of the form \( A^p \). If \( M \) is an \( A \)-module of finite type, then we have an exact sequence \( A^p \to A^q \to M \to 0 \), hence, by virtue of the right exactness of the functors \( M \otimes_A \hat{A} \) and \( \hat{M} \) (in \( M \)) on the category of \( A \)-modules of finite type, we have the commutative diagram

\[
\begin{array}{ccc}
A^p \otimes_A \hat{A} & \longrightarrow & A^q \otimes_A \hat{A} \\
\downarrow & & \downarrow \\
A^p & \longrightarrow & \hat{M} \longrightarrow 0,
\end{array}
\]

where the two rows are exact and the first two vertical arrows are isomorphisms; this immediately finishes the proof. \( \square \)

Corollary (7.3.4). — Let \( A \) be a Noetherian ring, \( \mathfrak{J} \) an ideal of \( A \), \( M \) and \( N \) two \( A \)-modules of finite type; we have the canonical functorial isomorphisms

\[
(M \otimes_A N)^\wedge \simeq \hat{M} \otimes_A \hat{N}, \quad (\text{Hom}_A(M, N))^\wedge \simeq \text{Hom}_A(\hat{M}, \hat{N}).
\]

Proof. This follows from Corollary (7.3.3), (6.2.1), and (6.2.2). \( \square \)

Corollary (7.3.5). — Let \( A \) be a Noetherian ring, \( \mathfrak{J} \) an ideal of \( A \). The following conditions are equivalent:

(a) \( \mathfrak{J} \) is contained in the radical of \( A \).

(b) \( \hat{A} \) is a faithfully flat \( A \)-module (6.4.1).

(c) Each \( A \)-module of finite type is separated for the \( \hat{3} \)-adic topology.

(d) Each submodule of an \( A \)-module of finite type is closed for the \( \hat{3} \)-adic topology.

Proof. As \( \hat{A} \) is a flat \( A \)-module, the conditions (b) and (c) are equivalent, since (b) is equivalent to saying that if \( M \) is an \( A \)-module of finite type, then the canonical map \( M \to \hat{M} = M \otimes_A \hat{A} \) is injective (6.6.1, c). It is immediate that (c) implies (d), since if \( N \) is a submodule of an \( A \)-module \( M \) of finite type, then \( M/N \) is separated for the \( \hat{3} \)-adic topology, so \( N \) is closed in \( M \). We show that (d) implies (a): if \( m \) is a maximal ideal of \( A \), then \( m \) is closed in \( A \) for the \( \hat{3} \)-adic topology, so \( m = \bigcap_{n \geq 0} (m + \mathfrak{J}^n) \), and as \( m + \mathfrak{J}^p \) is necessarily equal to \( A \) or to \( m \), we have that \( m + \mathfrak{J}^p = m \) for large enough \( p \), hence \( \mathfrak{J}^p \subseteq m \), and \( \mathfrak{J} \subseteq m \) when \( m \) is prime. Finally, (a) implies (b): indeed, let \( P \) be the closure of \( \{0\} \) in an \( A \)-module \( M \) of finite type, for the \( \hat{3} \)-adic topology; according to Krull’s Theorem (7.3.2), the topology induced on \( P \) by the \( \hat{3} \)-adic topology of \( M \) is the \( \hat{3} \)-adic topology of \( P \), so \( \mathfrak{J}^p = P \); as \( P \) is of finite type, it follows from Nakayama’s Lemma that \( P = 0 \) (\( \mathfrak{J} \) being contained in the radical of \( A \)). \( \square \)

We note that the conditions of Corollary (7.3.5) are satisfied when \( A \) is a local Noetherian ring and \( \mathfrak{J} \neq A \) is any ideal of \( A \).
Corollary (7.3.6). — If $A$ is a $\mathfrak{a}$-preadic Noetherian ring, then each $A$-module of finite type is separated and complete for the $\mathfrak{a}$-preadic topology.

PROOF. As we then have $\hat{A} = A$, this follows immediately from Corollary (7.3.3). □

We conclude that Proposition (7.2.9) gives the description of all the modules of finite type over an adic Noetherian ring.

Corollary (7.3.7). — Under the hypotheses of (7.3.2), the kernel of the canonical map $M \to \hat{M} = M \otimes_A \hat{A}$ is the set of the $x \in M$ killed by an element of $1 + \mathfrak{a}$.

PROOF. For each $x \in M$ in this kernel, it is necessary and sufficient that the separated completion of the submodule $Ax$ is 0 (by Krull’s Theorem (7.3.2)), in other words, that $x \in \mathfrak{a}x$. □

7.4. Quasi-finite modules over local rings

Definition (7.4.1). — Given a local ring $A$, with maximal ideal $\mathfrak{m}$, we say that an $A$-module $M$ is quasi-finite (over $A$) if $M/\mathfrak{m}M$ is of finite rank over the residue field $k = A/\mathfrak{m}$.

When $A$ is Noetherian, the separated completion $\hat{M}$ of $M$ for the $\mathfrak{m}$-preadic topology is then an $\hat{A}$-module of finite type; indeed, as $\mathfrak{m}/\mathfrak{m}^2$ is then an $A$-module of finite type, this follows from Example (7.2.12) and from the hypothesis on $M/\mathfrak{m}M$.

In particular, if we suppose that in addition $A$ is complete and $M$ is separated for the $\mathfrak{m}$-preadic topology (in other words, $\bigcap_n \mathfrak{m}^n M = 0$), then $M$ is also an $A$-module of finite type: indeed, $\hat{M}$ is then an $A$-module of finite type, and as $M$ identifies with a submodule of $\hat{M}$, $M$ is also of finite type (and is indeed identical to its completion according to Corollary (7.3.6)).

Proposition (7.4.2). — Let $A$, $B$ be two local rings, $\mathfrak{m}$, $\mathfrak{n}$ their maximal ideals, and suppose that $B$ is Noetherian. Let $\phi : A \to B$ be a local homomorphism, $M$ a $B$-module of finite type. If $M$ is a quasi-finite $A$-module, then the $\mathfrak{m}$-preadic and $\mathfrak{n}$-preadic topologies on $M$ are identical, thus separated.

PROOF. We note that by hypothesis $M/\mathfrak{m}M$ is of finite length as an $A$-module, thus also a fortiori as a $B$-module. We conclude that $n$ is the unique prime ideal of $B$ containing the annihilator of $M/\mathfrak{m}M$: indeed, we immediately reduce (according to (1.7.4) and (1.7.2)) to the case where $M/\mathfrak{m}M$ is simple, thus necessarily isomorphic to $B/n$, and our assertion is evident in this case. On the other hand, as $M$ is a $B$-module of finite type, the prime ideals which contain the annihilator of $M/\mathfrak{m}M$ are those which contain $\mathfrak{m}B + \mathfrak{n}$, where we denote by $\mathfrak{n}$ the annihilator of the $B$-module $M$ (1.7.5). As $B$ is Noetherian, we conclude ([Sam53b, p. 127, Cor. 4]) that $\mathfrak{m}B + \mathfrak{n}$ is an ideal of definition for $B$, in other words there exists a $k > 0$ such that $n^k \subset \mathfrak{m}B + \mathfrak{n} \subset n$; as a result, for each $h > 0$, we have $n^{hk} \subset (\mathfrak{m}B + \mathfrak{n})^h M = \mathfrak{m}^h M \subset n^h M$, which proves that the $\mathfrak{m}$-preadic and $\mathfrak{n}$-preadic topologies on $M$ are the same; the second is separated according to Corollary (7.3.5). □

Corollary (7.4.3). — Under the hypotheses of Proposition (7.4.2), if in addition $A$ is Noetherian and complete for the $\mathfrak{m}$-preadic topology, then $M$ is an $A$-module of finite type.

PROOF. Indeed, $M$ is then separated for the $\mathfrak{m}$-preadic topology, and our assertion follows from the remark after Definition (7.4.1). □

(7.4.4). The most important case of Proposition (7.4.2) is when $B$ is a quasi-finite $A$-module, i.e., $B/\mathfrak{m}B$ is an algebra of finite rank over $k = A/\mathfrak{m}$; furthermore, this condition can be broken down into the combination of the following:

(i) $\mathfrak{m}B$ is an ideal of definition for $B$;
(ii) $B/\mathfrak{n}$ is an extension of finite rank of the field $A/\mathfrak{m}$.

When this is so, every $B$-module of finite type is evidently a quasi-finite $A$-module.

Corollary (7.4.5). — Under the hypotheses of Proposition (7.4.2), if $\mathfrak{n}$ is the annihilator of the $B$-module $M$, then $B/\mathfrak{n}$ is a quasi-finite $A$-module.
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Proof. Suppose $M \neq 0$ (otherwise the corollary is evident). We can consider $M$ as a module over the local Noetherian ring $B/b$; its annihilator then being 0, the proof of Proposition (7.4.2) shows that $m(B/b)$ is an ideal of definition for $B/b$. On the other hand, $M/nM$ is a vector space of finite rank over $A/m$, being a quotient of $M/nM$, which is by hypothesis of finite rank over $A/m$; as $M \neq 0$, we have $M \neq nM$ by virtue of Nakayama’s Lemma; as $M/nM$ is a vector space $\neq 0$ over $B/n$, the fact that it is of finite rank over $A/m$ implies that $B/n$ is also of finite rank over $A/m$; the conclusion follows from (7.4.4) applied to the ring $B/b$. □

7.5. Rings of restricted formal series

(7.5.1). Let $A$ be a topological ring, linearly topologized, separated and complete; let $(\mathfrak{Z}_\lambda)$ be a fundamental system of neighborhoods of 0 in $A$ consisting of (open) ideals, such that $A$ canonically identifies with $\lim_\mathfrak{Z}_\lambda A$ (7.2.1). For each $\lambda$, let $B_\lambda = (A/\mathfrak{Z}_\lambda)[T_1, \ldots, T_r]$, where the $T_i$ are indeterminates; it is clear that the $B_\lambda$ form a projective system of discrete rings. We set $\lim B_\lambda = A\{T_1, \ldots, T_r\}$, and we will see that this topological ring is independent of the fundamental system of ideals $(\mathfrak{Z}_\lambda)$ considered. More precisely, let $A'$ be the subring of the ring of formal series $A[[T_1, \ldots, T_r]]$ consisting of formal series $\sum \alpha T^\alpha$ (with $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$) such that $c_{\alpha} = 0$ (according to the filter by compliments of finite subsets of $\mathbb{N}^r$); we say that these series are the restricted formal series in the $T_i$, with coefficients in $A$. For each neighborhood $V$ of 0 in $A$, let $V'$ be the set of $x = \sum \alpha c_{\alpha} T^\alpha \in A'$ such that $c_{\alpha} \in V$ for all $\alpha$. We verify immediately that the $V'$ form a fundamental system of neighborhoods of 0 defining on $A'$ a separated ring topology; we will canonically define a topological isomorphism from the ring $A\{T_1, \ldots, T_r\}$ to $A'$. For each $\alpha \in \mathbb{N}^r$ and each $\lambda$, let $\phi_{\lambda, \alpha}$ be the map from $(A/\mathfrak{Z}_\lambda)[T_1, \ldots, T_r]$ to $A/\mathfrak{Z}_\lambda$ which sends each polynomial in the first ring to coefficient of $T^\alpha$ in that polynomial. It is clear that the $\phi_{\lambda, \alpha}$ form a projective system of homomorphisms of $A/\mathfrak{Z}_\lambda$-modules, so their projective limit is a continuous homomorphism $\phi_\lambda : A\{T_1, \ldots, T_r\} \to A$; we will see that, for each $y \in A\{T_1, \ldots, T_r\}$, the formal series $\sum_\alpha \phi_\lambda(y) T^\alpha$ is restricted. Indeed, if $y_\lambda$ is the component of $y$ in $B_\lambda$, and if we denote by $H_\lambda$ the finite set of the $\alpha \in \mathbb{N}^r$ for which the coefficients of the polynomial $y_\lambda$ are nonzero, then we have $\phi_{\lambda, \alpha}(y_\lambda) \in \mathfrak{Z}_\lambda$ for $\mathfrak{Z}_\lambda \subset \mathfrak{Z}_\lambda$ and $\alpha \not\in H_\lambda$, and by passing to the limit, $\phi_\lambda(y) \in \mathfrak{Z}_\lambda$ for $\alpha \not\in H_\lambda$. We thus define a ring homomorphism $\phi : A\{T_1, \ldots, T_r\} \to A'$ by setting $\phi(y) = \sum \phi_\lambda(y) T^\alpha$, and it is immediate that $\phi$ is continuous. Conversely, if $\theta_\lambda$ is the canonical homomorphism $A \to A/\mathfrak{Z}_\lambda$, then for each element $z = \sum_\alpha c_{\alpha} T^\alpha \in A'$ and each $\lambda$, there are only a finite number of indices $\alpha$ such that $\theta_\lambda(c_{\alpha}) \neq 0$, and as a result $\psi_\lambda(z) = \sum_\alpha \theta_\lambda(c_{\alpha}) T^\alpha$ is in $B_\lambda$; the $\phi_\lambda$ are continuous and form a projective system of homomorphisms whose projective limit is a continuous homomorphism $\psi : A' \to A\{T_1, \ldots, T_r\}$; it remains to verify that $\phi \circ \psi$ and $\psi \circ \phi$ are the identity automorphisms, which is immediate.

(7.5.2). We identify $A\{T_1, \ldots, T_r\}$ with the ring $A'$ of restricted formal series by means of the isomorphisms defined in (7.5.1). The canonical isomorphisms

$$((A/\mathfrak{Z}_\lambda)[T_1, \ldots, T_r])[T_{r+1}, \ldots, T_s] \simeq (A/\mathfrak{Z}_\lambda)[T_1, \ldots, T_s]$$

define, by passing to the projective limit, a canonical isomorphism

$$(A\{T_1, \ldots, T_r\})\{T_{r+1}, \ldots, T_s\} \simeq A\{T_1, \ldots, T_r\}.$$

(7.5.3). For every continuous homomorphism $u : A \to B$ from $A$ to a linearly topologized ring $B$, separated and complete, and each system $(b_1, \ldots, b_r)$ of $r$ elements of $B$, there exists a unique continuous homomorphism $\overline{u} : A\{T_1, \ldots, T_r\} \to B$, such that $\overline{u}(a) = u(a)$ for all $a \in A$ and $\overline{u}(T_j) = b_j$ for $1 \leq j \leq r$. It suffices to set

$$\overline{u}\left(\sum \alpha c_{\alpha} T^\alpha\right) = \sum \alpha u(c_{\alpha}) b_1^{\alpha_1} \cdots b_r^{\alpha_r};$$

the verification of the fact that the family $(u(c_{\alpha}) b_1^{\alpha_1} \cdots b_r^{\alpha_r})$ is summable in $B$ and that $\overline{u}$ is continuous are immediate and left to the reader. We note that this property (for arbitrary $B$ and $b_j$) characterize the topological ring $A\{T_1, \ldots, T_r\}$ up to unique isomorphism.

Proposition (7.5.4).

(i) If $A$ is an admissible ring, then so is $A' = A\{T_1, \ldots, T_r\}$. 
Let \( A \) be an adic ring, \( \mathfrak{J} \) an ideal of definition for \( A \) such that \( \mathfrak{J}/\mathfrak{J}^2 \) is of finite type over \( A/\mathfrak{J} \). If we set \( \mathfrak{J}' = \mathfrak{J}A' \), then \( A' \) is also a \( \mathfrak{J}' \)-adic ring, and \( \mathfrak{J}'/\mathfrak{J}'^2 \) is of finite type over \( A'/\mathfrak{J}' \). If in addition \( A \) is Noetherian, then so is \( A' \).

**Proof.**

(i) If \( \mathfrak{J} \) is an ideal of \( A \), \( \mathfrak{J}' \) the ideal of \( A' \) consisting of the \( \sum \alpha \mathfrak{J} \) such that \( \alpha \in \mathfrak{J} \) for all \( \alpha \), then \( (\mathfrak{J}')^n \subset (\mathfrak{J}^n) \); if \( \mathfrak{J} \) is an ideal of definition for \( A \), then \( \mathfrak{J}' \) is also an ideal of definition for \( A' \).

(ii) Set \( A_1 = A/\mathfrak{J}^{i+1} \), and for \( i \leq j \), let \( u_{ij} \) be the canonical homomorphism \( A/\mathfrak{J}^{i+1} \to A/\mathfrak{J}^{i+1} \); set \( A_i' = A_i[T_1, \ldots, T_j] \), and let \( u_{ij}' \) be the homomorphism \( A_i' \to A_i' (i \leq j) \) obtained by applying \( u_{ij} \) to the coefficients of the polynomials in \( A_i' \). We will show that the projective system \( (A_i', u_{ij}') \) satisfies the conditions of Proposition (7.2.7); as \( \mathfrak{J}' \) is the kernel of \( A' \to A'_0 \), this proves the first assertion of (ii). It is clear that the \( u_{ij}' \) are surjective; the kernel \( \mathfrak{J}'_i \) of \( u_{ij} \) is the set of polynomials in \( A_i[T_1, \ldots, T_j] \) whose coefficients are in \( \mathfrak{J}/\mathfrak{J}^{i+1} \); in particular, \( \mathfrak{J}_i' \) is the set of polynomials in \( A_i[T_1, \ldots, T_j] \) whose coefficients are in \( \mathfrak{J}/\mathfrak{J}^2 \). As \( \mathfrak{J}/\mathfrak{J}^2 \) is of finite type over \( A_1 = A/\mathfrak{J}^2 \), we see that \( \mathfrak{J}_i'/\mathfrak{J}_i^{2} \) is a module of finite type over \( A_i' \) (or equivalently, over \( A'_0 = A'_1/\mathfrak{J}_1' \)). We will now show that the kernel of \( u_{ij} \) is \( \mathfrak{J}_j^{i+1} \). It is evident that \( \mathfrak{J}_j^{i+1} \) is contained in this kernel. On the other hand, let \( a_1, \ldots, a_m \) be the elements of \( \mathfrak{J} \) whose classes mod \( \mathfrak{J}^2 \) generate \( \mathfrak{J}/\mathfrak{J}^2 \); we verify immediately that the classes mod \( \mathfrak{J}^{i+1} \) of monomials of degree \( \leq j \) in the \( a_k (1 \leq k \leq m) \) generate \( \mathfrak{J}/\mathfrak{J}^{i+1} \), and the classes of monomials of degree \( i \) mod \( \mathfrak{J}/\mathfrak{J}^{i+1} \); a monomial in the \( T_k \) having such an element for a coefficient is thus a product of \( i + 1 \) elements of \( \mathfrak{J}' \), which establishes our assertion. Finally, if \( A \) is Noetherian, then so is \( A'/\mathfrak{J}' = (A/\mathfrak{J})[T_1, \ldots, T_j] \), hence \( A' \) is Noetherian (7.2.8).

**Proposition (7.5.5).** — Let \( A \) be a Noetherian \( \mathfrak{J} \)-adic ring, \( B \) an admissible topological ring, \( \phi : A \to B \) a continuous homomorphism, making \( A \) and \( B \)-algebra. The following conditions are equivalent:

(a) \( B \) is Noetherian and \( \mathfrak{J}B \)-adic, and \( B/\mathfrak{J}B \) is an algebra of finite type over \( A/\mathfrak{J} \).

(b) \( B \) is topologically \( A \)-isomorphic to \( \lim B_n \), where \( B_n = B_m/\mathfrak{J}^{n+1}B_m \) for \( m \geq n \), and \( B_1 \) is an algebra of finite type over \( A_1 = A/\mathfrak{J}^2 \).

(c) \( B \) is topologically \( A \)-isomorphic to a quotient of an algebra of the form \( A[T_1, \ldots, T_j] \) by an ideal (necessarily closed according to Corollary (7.3.6) and Proposition (7.5.4), ii).

**Proof.** As \( A \) is Noetherian, so is \( A' = A[T_1, \ldots, T_j] \) (7.5.4), so (c) implies that \( B \) is Noetherian; as \( \mathfrak{J}' = \mathfrak{J}A' \) is an open neighborhood of 0 in \( A' \) such that the \( \mathfrak{J}'^n \) form a fundamental system of neighborhoods of 0, the images \( \mathfrak{J}'^nB \) of the \( \mathfrak{J}'^n \) form a fundamental system of neighborhoods of 0 in \( B \), and as \( B \) is separated and complete, \( B \) is a \( \mathfrak{J}B \)-adic ring. Finally, \( B/\mathfrak{J}B \) in an algebra (over \( A/\mathfrak{J} \)) quotient of \( A'/\mathfrak{J}A' = (A/\mathfrak{J})[T_1, \ldots, T_j] \), so it is of finite type, which proves that (c) implies (a).

If \( B \) is \( \mathfrak{J}B \)-adic and Noetherian, then \( B \) is isomorphic to \( \lim B_n \), where \( B_n = B/\mathfrak{J}^{n+1}B \) (7.2.11), and \( B/\mathfrak{J}B \) is a module of finite type over \( B/\mathfrak{J}B \). Let \( (a_{ij})_{1 \leq i, j \leq s} \) be a system of generators for the \( B/\mathfrak{J}B \)-module \( \mathfrak{J}B/\mathfrak{J}^2B \), and let \( (c_{ij})_{1 \leq i, j \leq s} \) be a system of elements of \( B/\mathfrak{J}^2B \) such that the classes mod \( B/\mathfrak{J}^2B \) form a system of generators for the \( A/\mathfrak{J} \)-algebra \( B/\mathfrak{J}B \); we see immediately that the \( c_{ij} \) form a system of generators for the \( A/\mathfrak{J} \)-algebra \( B/\mathfrak{J}^2B \), hence (a) implies (b).

It remains to prove that (b) implies (c). The hypotheses imply that \( B_1 \) is a Noetherian ring, and as \( B_1 = B_2/\mathfrak{J}^2B_2 \), we have \( \mathfrak{J}^2B_1 = 0 \), hence \( \mathfrak{J}B_1 = \mathfrak{J}B_1/\mathfrak{J}^2B_1 \) is a \( B_1 \)-module of finite type. The conditions of Proposition (7.2.7) are thus satisfied by the projective system \( (B_n) \) and \( B \) is a \( \mathfrak{J}B \)-adic ring. Let \( (c_{ij})_{1 \leq i, j \leq s} \) be a finite system of elements of \( B \) whose classes mod \( \mathfrak{J}B \) generate the \( A/\mathfrak{J} \)-algebra \( B/\mathfrak{J}B \), and whose linear combinations with coefficients in \( \mathfrak{J} \) are such that their classes mod \( \mathfrak{J}^2B \) generate the \( B_1 \)-module \( \mathfrak{J}B/\mathfrak{J}^2B \). There exists a continuous \( A \)-homomorphism \( u \) from \( A' = A[T_1, \ldots, T_j] \) to \( B \) which reduces to \( \phi \) on \( A \) and is such that \( u(T_i) = c_i \) for \( 1 \leq i \leq r \) (7.5.3); if we prove that \( u \) is surjective, then (c) will be established, since from \( u(A') = B \) we deduce that \( u(\mathfrak{J}^nA') = \mathfrak{J}^nB \), in other words that \( u \) is a strict morphism of topological rings and \( B \) is this.
isomorphic to a quotient of \( A' \) by a closed ideal. As \( B \) is complete for the \( \mathfrak{J}B \)-adic topology, it suffices ([CC, p. 18–07]) to show that the homomorphism \( \text{grad}(A') \to \text{grad}(B) \), induced canonically by \( u \) for the \( \mathfrak{J} \)-adic filtrations on \( A' \) and \( B \), is surjective. But by definition, the homomorphisms \( A'/\mathfrak{J}A' \to B/\mathfrak{J}B \) and \( \mathfrak{J}^nA'/\mathfrak{J}^{n+1}A' \to B/\mathfrak{J}^nB \) induced by \( u \) are surjective; by induction on \( n \), we immediately deduce that so is \( \mathfrak{J}^nA'/\mathfrak{J}^{n+1}A' \to B/\mathfrak{J}^nB \), and a fortiori so is \( \mathfrak{J}^nA'/\mathfrak{J}^{n+1}A' \to \mathfrak{J}^nB/\mathfrak{J}^{n+1}B \), which finishes the proof.

\[ \square \]

### 7.6. Completed rings of fractions

**Proposition (7.6.2).** — The ring \( A\{S^{-1}\} \) is topologically isomorphic to the separated completion of the ring \( S^{-1}A \) for the topology which has a fundamental system of neighborhoods of \( 0 \) consisting of \( S \)-adically open sets.

**Proof.** If \( v_1 \) is the canonical homomorphism \( S^{-1}A \to S^{-1}_{A,1}A_{A,1} \) induced by \( u_{1,\lambda} \), then the kernel of \( v_1 \) is surjective, hence the proposition (7.2.1).

**Corollary (7.6.3).** — If \( S' \) is the canonical image of \( S \) in the separated completion \( \hat{A} \) of \( A \), then \( A\{S'^{-1}\} \) canonically identifies with \( \hat{A}\{S'^{-1}\} \).

We note that if \( A \) is separated and complete, then it is not necessarily the same for \( S^{-1}A \) with the topology defined by the \( S^{-1}J_{\lambda} \), as we see for example by taking \( S \) to be the set of the \( f^n \) (\( n \geq 0 \)), where \( f \) is topologically nilpotent but not nilpotent: indeed, \( S^{-1}A \) is not 0 and on the other hand, for each \( \lambda \) there exists an \( n \) such that \( f^n \in J_{\lambda} \), so \( 0 = f^n/f^n \in S^{-1}J_{\lambda} \) and \( S^{-1}J_{\lambda} = S^{-1}A \).

**Corollary (7.6.4).** — If, in \( A \), 0 does not belong to \( S \), then the ring \( A\{S^{-1}\} \) is not 0.

**Proof.** Indeed, 0 does not belong to \( \{1\} \) in the ring \( S^{-1}A \); otherwise, we would have that \( 1 \in S^{-1}J_{\lambda} \) for each open ideal \( J_{\lambda} \) of \( A \), and it follows that \( J_{\lambda} \cap S \neq \emptyset \) for all \( \lambda \), contradicting the hypothesis.

**Proposition (7.6.5).** We say that \( A\{S^{-1}\} \) is the completed ring of fractions of \( A \) with denominators in \( S \). With the above notation, it is clear that the inverse image of \( S^{-1}J_{\lambda} \) in \( A \) contains \( J_{\lambda} \), hence the canonical map \( A \to S^{-1}A \) is continuous, and if we compose it with the canonical map \( S^{-1}A \to A\{S^{-1}\} \), we obtain a canonical continuous homomorphism \( A \to A\{S^{-1}\} \), the projective limit of the homomorphisms \( A \to S_{A,1}^{-1}A_{A,1} \).

**Proposition (7.6.6).** The couple consisting of \( A\{S^{-1}\} \) and the canonical homomorphism \( A \to A\{S^{-1}\} \) are characterized by the following universal property: every continuous homomorphism \( u \) from \( A \) to a linearly topologized ring \( B \), separated and complete, such that \( u(S) \) consists of the invertible elements of \( B \), uniquely factorizes as \( A \to A\{S^{-1}\} \overset{u'}{\to} B \), where \( u' \) is continuous. Indeed, \( u \) uniquely factorizes as \( A \to S^{-1}A \overset{v'}{\to} B \); as for each open ideal \( J_{\lambda} \) of \( B \) we have that \( u^{-1}(J_{\lambda}) \) contains \( J_{\lambda} \), \( v'^{-1}(J_{\lambda}) \) necessarily contains \( S^{-1}J_{\lambda} \), so \( v' \) is continuous; since \( B \) is separated and complete, \( v' \) uniquely factorizes as \( S^{-1}A \to A\{S^{-1}\} \overset{v'}{\to} B \), where \( u' \) is continuous; hence our assertion.

**Proposition (7.6.7).** Let \( B \) be a second linearly topologized ring, \( T \) a multiplicative subset of \( B \), \( \phi : A \to B \) a continuous homomorphism such that \( \phi(S) \subset T \). According to the above, the continuous homomorphism \( A \overset{\phi}{\to} B \to B\{T^{-1}\} \) uniquely factorizes as \( A \to A\{S^{-1}\} \overset{\phi'}{\to} B\{T^{-1}\} \), where \( \phi' \) is continuous. In particular, if \( B = A \) and if \( \phi \) is the identity, we see that for \( S \subset T \) we have a continuous homomorphism \( \rho^{T,S} : A\{S^{-1}\} \to A\{T^{-1}\} \) obtained by passing to the separated completion from \( S^{-1}A \to T^{-1}A \); if \( U \) is a third multiplicative subset of \( A \) such that \( S \subset T \subset U \), then we have \( \rho^{U,S} = \rho^{U,T} \circ \rho^{T,S} \).
(7.6.8). Let $S_1$, $S_2$ be two multiplicative subsets of $A$, and let $S'_2$ be the canonical image of $S_2$ in $A\{S_1^{-1}\}$; we then have a canonical topological isomorphism $A\{S_1S_2^{-1}\} \simeq A\{S_1^{-1}\}\{S'_2\}$, as we see from the canonical isomorphism $(S_1S_2)^{-1}A \simeq S'_2^{-1}(S_1^{-1}A)$ (where $S'_2$ is the canonical image of $S_2$ in $S_1^{-1}A$), which is bicontinuous.

(7.6.9). Let $a$ be an open ideal of $A$; we can assume that $\mathfrak{A}_\lambda \subset a$ for all $\lambda$, and as a result $S^{-1}\mathfrak{A}_\lambda \subset S^{-1}a$ in the ring $S^{-1}A$; in other words, $S^{-1}a$ is an open ideal of $S^{-1}A$; we denote by $a\{S^{-1}\}$ its separated completion, equal to $\lim_{\lambda} (S^{-1}a/S^{-1}\mathfrak{A}_\lambda)$, which is an open ideal of $A\{S^{-1}\}$, isomorphic to the closure of the canonical image of $S^{-1}a$. In addition, the discrete ring $A\{S^{-1}\}/a\{S^{-1}\}$ is canonically isomorphic to $S^{-1}A/S^{-1}a = S^{-1}(A/a)$. Conversely, if $a'$ is an open ideal of $A\{S^{-1}\}$, then $a'$ contains an ideal of the form $S^{-1}\mathfrak{A}_\lambda$ which is the inverse image of an ideal of $S^{-1}A/S^{-1}\mathfrak{A}_\lambda$, which is necessarily (1.2.6) of the form $S^{-1}a$, where $a \supset \mathfrak{A}_\lambda$. We conclude that $a' = a\{S^{-1}\}$. In particular (1.2.6):

**Proposition (7.6.10).** — The map $p \mapsto p\{S^{-1}\}$ is an increasing bijection from the set of open prime ideals $p$ of $A$ such that $p \cap S = \emptyset$ to the set of open prime ideals of $A\{S^{-1}\}$; in addition, the field of fractions of $A\{S^{-1}\}/p\{S^{-1}\}$ is canonically isomorphic to that of $A/p$.

**Proposition (7.6.11).** —

(i) If $A$ is an admissible ring, then so is $A' = A\{S^{-1}\}$, and for every ideal of definition $\mathfrak{A}$ for $A$, $\mathfrak{A}' = \mathfrak{A}\{S^{-1}\}$ is an ideal of definition for $A'$.

(ii) Let $A$ be an adic ring, $\mathfrak{A}$ an ideal of definition for $A$ such that $\mathfrak{A}/\mathfrak{A}^2$ is of finite type over $A/\mathfrak{A}$; then $A'$ is a $\mathfrak{A}'$-adic ring and $\mathfrak{A}'/\mathfrak{A}'^2$ is of finite type over $A'/\mathfrak{A}'$. If in addition $A$ is Noetherian, then so is $A'$.

**Proof.**

(i) If $\mathfrak{A}$ is an ideal of definition for $A$, then it is clear that $S^{-1}\mathfrak{A}$ is an ideal of definition for the topological ring $S^{-1}A$, since we have $(S^{-1}\mathfrak{A})^n = S^{-1}\mathfrak{A}^n$. Let $A''$ be the separated ring associated to $S^{-1}A$, $\mathfrak{A}''$ the image of $S^{-1}\mathfrak{A}$ in $A''$; the image of $S^{-1}\mathfrak{A}^n$ is $\mathfrak{A}''^n$, so $\mathfrak{A}''^n$ tends to 0 in $A''$; as $\mathfrak{A}'$ is the closure of $\mathfrak{A}''$ in $A'$, $\mathfrak{A}''^n$ is contained in the closure of $\mathfrak{A}''$, hence tends to 0 in $A'$.

(ii) Set $A_i = A/\mathfrak{A}_{i+1}^i$, and for $i < j$, let $u_{ij}$ be the canonical homomorphism $A/\mathfrak{A}_{i+1}^i \rightarrow A/\mathfrak{A}_{j+1}^j$; let $S_i$ be the canonical image of $S$ in $A_i$, and set $A'_i = S_i^{-1}A_i$; finally, let $u'_{ij} : A'_i \rightarrow A'_j$ be the homomorphism canonically induced by $u_{ij}$. We show that the projective system $(A'_i, u'_{ij})$ satisfies the conditions of Proposition (7.2.7): it is clear that the $u'_{ij}$ are surjective; on the other hand, the kernel of $u'_{ij}$ is $S_i^{-1}(\mathfrak{A}^{-1}_{j+1}/\mathfrak{A}_{i+1}^i)$ (1.3.2), equal to $\mathfrak{A}'_{i+1}^j$, where $\mathfrak{A}'_{i+1}^j = S_1^{-1}(\mathfrak{A}/\mathfrak{A}^j_{i+1})$; finally, $\mathfrak{A}'_{i}/\mathfrak{A}'_{i}^2 = S_1^{-1}(\mathfrak{A}/\mathfrak{A}^2)$, and as $\mathfrak{A}/\mathfrak{A}^2$ is of finite type over $A/\mathfrak{A}^2$, $\mathfrak{A}'_{i}/\mathfrak{A}'_{i}^2$ is of finite type over $A'_i$. Finally, if $A$ is Noetherian, then so is $A'_0 = S_0^{-1}(A/\mathfrak{A})$, which finishes the proof (7.2.8).

**Corollary (7.6.12).** — Under the hypotheses of Proposition (7.6.11, ii), we have $(\mathfrak{A}\{S^{-1}\})^n = \mathfrak{A}^n\{S^{-1}\}$.

**Proof.** This follows from Proposition (7.2.7) and the proof of Proposition (7.6.11).

**Proposition (7.6.13).** — Let $A$ be an adic Noetherian ring, $S$ a multiplicative subset of $A$; then $A\{S^{-1}\}$ is a flat $A$-module.

**Proof.** If $\mathfrak{A}$ is an ideal of definition for $A$, then $A\{S^{-1}\}$ is the separated completion of the Noetherian ring $S^{-1}A$ equipped with the $S^{-1}\mathfrak{A}$-preadic topology; as a result (7.3.3) $A\{S^{-1}\}$ is a flat $S^{-1}A$-module; as $S^{-1}A$ is a flat $A$-module (6.3.1), the proposition follows from the transitivity of flatness (6.2.1).

**Corollary (7.6.14).** — Under the hypotheses of Proposition (7.6.13), let $S' \subset S$ be a second multiplicative subset of $A$; then $A\{S^{-1}\}$ is a flat $A\{S'^{-1}\}$-module.
Proof. By (7.6.8), \(A\{S^{-1}\}\) canonically identifies with \(A\{S'^{-1}\}\{S_0^{-1}\}\), where \(S_0\) is the canonical image of \(S\) in \(A\{S'^{-1}\}\), and \(A\{S'^{-1}\}\) is Noetherian (7.6.11). \(\square\)

(7.6.15). For each element \(f\) of a linearly topologized ring \(A\), we denote by \(A_{(f)}\) the completed ring of fractions \(A\{S_f^{-1}\}\), where \(S_f\) is the multiplicative set of the \(f^n\) \((n \geq 0)\); for each open ideal \(a\) of \(A\), we write \(a_{(f)}\) for \(a\{S_f^{-1}\}\). If \(g\) is a second element of \(A\), then we have a canonical continuous homomorphism \(A_{(f)} \rightarrow A_{(g)}\) (7.6.7). When \(f\) varies over a multiplicative subset \(S\) of \(A\), the \(A_{(f)}\) form a filtered inductive system with the above homomorphisms; we set \(A_S = \lim_{f \in S} A_{(f)}\). For every \(f \in S\), we have a homomorphism \(A_{(f)} \rightarrow A\{S^{-1}\}\) (7.6.7), and these homomorphisms form an inductive system; by passing to the inductive limit, they thus define a canonical homomorphism \(A_S \rightarrow A\{S^{-1}\}\).

Proposition (7.6.16). — If \(A\) is a Noetherian ring, then \(A\{S^{-1}\}\) is a flat module over \(A_S\).

Proof. By (7.6.14), \(A\{S^{-1}\}\) is flat for each of the rings \(A_{(f)}\) for \(f \in S\), and the conclusion follows from (6.2.3). \(\square\)

Proposition (7.6.17). — Let \(p\) be an open prime ideal in an admissible ring \(A\), and let \(S = A - p\). Then the rings \(A\{S^{-1}\}\) and \(A_S\) are local rings, the canonical homomorphism \(A_S \rightarrow A\{S^{-1}\}\) is local, and the residue fields of \(A_S\) and \(A\{S^{-1}\}\) are canonically isomorphic to the field of fractions of \(A/p\).

Proof. Let \(\mathfrak{a} \subset p\) be an ideal of definition for \(A\); we have \(S^{-1}\mathfrak{a} \subset S^{-1}p\) = \(pA_p\), so \(A_p/S^{-1}\mathfrak{a}\) is a local ring; we conclude from Corollary (7.1.12), (7.6.9), and Proposition (7.6.11, i) that \(A\{S^{-1}\}\) is a local ring. Set \(m = \lim_{f \in S} \mathfrak{p}_f\), which is an ideal in \(A_S\); we will see that each element in \(A_S\) not in \(m\) is invertible. Indeed, such an element is the image in \(A_S\) of an element \(z \in A_{(f)}\) not in \(p_{(f)}\), for an \(f \in S\); its canonical image \(z_0\) in \(A_{(f)}/\mathfrak{a}_f = S_f^{-1}(A/\mathfrak{a})\) therefore is not in \(S_f^{-1}(p/\mathfrak{a})\) (7.6.9), which means that \(z_0 = x/f^k\), where \(x \in A\), \(f\) are the classes of \(x, f\) mod \(\mathfrak{a}\). As \(x \in S\), we have \(g = xf \in S\), and in \(S_f^{-1}A\), the canonical image \(y_0 = x^{k+1}/g^k\) of \(x/f^k\) in \(S_f^{-1}A\) admits an inverse \(x^{k-1}f^{2k}/g^k\). This implies \(a\) fortiori that the image of \(y_0\) in \(S_f^{-1}A/S_f^{-1}\mathfrak{a}\) is invertible, so (7.6.9) and Corollary (7.1.12)) the canonical image \(y\) of \(z\) in \(A_S\) is invertible; the image of \(z\) in \(A_S\) (equal to that of \(y\)) is a result invertible. We thus see that \(A_S\) a local ring with maximal ideal \(m\); in addition, the image of \(p_{(f)}\) in \(A\{S^{-1}\}\) is contained in the maximal ideal \(p\{S^{-1}\}\) of this ring; \(a\) fortiori, the image of \(m\) in \(A\{S^{-1}\}\) is contained in \(p\{S^{-1}\}\), so the canonical homomorphism \(A_S \rightarrow A\{S^{-1}\}\) is local. Finally, as each element of \(A\{S^{-1}\}/p\{S^{-1}\}\) is the image of an element in the ring \(S_f^{-1}A\) for a suitable \(f \in S\), the homomorphism \(A_S \rightarrow A\{S^{-1}\}/p\{S^{-1}\}\) is surjective, and gives an isomorphism of the residue fields by passing to quotients. \(\square\)

Corollary (7.6.18). — Under the hypotheses of Proposition (7.6.17), if we suppose also that \(A\) is an adic Noetherian ring, then the local rings \(A\{S^{-1}\}\) and \(A_S\) are Noetherian, and \(A\{S^{-1}\}\) is a faithfully flat \(A_S\)-module.

Proof. We know from before (7.6.11, ii) that \(A\{S^{-1}\}\) is Noetherian and \(A_S\)-flat (7.6.16); as the homomorphism \(A_S \rightarrow A\{S^{-1}\}\) is local, we conclude that \(A\{S^{-1}\}\) is a faithfully flat \(A_S\)-module (6.6.2), and as a result that \(A_S\) is Noetherian (6.5.2). \(\square\)
7.7. Completed tensor products

(7.7.1). Let $A$ be a linearly topologized ring, $M$, $N$ two linearly topologized $A$-modules. Let $\mathfrak{J}$, $V$, $W$ be open neighborhoods of $0$ in $A$, $M$, $N$ respectively, which are $A$-modules, and such that $\mathfrak{J}M \subset V$, $\mathfrak{J}N \subset W$, so that $M/V$ and $N/W$ can be considered as $A/\mathfrak{J}$-modules. When $\mathfrak{J}$, $V$, $W$ vary over

the systems of open neighborhoods satisfying these properties, it is immediate that the modules $(M/V) \otimes_{A/\mathfrak{J}} (N/W)$ form a projective system of modules over the projective system of rings $A/\mathfrak{J}$; by passing to the projective limit, we thus obtain a module over the separated completion $\tilde{A}$ of $A$, which we call the completed tensor product of $M$ and $N$ and denote by $(M \otimes_A N)^\wedge$. If we have that $M/V$ is canonically isomorphic to $\tilde{M}/\tilde{V}$, where $\tilde{M}$ is the separated completion of $M$ and $\tilde{V}$ the closure in $\tilde{M}$ of the image of $V$, then we see that the completed tensor product $(M \otimes_A N)^\wedge$ canonically identifies with $(M \otimes_{\tilde{A}} \tilde{N})^\wedge$, which we denote by $\tilde{M} \otimes_{\tilde{A}} \tilde{N}$.

(7.7.2). With the above notation, the tensor products $(M/V) \otimes_A (N/W)$ and $(M/V) \otimes_{A/\mathfrak{J}} (N/W)$ identify canonically; they identify with $(M \otimes_A N)/(\text{Im}(V \otimes_A N) + \text{Im}(M \otimes_A W))$. We conclude that $(M \otimes_A N)^\wedge$ is the separated completion of the $A$-module $M \otimes_A N$, equipped with the topology for which the submodules

$$\text{Im}(V \otimes_A N) + \text{Im}(M \otimes_A W)$$

form a fundamental system of neighborhoods of $0$ ($V$ and $W$ varying over the set of open submodules of $M$ and $N$ respectively); we say for brevity that this topology is the tensor product of the given topologies on $M$ and $N$.

(7.7.3). Let $M'$, $N'$ be two linearly topologized $A$-modules, $u : M \to M'$, $v : N \to N'$ two continuous homomorphisms; it is immediate that $u \otimes v$ is continuous for the tensor product topologies on $M \otimes_A N$ and $M' \otimes_A N'$ respectively; by passing to the separated completions, we obtain a continuous homomorphism $(M \otimes_A N)^\wedge \to (M' \otimes_A N')^\wedge$, which we denote by $u \otimes v$; $(M \otimes_A N)^\wedge$ is thus a bifunctor in $M$ and $N$ on the category of linearly topologized $A$-modules.

(7.7.4). We similarly define the completed tensor product of any finite number of linearly topologized $A$-modules; it is immediate that this product has the usual properties of associativity and commutativity.

(7.7.5). If $B$, $C$ are two linearly topologized $A$-algebras, then the tensor product topology on $B \otimes_A C$ has for a fundamental system of neighborhoods of $0$ the ideals $\text{Im}(\mathfrak{r} \otimes_A C) + \text{Im}(B \otimes_A \mathfrak{L})$ of the algebra $B \otimes_A C$, $\mathfrak{r}$ (resp. $\mathfrak{L}$) varying over the set of open ideals of $B$ (resp. $C$). As a result, $(B \otimes_A C)^\wedge$ is equipped with the structure of a topological $A$-algebra, the projective limit of the projective system of $A/\mathfrak{J}$-algebras $(B/\mathfrak{r}) \otimes_{A/\mathfrak{J}} (C/\mathfrak{L})$ (the open ideal of $A$ such that $\mathfrak{J}B \subset \mathfrak{r}$, $\mathfrak{J}C \subset \mathfrak{L}$; it always exists). We say that this algebra is the completed tensor product of the algebras $B$ and $C$.

(7.7.6). The $A$-algebra homomorphisms $b \mapsto b \otimes 1$, $c \mapsto 1 \otimes c$ from $B$ and $C$ to $B \otimes_A C$ are continuous when we equip the latter algebra with the tensor product topology; by composing with the canonical homomorphism from $B \otimes_A C$ to its separated completion, they give canonical homomorphisms $\rho : B \to (B \otimes_A C)^\wedge$, $\sigma : C \to (B \otimes_A C)^\wedge$. The algebra $(B \otimes_A C)^\wedge$ and the homomorphisms $\rho$ and $\sigma$ have in addition the following universal property: for every linearly topologized $A$-algebra $D$, separated and complete, and each pair of continuous $A$-homomorphisms $u : B \to D$, $v : C \to D$, there exists a unique continuous $A$-homomorphism $w : (B \otimes_A C)^\wedge \to D$ such that $u = w \circ \rho$ and $v = w \circ \sigma$. Indeed, there already exists a unique $A$-homomorphism $w_0 : B \otimes_A C \to D$ such that $u(b) = w_0(b \otimes 1)$ and $v(c) = w_0(1 \otimes c)$, and it remains to prove that $w_0$ is continuous, since it then gives a continuous homomorphism $w$ by passing to the separated completion. If $\mathfrak{M}$ is an open ideal of $D$, then there exists by hypothesis open ideals $\mathfrak{R} \subset B$, $\mathfrak{L} \subset C$ such that $u(\mathfrak{R}) \subset \mathfrak{M}$, $v(\mathfrak{L}) \subset \mathfrak{M}$; the image under $w_0$ of $\text{Im}(\mathfrak{R} \otimes_A C) + \text{Im}(B \otimes_A \mathfrak{L})$ is again contained in $\mathfrak{M}$, hence our assertion.

Proposition (7.7.7). — If $B$ and $C$ are two preadmissible $A$-algebras, then $(B \otimes_A C)^\wedge$ is admissible, and if $\mathfrak{r}$ (resp. $\mathfrak{L}$) is an ideal of definition for $B$ (resp. $C$), then the closure in $(B \otimes_A C)^\wedge$ of the canonical image of $\mathfrak{r} = \text{Im}(\mathfrak{r} \otimes_A C) + \text{Im}(B \otimes_A \mathfrak{L})$ is an ideal of definition.

Proof. It suffices to show that $\mathfrak{r}^n$ tends to $0$ for the tensor product topology, which follows immediately from the inclusion

$$\mathfrak{r}^{2n} \subset \text{Im}(\mathfrak{r}^n \otimes_A C) + \text{Im}(B \otimes_A \mathfrak{L}^n).$$
Proposition (7.7.8). — Let $A$ be a preadic ring, $\mathfrak{J}$ an ideal of definition for $A$, $M$ an $A$-module of finite type, equipped with the $\mathfrak{J}$-adic topology. For every topological adic Noetherian $A$-algebra $B$, $B \otimes_A M$ identifies with the completed tensor product $(B \otimes_A M)^\wedge$.

Proof. If $\mathfrak{J}$ is an ideal of definition for $B$, there exists by hypothesis an integer $m$ such that $\mathfrak{J}^m B \subset \mathfrak{J}$, so $\text{Im}(B \otimes_A \mathfrak{J}^m M) = \text{Im}(\mathfrak{J}^m B \otimes_A M) \subset \text{Im}(\mathfrak{J}^m (B \otimes_A M) = \mathfrak{J}^m (B \otimes_A M)$; we conclude that over $B \otimes_A M$, the tensor products of the topologies of $B$ and $M$ is the $\mathfrak{J}$-adic topology. As $B \otimes_A M$ is a $B$-module of finite type, the proposition follows from Corollary (7.3.6).

7.8. Topologies on modules of homomorphisms

(7.8.1). Let $A$ be a Noetherian $\mathfrak{J}$-adic ring, $M$ and $N$ two $A$-modules of finite type, equipped with the $\mathfrak{J}$-adic topology; we know (7.3.6) that they are separated and complete; in addition, every $A$-homomorphism $M \to N$ is automatically continuous, and the $A$-module $\text{Hom}_A(M, N)$ is of finite type. For every integer $i \geq 0$, set $A_i = A/\mathfrak{J}^{i+1}$, $M_i = M/\mathfrak{J}^{i+1}M$, $N_i = N/\mathfrak{J}^{i+1}N$; for $i \leq j$, every homomorphism $u_i : M_i \to N_i$ maps $\mathfrak{J}^{i+1}M_i$ to $\mathfrak{J}^{i+1}N_i$, thus giving by passage to quotients a homomorphism $u_i : M_i \to N_i$, which defines a canonical homomorphism $\text{Hom}_{A_i}(M_i, N_i) \to \text{Hom}_{A_j}(M_j, N_j)$; in addition, the $\text{Hom}_{A_i}(M_i, N_i)$ form a projective system for these homomorphisms, and it follows from Corollary (7.2.10) that there is a canonical isomorphism $\phi : \text{Hom}_A(M, N) \to \varprojlim \text{Hom}_{A_i}(M_i, N_i)$. In addition:

Proposition (7.8.2). — If $M$ and $N$ are modules of finite type over a $\mathfrak{J}$-adic Noetherian ring $A$, then the submodules $\text{Hom}_A(M, \mathfrak{J}^{i+1}N)$ form a fundamental system of neighborhoods of $0$ in $\text{Hom}_A(M, N)$ for the $\mathfrak{J}$-adic topology, and the canonical isomorphism $\phi : \text{Hom}_A(M, N) \to \varprojlim \text{Hom}_{A_i}(M_i, N_i)$ is a topological isomorphism.

Proof. We can consider $M$ as the quotient of a free $A$-module $L$ of finite type, and as a result identify $\text{Hom}_A(M, N)$ as a submodule of $\text{Hom}_A(L, N)$; in this identification, $\text{Hom}_{A}(M, \mathfrak{J}^{i+1}N)$ is the intersection of $\text{Hom}_A(M, N)$ and $\text{Hom}_A(L, \mathfrak{J}^{i+1}N)$; as the induced topology on $\text{Hom}_A(M, N)$ by the $\mathfrak{J}$-adic topology of $\text{Hom}_A(M, N)$ is the $\mathfrak{J}$-adic (7.3.2.1), we have reduced to proving the first assertion for $M = L = A^m$; but then $\text{Hom}_A(L, N) = N^m$, $\text{Hom}_A(L, \mathfrak{J}^{i+1}N) = (\mathfrak{J}^{i+1}N)^m = \mathfrak{J}^{i+1}N^m$, and the result is evident. To establish the second assertion, we note that the image of $\text{Hom}_A(M, \mathfrak{J}^{i+1}N)$ in $\text{Hom}_{A_j}(M_j, N_j)$ is zero for $j \leq i$, hence $\phi$ is continuous; conversely, the inverse image in $\text{Hom}_A(M, N)$ of $0$ of $\text{Hom}_{A_i}(M_i, N_i)$ is $\text{Hom}_{A_j}(M_j, N_j)$, so $\phi$ is bicontinuous.

If we only suppose that $A$ is a Noetherian $\mathfrak{J}$-preadiic ring, $M$ and $N$ two $A$-modules of finite type, separated for the $\mathfrak{J}$-adic topology, then the following proof shows that the first assertion of Propositon (7.8.2) remains valid, and that $\phi$ is a topological isomorphism from $\text{Hom}_A(M, N)$ to a submodule of $\varprojlim \text{Hom}_{A_i}(M_i, N_i)$.

Proposition (7.8.3). — Under the hypotheses of Proposition (7.8.2), the set of injective (resp. surjective, bijective) homomorphisms from $M$ to $N$ is an open subset of $\text{Hom}_A(M, N)$.

Proof. According to Corollaries (7.3.5) and (7.1.14), for $u$ to be injective, it is necessary and sufficient that the corresponding homomorphism $u_0 : M/\mathfrak{J}^iM \to N/\mathfrak{J}^iN$ is, and the set of surjective homomorphisms from $M$ to $N$ is thus the inverse image under the continuous map $\text{Hom}_A(M, N) \to \text{Hom}_{A_0}(M_0, N_0)$ of a subset of a discrete space. We now show that the set of injective homomorphisms is open; let $\nu$ be such a homomorphism and set $M' = \nu(M)$; by the Artin–Rees Lemma (7.3.2.1), there exists an integer $k \geq 0$ such that $M' \cap \mathfrak{J}^{m+k}N \subset \mathfrak{J}^m M'$ for all $m > 0$; we will see that for all $w \in \mathfrak{J}^{k+1} \text{Hom}_A(M, N)$, $u + w$ is injective, which will finish the proof. Indeed, let $x \in M$ be such that $u(x) = 0$; we prove that for every $i \geq 0$ the relation $x \in \mathfrak{J}^iM$ implies that $x \in \mathfrak{J}^{i+1}M$; this follows from $x \in \bigcap_{i \geq 0} \mathfrak{J}^iM = (0)$. Indeed, we then have $w(x) \in \mathfrak{J}^{i+k+1}N$, and as a result $u(x) = -w(x) \in M'$, so $u(x) \in M' \cap \mathfrak{J}^{i+k+1}N \subset \mathfrak{J}^{i+1}M'$, and as $u$ is an isomorphism from $M$ to $M'$, $x \in \mathfrak{J}^{i+1}M$, q.e.d.
§8. REPRESENTABLE FUNCTORS

8.1. Representable functors

(8.1.1). We denote by \( \text{Set} \) the category of sets. Let \( \mathcal{C} \) be a category; for two objects \( X, Y \) of \( \mathcal{C} \), we set \( h_X(Y) = \text{Hom}(Y, X) \); for each morphism \( u : Y \rightarrow Y' \) in \( \mathcal{C} \), we denote by \( h_X(u) \) the map \( v \mapsto vu \) from \( \text{Hom}(Y', X) \) to \( \text{Hom}(Y, X) \). It is immediate that with these definitions, \( h_X : \mathcal{C} \rightarrow \text{Set} \) is a contravariant functor, i.e., an object of the category \( \text{Hom}(\mathcal{C}^{\text{op}}, \text{Set}) \), of covariant functors from the category \( \mathcal{C}^{\text{op}} \) (the dual of the category \( \mathcal{C} \)) to the category \( \text{Set} \) (T, 1.7, (d) and [Car]).

(8.1.2). Now let \( w : X \rightarrow X' \) be a morphism in \( \mathcal{C} \); for each \( Y \in \mathcal{C} \) and each \( v \in \text{Hom}(Y, X) \), we have \( vw \in \text{Hom}(Y, X' \prime) \); we denote by \( hw(Y) \) the map \( v \mapsto vw \) from \( h_X(Y) \) to \( h_X(X' \prime) \). It is immediate that for each morphism \( u : Y \rightarrow Y' \) in \( \mathcal{C} \), the diagram

\[
\begin{array}{ccc}
h_X(Y') & \xrightarrow{h_X(u)} & h_X(Y) \\
\downarrow{h_w(Y')} & & \downarrow{h_w(Y)} \\
h_X(X' \prime) & \xrightarrow{h_X(u)} & h_X(X) \\
\end{array}
\]

is commutative; in other words, \( h_w \) is a natural transformation (or functorial morphism) \( h_X \rightarrow h_X(1, \mathcal{C}, \text{op}) \), (T, 1.2), also a morphism in the category \( \text{Hom}(\mathcal{C}^{\text{op}}, \text{Set}) \) (T, 1.7, (d)). The definitions of \( h_X \) and of \( h_w \) therefore constitute the definition of a canonical covariant functor

\[
h : \mathcal{C} \rightarrow \text{Hom}(\mathcal{C}^{\text{op}}, \text{Set}), \quad X \mapsto h_X.
\]

(8.1.3). Let \( X \) be an object in \( \mathcal{C} \), \( F \) a contravariant functor from \( \mathcal{C} \) to \( \text{Set} \) (an object of \( \text{Hom}(\mathcal{C}^{\text{op}}, \text{Set}) \)). Let \( g : h_X \rightarrow F \) be a natural transformation: for all \( Y \in \mathcal{C} \), \( g(Y) \) is thus a map \( h_X(Y) \rightarrow F(Y) \) such that for each morphism \( u : Y \rightarrow Y' \) in \( \mathcal{C} \), the diagram

\[
\begin{array}{ccc}
h_X(Y') & \xrightarrow{h_X(u)} & h_X(Y) \\
\downarrow{g(Y')} & & \downarrow{g(Y)} \\
F(Y') & \xrightarrow{F(u)} & F(Y) \\
\end{array}
\]

is commutative. In particular, we have a map \( g(X) : h_X(X) = \text{Hom}(X, X) \rightarrow F(X) \), hence an element

\[
\alpha(g) = (g(X))(1_X) \in F(X)
\]

and as a result a canonical map

\[
\alpha : \text{Hom}(h_X, F) \rightarrow F(X).
\]

Conversely, consider an element \( \xi \in F(X) \); for each morphism \( v : Y \rightarrow X \) in \( \mathcal{C} \), \( F(v) \) is a map \( F(X) \rightarrow F(Y) \); consider the map

\[
v \mapsto (F(v))(\xi)
\]

from \( h_X(Y) \) to \( F(Y) \); if we denote by \( (\beta(\xi))(Y) \) this map,

\[
\beta(\xi) : h_X \rightarrow F
\]

is a natural transformation, since for each morphism \( u : Y \rightarrow Y' \) in \( \mathcal{C} \) we have \( (F(u))(\xi) = (F(v) \circ F(\xi))(\xi) \), which makes (8.1.3.1) commutative for \( \xi = \beta(\xi) \). We have thus defined a canonical map

\[
\beta : F(X) \rightarrow \text{Hom}(h_X, F).
\]

Proposition (8.1.4). — The maps \( \alpha \) and \( \beta \) are the inverse bijections of each other.

Proof. We calculate \( \alpha(\beta(\xi)) \) for \( \xi \in F(X) \); for each \( Y \in \mathcal{C} \), \( (\beta(\xi))(Y) \) is a map \( g_1(Y) : v \mapsto (F(v))(\xi) \) from \( h_X(Y) \) to \( F(Y) \). We thus have

\[
\alpha(\beta(\xi)) = g_1(X)(1_X) = (F(1_X))(\xi) = 1_{F(X)}(\xi) = \xi.
\]

We now calculate \( \beta(\alpha(g)) \) for \( g \in \text{Hom}(h_X, F) \); for each \( Y \in \mathcal{C} \), \( (\beta(\alpha(g)))(Y) \) is the map \( v \mapsto (F(v))(g(X))(1_X) \); according to the commutativity of (8.1.3.1), this map is none other than \( v \mapsto \)
(8.1.5). Recall that a subcategory \( \mathcal{C}' \) of a category \( \mathcal{C} \) is defined by the condition that its objects are objects of \( \mathcal{C} \), and that if \( X', Y' \) are two objects of \( \mathcal{C}' \), then the set \( \text{Hom}_{\mathcal{C}'}(X', Y') \) of morphisms \( X' \to Y' \) in \( \mathcal{C}' \) is a subset of the set \( \text{Hom}_{\mathcal{C}}(X', Y') \) of morphisms \( X' \to Y' \) in \( \mathcal{C} \), the canonical map of “composition of morphisms”

\[
\text{Hom}_{\mathcal{C}'}(X', Y') \times \text{Hom}_{\mathcal{C}'}(Y', Z') \to \text{Hom}_{\mathcal{C}'}(X', Z')
\]

being the restriction of the canonical map

\[
\text{Hom}_{\mathcal{C}}(X', Y') \times \text{Hom}_{\mathcal{C}}(Y', Z') \to \text{Hom}_{\mathcal{C}}(X', Z').
\]

We say that \( \mathcal{C}' \) is a full subcategory of \( \mathcal{C} \) if \( \text{Hom}_{\mathcal{C}'}(X', Y') = \text{Hom}_{\mathcal{C}}(X', Y') \) for every pair of objects in \( \mathcal{C}' \). The subcategory \( \mathcal{C}' \) of \( \mathcal{C} \) consisting of the objects of \( \mathcal{C} \) isomorphic to objects of \( \mathcal{C}' \) is then again a full subcategory of \( \mathcal{C} \), equivalent (T.1.2) to \( \mathcal{C}' \) as we verify easily.

A covariant functor \( F : \mathcal{C}_1 \to \mathcal{C}_2 \) is called fully faithful if for every pair of objects \( X_1, Y_1 \) of \( \mathcal{C}_1 \), the map \( u \mapsto F(u) \) from \( \text{Hom}(X_1, Y_1) \) to \( \text{Hom}(F(X_1), F(Y_1)) \) is bijective; this implies that the subcategory \( F(\mathcal{C}_1) \) of \( \mathcal{C}_2 \) is full. In addition, if two objects \( X_1, X'_1 \) have the same image \( X_2 \), then there exists a unique isomorphism \( u : X_1 \to X'_1 \) such that \( F(u) = 1_{X_1} \). For each object \( X_2 \) of \( \mathcal{C}_1 \), let \( G(X_2) \) be one of the objects \( X_1 \) of \( \mathcal{C}_1 \) such that \( F(X_1) = X_2 \) (\( G \) is defined by means of the axiom of choice); for each morphism \( v : X_2 \to Y_2 \) in \( \mathcal{C}_1 \), \( G(v) \) will be the unique morphism \( u : G(X_2) \to G(Y_2) \) such that \( F(u) = v \); \( G \) is then a functor from \( \mathcal{C}_1 \) to \( \mathcal{C}_2 \); \( FG \) is the identity functor on \( \mathcal{C}_1 \), and the above shows that there exists an isomorphism of functors \( \phi : 1_{\mathcal{C}_1} \to GF \) such that \( F, G, \phi \), and the identity \( 1_{F(\mathcal{C}_1)} \to FG \) defines an equivalence between the category \( \mathcal{C}_1 \) and the full subcategory \( F(\mathcal{C}_1) \) of \( \mathcal{C}_2 \) (T.1.2).

(8.1.6). We apply Proposition (8.1.4) to the case where \( F \) is \( h_X' \), \( X' \) being any object of \( \mathcal{C} \); the map \( \beta : \text{Hom}(X, X') \to \text{Hom}(h_X, h_X') \) is none other than the map \( w \mapsto w \) defined in (8.1.2); this map being bijective, we see with the terminology of (8.1.5) that:

**Proposition (8.1.7).** — The canonical functor \( h : \mathcal{C} \to \text{Hom}(\mathcal{C}^\text{op}, \text{Set}) \) is fully faithful.

(8.1.8). Let \( F \) be a contravariant functor from \( \mathcal{C} \) to \( \text{Set} \); we say that \( F \) is representable if there exists an object \( X \in \mathcal{C} \) such that \( F \) is isomorphic to \( h_X \); it follows from Proposition (8.1.7) that the data of an \( X \in \mathcal{C} \) and an isomorphism of functors \( g : h_X \to F \) determines \( X \) up to unique isomorphism. Proposition (8.1.7) then implies that \( g \) defines an equivalence between \( \mathcal{C} \) and the full subcategory of \( \text{Hom}(\mathcal{C}^\text{op}, \text{Set}) \) consisting of the contravariant representable functors. It follows from Proposition (8.1.4) that the data of a natural transformation \( g : h_X \to F \) is equivalent to that of an element \( \xi \in F(X) \); to say that \( g \) is an isomorphism is equivalent to the following condition on \( \xi \): for every object \( Y \in \mathcal{C} \) the map \( v \mapsto (F(v))(\xi) \) from \( \text{Hom}(Y, X) \) to \( F(Y) \) is bijective. When \( \xi \) satisfies this condition, we say that the pair \( (X, \xi) \) represents the representable functor \( F \). By abuse of language, we also say that the object \( X \in \mathcal{C} \) represents \( F \) if there exists a \( \xi \in F(X) \) such that \( (X, \xi) \) represents \( F \), in other words if \( h_X \) is isomorphic to \( F \).

Let \( F, F' \) be two contravariant representable functors from \( \mathcal{C} \) to \( \text{Set} \), \( h_X \to F \) and \( h_{X'} \to F' \) two isomorphisms of functors. Then it follows from (8.1.6) that there is a canonical bijective correspondence between \( \text{Hom}(X, X') \) and the set \( \text{Hom}(F, F') \) of natural transformations \( F \to F' \).

(8.1.9). Example 1. Projective limits. The notion of a contravariant representable functor covers in particular the “dual” notion of the usual notion of a “solution to a universal problem”. More generally, we will see that the notion of the projective limit is a special case of the notion of a representable functor. Recall that in a category \( \mathcal{C} \), we define a projective system by the data of a preordered set \( I \), a family \( (A_\alpha)_{\alpha \in I} \) of objects of \( \mathcal{C} \), and for every pair of indices \( (\alpha, \beta) \) such that \( \alpha \leq \beta \), a morphism \( u_{\alpha\beta} : A_\beta \to A_\alpha \). A projective limit of this system in \( \mathcal{C} \) consists of an object \( B \in \mathcal{C} \) (denoted \( \lim \{ A_\alpha \}_\alpha \)), and for each \( \alpha \in I \), a morphism \( u_\alpha : B \to A_\alpha \) such that \( 1^\text{st} \). \( u_\alpha = u_{\alpha\beta}u_\beta \) for \( \alpha \leq \beta \); \( 2^\text{nd} \). for every object \( X \) of \( \mathcal{C} \) and every family \( (v_\alpha)_{\alpha \in I} \) of morphisms \( v_\alpha : X \to A_\alpha \) such that \( v_\alpha = u_{\alpha\beta}v_\beta \) for \( \alpha \leq \beta \), there exists a unique morphism \( v : X \to B \) (denoted \( \lim v_\alpha \)) such that \( v_\alpha = u_\alpha v \) for all \( \alpha \in I \) (T.1.8). This can be interpreted in the following way: the \( u_{\alpha\beta} \) canonically define maps

\[
\bar{u}_{\alpha\beta} : \text{Hom}(X, A_\beta) \to \text{Hom}(X, A_\alpha)
\]
which define a projective system of sets \( (\text{Hom}(X, A_k), \pi_{ak}) \), and \((v_a)\) is by definition an element of the set \( \lim \text{Hom}(X, A_a) \); it is clear that \( X \mapsto \lim \text{Hom}(X, A_a) \) is a contravariant functor from \( \mathcal{C} \) to \( \text{Set} \), and the existence of the projective limit \( B \) is equivalent to saying that \((v_a) \mapsto \lim v_a \) is an isomorphism of functors in \( X \)

\[
\lim \text{Hom}(X, A_a) \cong \text{Hom}(X, B),
\]

in other words, that the functor \( X \mapsto \lim \text{Hom}(X, A_a) \) is representable.

\(8.1.10\). Example II. Final objects. Let \( \mathcal{C} \) be a category, \( \{a\} \) a singleton set. Consider the contravariant functor \( F : \mathcal{C} \to \text{Set} \) which sends every object \( X \) of \( \mathcal{C} \) to the set \( \{a\} \), and every morphism \( X \to X' \) in \( \mathcal{C} \) to the unique map \( \{a\} \to \{a\} \). To say that this functor is representable means that there exists an object \( e \in \mathcal{C} \) such that for every \( Y \in \mathcal{C} \), \( \text{Hom}(Y, e) = h_e(Y) \) is a singleton set; we say that \( e \) is an final object of \( \mathcal{C} \), and it is clear that two final objects of \( \mathcal{C} \) are isomorphic (which allows us to define, in general with the axiom of choice, one final object of \( \mathcal{C} \) which we then denote \( e_c \)). For example, in the category \( \text{Set} \), the final objects are the singleton sets; in the category of augmented algebras over a field \( K \) (where the morphisms are the algebra homomorphisms compatible with the augmentation), \( K \) is a final object; in the category of \( S \)-preschemes \( (I, 2.5.1) \), \( S \) is a final object.

\(8.1.11\). For two objects \( X \) and \( Y \) of a category \( \mathcal{C} \), set \( h'_e(Y) = \text{Hom}(X, Y) \), and for every morphism \( u : Y \to Y' \), let \( h'_e(u) \) be the map \( v \mapsto vu \) from \( \text{Hom}(X, Y) \) to \( \text{Hom}(X, Y') \); \( h'_e \) is then a covariant functor \( \mathcal{C} \to \text{Set} \), so we deduce as in \(8.1.2\) the definition of a canonical covariant functor \( h' : \mathcal{C}^{\text{op}} \to \text{Hom}(\mathcal{C}, \text{Set}) \); a covariant functor \( F \) from \( \mathcal{C} \) to \( \text{Set} \), in other words an object of \( \text{Hom}(\mathcal{C}, \text{Set}) \), is then representable if there exists an object \( X \in \mathcal{C} \) (necessarily unique up to unique isomorphism) such that \( F \) is isomorphic to \( h'_e \); we leave it to the reader to develop the “dual” notions of the above, which this time cover the notion of an inductive limit, and in particular the usual notion of a “solution to a universal problem”.

8.2. Algebraic structures in categories

\(8.2.1\). Given two contravariant functors \( F \) and \( F' \) from \( \mathcal{C} \) to \( \text{Set} \), recall that for every object \( Y \in \mathcal{C} \), we set \( (F \times F')(Y) = F(Y) \times F'(Y) \), and for every morphism \( u : Y \to Y' \) in \( \mathcal{C} \), we set \( (F \times F')(u) = F(u) \times F'(u) \), which is the map \( (t, t') \mapsto (F(u)(t), F'(u)(t')) \) from \( (F(Y') \times F'(Y')) \) to \( F(Y) \times F'(Y) \); \( F \times F' : \mathcal{C} \to \text{Set} \) is thus a contravariant functor (which is none other than the product of the objects \( F \) and \( F' \) in the category \( \text{Hom}(\mathcal{C}^{\text{op}}, \text{Set}) \)). Given an object \( X \in \mathcal{C} \), we call an internal composition law on \( X \) a natural transformation

\[
\gamma_X : h_X \times h_X \to h_X.
\]

In other words (T, 1.2), for every object \( Y \in \mathcal{C} \), \( \gamma_X(Y) \) is a map \( h_X(Y) \times h_X(Y) \to h_X(Y) \) (thus by definition an internal composition law on the set \( h_X(Y) \)) with the condition that for every morphism \( u : Y \to Y' \) in \( \mathcal{C} \), the diagram

\[
\begin{array}{ccc}
h_X(Y') \times h_X(Y') & \xrightarrow{h_X(u) \times h_X(u)} & h_X(Y) \times h_X(Y) \\
\downarrow{\gamma_X(Y')} & & \downarrow{\gamma_X(Y)} \\
\downarrow{h_X(Y')} & & \downarrow{h_X(Y)} \\
h_X(Y') & \xrightarrow{h_X(u)} & h_X(Y)
\end{array}
\]

is commutative; this implies that for the composition laws \( \gamma_X(Y) \) and \( \gamma_X(Y') \), \( h_X(u) \) is a homomorphism from \( h_X(Y') \) to \( h_X(Y) \).

In a similar way, given two objects \( Z \) and \( X \) of \( \mathcal{C} \), we call an external composition law on \( X \), with \( Z \) as its domain of operators a natural transformation

\[
\omega_{X,Z} : h_Z \times h_X \to h_X.
\]

We see as above that for every \( Y \in \mathcal{C} \), \( \omega_{X,Z}(Y) \) is an external composition law on \( h_X(Y) \), with \( h_Z(Y) \) as its domain of operators and such that for every morphism \( u : Y \to Y' \), \( h_X(u) \) and \( h_Z(u) \) form a di-homomorphism from \( (h_Z(Y'), h_X(Y')) \) to \( (h_X(Y), h_X(Y)) \).
(8.2.2). Let $X'$ be a second object of $\mathcal{C}$, and suppose we are given an internal composition law $\gamma_{X'}$ on $X'$; we say that a morphism $w : X \to X'$ in $\mathcal{C}$ is a homomorphism for the composition laws if for every $Y \in \mathcal{C}$, $h_w(Y) : h_X(Y) \to h_{X'}(Y)$ is a homomorphism for the composition laws $\gamma_X(Y)$ and $\gamma_{X'}(Y)$. If $X''$ is a third object of $\mathcal{C}$ equipped with an internal composition law $\gamma_{X''}$ and $w' : X' \to X''$ is a morphism in $\mathcal{C}$ which is a homomorphism for $\gamma_{X'}$ and $\gamma_{X''}$, then it is clear that the morphism $w'w : X \to X''$ is a homomorphism for the composition laws $\gamma_X$ and $\gamma_{X''}$. An isomorphism $w : X \cong X'$ in $\mathcal{C}$ is called an isomorphism for the composition laws $\gamma_X$ and $\gamma_{X'}$ if $w$ is a homomorphism for these composition laws, and if its inverse morphism $w^{-1}$ is a homomorphism for the composition laws $\gamma_{X'}$ and $\gamma_X$.

We define in a similar way the di-homomorphisms for pairs of objects of $\mathcal{C}$ equipped with external composition laws.

(8.2.3). When an internal composition law $\gamma_X$ on an object $X \in \mathcal{C}$ is such that $\gamma_X(Y)$ is a group law on $h_X(Y)$ for every $Y \in \mathcal{C}$, we say that $X$, equipped with this law, is a $\mathcal{C}$-group or a group object in $\mathcal{C}$. We similarly define $\mathcal{C}$-rings, $\mathcal{C}$-modules, etc.

(8.2.4). Suppose that the product $X \times X$ of an object $X \in \mathcal{C}$ by itself exists in $\mathcal{C}$; by definition, we then have $h_{X \times X} = h_X \times h_X$ up to canonical isomorphism, since it is a particular case of the projective limit (8.1.9); an internal composition law on $X$ can thus be considered as a functorial morphism $\gamma_X : h_{X \times X} \to h_X$, and thus canonically determine (8.1.6) an element $c_X \in \text{Hom}(X \times X, X)$ such that $h_{c_X} = \gamma_X$; in this case, the data of an internal composition law on $X$ is equivalent to the data of a morphism $X \times X \to X$; when $\mathcal{C}$ is the category $\text{Set}$, we recover the classical notion of an internal composition law on a set. We have an analogous result for an external composition law when the product $Z \times X$ exists in $\mathcal{C}$.

(8.2.5). With the above notation, suppose that in addition $X \times X \times X$ exists in $\mathcal{C}$; the characterization of the product as an object representing a functor (8.1.9) implies the existence of canonical isomorphisms

$$
(X \times X) \times X \cong X \times X \times X \cong X \times (X \times X);
$$

if we canonically identify $X \times X \times X$ with $(X \times X) \times X$, then the map $\gamma_X(Y) \times 1_{h_X(Y)}$ identifies with $h_{c_X \times 1_X}(Y)$ for all $Y \in \mathcal{C}$. As a result, it is equivalent to say that for every $Y \in \mathcal{C}$, the internal law $\gamma_X(Y)$ is associative, or that the diagram of maps

$$
\begin{array}{ccc}
h_X(Y) \times h_X(Y) & \xrightarrow{\gamma_X(Y) \times 1} & h_X(Y) \\
\downarrow{1 \times \gamma_X(Y)} & & \downarrow{\gamma_X(Y)} \\
h_X(Y) \times h_X(Y) & \xrightarrow{\gamma_X(Y)} & h_X(Y)
\end{array}
$$

is commutative, or that the diagram of morphisms

$$
\begin{array}{ccc}
X \times X \times X & \xrightarrow{c_X \times 1_X} & X \times X \\
\downarrow{1 \times c_X} & & \downarrow{c_X} \\
X \times X & \xrightarrow{c_X} & X
\end{array}
$$

is commutative.

(8.2.6). Under the hypotheses of (8.2.5), if we want to express, for every $Y \in \mathcal{C}$, the internal law $\gamma_X(Y)$ as a group law, then it is first necessary that it is associative, and second that there exists a map $a_X(Y) : h_X(Y) \to h_X(Y)$ having the properties of the inverse operation of a group; as for every morphism $u : Y \to Y'$ in $\mathcal{C}$, we have seen that $h_X(u)$ must be a group homomorphism $h_X(Y') \to h_X(Y)$, we first see that $a_X : h_X \to h_X$ must be a natural transformation. On the other hand,
one can express the characterist properties of the inverse $s \mapsto s^{-1}$ of a group $G$ without involving
the identity element: it suffices to check that the two composite maps
\[(s, t) \mapsto (s, s^{-1}, t) \mapsto (s, s^{-1}t) \mapsto s(s^{-1}t),
\[(s, t) \mapsto (s, s^{-1}, t) \mapsto (s, ts^{-1}) \mapsto (ts^{-1})s\]
are equal to the second projection $(s, t) \mapsto t$ from $G \times G$ to $G$. By (8.1.10), we have $\alpha_X = h_{\alpha_X}$, where $\alpha_X \in \text{Hom}(X, X)$; the first condition above then expresses that the composite morphism
\[X \times X \xrightarrow{(1_X, \alpha_X) \times 1_X} X \times X \times X \xrightarrow{1_X \times \varepsilon_X} X \times X \xrightarrow{\varepsilon_X} X\]
is the second projection $X \times X \to X$ in $\mathcal{C}$, and the second condition is similar.

(8.2.7). Now suppose that there exists a final object $e$ (8.1.10) in $\mathcal{C}$. Let us always assume that $\gamma_X(Y)$
is a group law on $\text{Hom}(Y)$ for every $Y \in \mathcal{C}$, and denote by $\eta_X(Y)$ the identity element of $\gamma_X(Y)$.
As, for every morphism $u : Y \to Y'$ in $\mathcal{C}$, $h_X(u)$ is a group homomorphism, we have $\eta_X(Y) = (h_X(u))(\eta_X(Y'))$; taking in particular $Y' = e$, in which case $u$ is the unique element $e$ of $\text{Hom}(Y, e)$, we see that the element $\eta_X(e)$ completely determines $\eta_X(Y)$ for every $Y \in \mathcal{C}$. Set $e_X = \eta_X(X)$, the identity element of the group $h_X(X) = \text{Hom}(X, X)$; the commutativity of the diagram
\[
\begin{array}{ccc}
\text{h}_X(e) & \xrightarrow{h_X(e)} & h_X(Y) \\
\text{h}_X(e) & \Downarrow & \text{h}_X(e) \\
\text{h}_X(e) & \xrightarrow{h_X(e)} & h_X(Y)
\end{array}
\]
(cf. (8.1.2)) shows that, on the set $h_X(Y)$, the map $h_{e_X}(Y)$ is none other than $s \mapsto \eta_X(Y)$ sending
every element to the identity element. We then verify that the fact that $\eta_X(Y)$ is the identity element
of $\gamma_X(Y)$ for every $Y \in \mathcal{C}$ is equivalent to saying that the composite morphism
\[X \xrightarrow{(1_X, 1_X)} X \times X \xrightarrow{1_X \times e_X} X \times X \xrightarrow{e_X} X,
\]
and the analog in which we swap $1_X$ and $e_X$, are both equal to $1_X$.

(8.2.8). One could of course easily extend the examples of algebraic structures in categories. The
example of groups was treated with enough detail, but latter on we will usually leave it to the reader
to develop analogus notions for the examples of algebraic structures we will encounter.

§9. CONSTRUCTIBLE SETS

9.1. Constructible sets

Definition (9.1.1). — We say that a continuous map $f : X \to Y$ is quasi-compact if for every quasi-compact open subset $U$ of $Y$, $f^{-1}(U)$ is quasi-compact. We say that a subset $Z$ of a topological space $X$ is retrocompact in $X$ if the canonical injection $Z \to X$ is quasi-compact, in other words, if for every quasi-compact open subset $U$ of $X$, $U \cap Z$ is quasi-compact.

A closed subset of $X$ is retrocompact in $X$, but a quasi-compact subset of $X$ is not necessarily
retrocompact in $X$. If $X$ is quasi-compact, every retrocompact open subset of $X$ is quasi-compact. It
is clear that every finite union of retrocompact sets in $X$ is retrocompact in $X$, as every finite union of
quasi-compact sets is quasi-compact. Every finite intersection of retrocompact open sets in $X$ is a
retrocompact open set in $X$. In a locally Noetherian space $X$, every quasi-compact set is a Noetherian
subspace, and as a result every subset of $X$ is retrocompact in $X$.

Definition (9.1.2). — Given a topological space $X$, we say that a subset of $X$ is constructible if it
belongs to the smallest set of subsets $\mathcal{S}$ of $X$ containing all the retrocompact open subsets of $\mathcal{S}$ and
is stable under finite intersections and complements (which implies that $\mathcal{S}$ is also stable under finite
unions).

Proposition (9.1.3). — For a subset of $X$ to be constructible, it is necessary and sufficient for it to be a finite
union of sets of the form $U \cap CV$, where $U$ and $V$ are retrocompact open sets in $X$. 

9.2. Local properties of constructible sets

Definition (9.2.1). — Let the $\mathcal{C}$ be a category, and $F$ a set of subsets of $X$. We say that
$F$ is stable under finite intersections and complements if $F$ is stable under finite intersections and complements (which implies that $F$ is also stable under finite
unions).
PROOF. It is clear that the condition is sufficient. To see that it is necessary, consider the set \( \mathcal{G} \) of finite unions of sets of the form \( U \cap \mathcal{C}V \), where \( U \) and \( V \) are retrocompact open sets in \( X \); it suffices to see that every complement of a set in \( \mathcal{G} \) is in \( \mathcal{G} \). Let \( Z = \bigcup_{i \in I} (U_i \cap \mathcal{C}V_i) \), where \( I \) is finite, \( U_i \) and \( V_i \) retrocompact open sets in \( X \); we have \( \mathcal{C}Z = \bigcap_{i \in I} (V_i \cup \mathcal{C}U_i) \), so \( Z \) is a finite union of sets which are intersections of a certain number of the \( V_i \) and of a certain number of the \( \mathcal{C}U_i \), thus of the form \( V \cap \mathcal{C}U \), where \( U \) is the union of a certain number of the \( U_i \) and \( V \) is the intersection of a certain number of the \( V_i \); but we have noted above that finite unions and intersections of retrocompact open sets in \( X \) are retrocompact open sets in \( X \), hence the conclusion. \( \square \)

Corollary (9.1.4). — Every constructible subset of \( X \) is retrocompact in \( X \).

PROOF. It suffices to show that if \( U \) and \( V \) are retrocompact open sets in \( X \), then \( U \cap \mathcal{C}V \) is retrocompact in \( X \); if \( W \) is a quasi-compact open set in \( X \), then \( W \cap U \cap \mathcal{C}V \) is closed in the quasi-compact space \( W \cap U \), hence it is quasi-compact. \( \square \)

In particular:

Corollary (9.1.5). — For an open subset \( U \) of \( X \) to be constructible, it is necessary and sufficient for it to be retrocompact in \( X \). For a closed subset \( F \) of \( X \) to be constructible, it is necessary and sufficient for the open set \( \mathcal{C}F \) to be retrocompact.

(9.1.6). An important case is when every quasi-compact open subset of \( X \) is retrocompact, in other words, when the intersection of two quasi-compact open subsets of \( X \) is quasi-compact (cf. (I, 5.5.6)). When \( X \) is also quasi-compact, this implies that the retrocompact open subsets of \( X \) are identical to the quasi-compact open subsets of \( X \), and the constructible subsets of \( X \) are finite unions of the form \( U \cap \mathcal{C}V \), where \( U \) and \( V \) are quasi-compact open sets.

Corollary (9.1.7). — For a subset of a Noetherian space to be constructible, it is necessary and sufficient for it to be a finite union of locally closed subsets of \( X \).

Proposition (9.1.8). — Let \( X \) be a topological space, \( U \) an open subset of \( X \).

(i) If \( T \) is a constructible subset of \( X \), then \( T \cap U \) is a constructible subset of \( U \).

(ii) In addition, suppose that \( U \) is retrocompact in \( X \). For a subset \( Z \) of \( U \) to be constructible in \( X \), it is necessary and sufficient for it to be constructible in \( U \).

PROOF.

(i) Using Proposition (9.1.3), we reduce to showing that if \( T \) is a retrocompact open set in \( X \), then \( T \cap U \) is a retrocompact open set in \( U \), in other words, for every quasi-compact open \( W \subseteq U \), \( T \cap U \cap W = T \cap W \) is quasi-compact, which immediately follows from the hypothesis.

(ii) The condition is necessary by (i), so it remains to show that it is sufficient. By Proposition (9.1.3), it suffices to consider the case where \( Z \) is a retrocompact open set in \( U \), because it will then follow that \( U \cap Z \) is constructible in \( X \), and if \( Z \) and \( Z' \) are two retrocompact opens in \( U \), then \( Z \cap (U - Z') \) will be constructible in \( X \). If \( W \) is a quasi-compact open set in \( X \) and \( Z \) a retrocompact open set in \( U \), then we have \( W \cap U = Z \cap (W \cap U) \), and by hypothesis \( W \cap U \) is a quasi-compact open set in \( U \); so \( W \cap U \) is quasi-compact, and as a result \( Z \) is a retrocompact open set in \( X \), and a fortiori constructible in \( X \). \( \square \)

Corollary (9.1.9). — Let \( X \) be a topological space, \( (U_i)_{i \in I} \) a finite cover of \( X \) by retrocompact open sets in \( X \). For a subset \( Z \) of \( X \) to be constructible in \( X \), it is necessary and sufficient for \( Z \cap U_i \) to be constructible in \( U_i \) for all \( i \in I \).

(9.1.10). In particular, suppose that \( X \) is quasi-compact and every point of \( X \) admits a fundamental system of retrocompact open neighborhoods in \( X \) (and a fortiori quasi-compact); then the condition for a subset \( Z \) of \( X \) to be constructible in \( X \) is of a local nature, in other words, it is necessary and sufficient that for every \( x \in X \), there exists an open neighborhood \( V \) of \( x \) such that \( V \cap Z \) is constructible in \( V \). Indeed, if this condition is satisfied, then there exists for every \( x \in X \) an open neighborhood \( V \) of \( x \) which is retrocompact in \( X \) and such that \( V \cap Z \) is constructible in \( V \), by the
hypotheses on X and by Proposition (9.1.8, i); it then suffices to cover X by a finite number of these neighborhoods and to apply Corollary (9.1.9).

**Definition (9.1.11).** — Let X be a topological space. We say that a subset T of X is locally constructible in X if for every x ∈ X there exists an open neighborhood V of x such that T ∩ V is constructible in V.

It follows from Proposition (9.1.8, i) that if V is such that V ∩ T is constructible in V, then for every open W ⊂ V, W ∩ T is constructible in W. If T is locally constructible in X, then for every open set U in X, T ∩ U is locally constructible in U, as a result of the above remark. The same remark shows that the set of locally constructible subsets of X is stable under finite unions and finite intersections; on the other hand, it is clear that it is also stable under taking complements.

**Proposition (9.1.12).** — Let X be a topological space. Every constructible set in X is locally constructible in X. The converse is true if X is quasi-compact and if its topology admits a basis formed by the retrocompact sets in X.

**Proof.** The first assertion follows from Definition (9.1.11) and the second from (9.1.10). □

**Corollary (9.1.13).** — Let X be a topological space whose topology admits a basis formed by the retrocompact sets in X. Then every locally constructible subset T of X is retrocompact in X.

**Proof.** Let U be a quasi-compact open set in X; T ∩ U is locally constructible in U, hence constructible in U by Proposition (9.1.12), and as a result quasi-compact by Corollary (9.1.4). □

### 9.2. Constructible subsets of Noetherian spaces

(9.2.1). We have seen (9.1.7) that in a Noetherian space X, the constructible subsets of X are the finite unions of locally closed subsets of X.

The inverse image of a constructible set in X by a continuous map from a Noetherian space Y' to X is constructible in X'. If Y is a constructible subset of a Noetherian space X, then the subsets of Y are constructible as subspaces of Y and are identical to those which are constructible as subspaces of X.

**Proposition (9.2.2).** — Let X be an irreducible Noetherian space, E a constructible subset of X. For E to be everywhere dense in X, it is necessary and sufficient for E to contain a nonempty open subset of X.

**Proof.** The condition is evidently sufficient, as every nonempty open set is dense in X. Conversely, let \( E = \bigcup_{i=1}^{n} (U_i \cap F_i) \) be a constructible subset of X, the \( U_i \) being nonempty open sets and the \( F_i \) closed in X; we then have \( E \subset \bigcup_{i=1}^{n} F_i \). As a result, if \( E = X \), then X is equal to one of the \( F_i \), hence \( E \supset U_i \), which finishes the proof. □

When X admits a generic point \( x \) (0.2.1.2), the condition of Proposition (9.2.2) is equivalent to the relation \( x \in E \).

**Proposition (9.2.3).** — Let X be a Noetherian space. For a subset E of X to be constructible, it is necessary and sufficient that, for every irreducible closed subset Y of X, \( E \cap Y \) is rare in Y or contains a nonempty open subset of Y.

**Proof.** The necessity of the condition follows from the fact that \( E \cap Y \) must be a constructible subset of Y and from Proposition (9.2.2), since a nondense subset of Y is necessarily rare in the irreducible space Y (0.2.1.1). To prove that the condition is sufficient, apply the principle of Noetherian induction (0.2.2.2) to the set \( \mathfrak{F} \) of closed subsets Y of X such that \( Y \cap E \) is constructible (relative to Y or relative to X, which are equivalent); we can thus assume that for every closed subset \( Y \neq X \) of X, \( E \cap Y \) is constructible. First suppose that X is not irreducible, and let \( X_i \) (\( 1 \leq i \leq m \)) are its irreducible components, necessarily of finite number (0.2.5); by hypothesis the \( E \cap X_i \) are constructible, hence their union \( E \) is as well. Suppose now that X is irreducible; then by hypothesis, if \( E \) is rare, then \( E \neq X \) and \( E = E \cap \bar{E} \) is constructible; if \( E \) contains a nonempty open subset \( U \) of X, then it is the union of \( U \) and \( E \cap (X - U) \); but \( X - U \) is a closed set distinct from X, so \( E \cap (X - U) \) is constructible; as a result, \( E \) is itself constructible, which finishes the proof. □
Corollary (9.2.4). — Let $X$ be a Noetherian space, $(E_{α})$ an increasing filtered family of constructible subsets of $X$, such that

1. $X$ is the union of the family $(E_{α})$.
2. Every irreducible closed subset of $X$ is contained in the closure of one of the $E_{α}$.

Then there exists an index $α$ such that $X = E_{α}$.

When every irreducible closed subset of $X$ admits a generic point, the hypothesis (1st) can be omitted.

Proof. We apply the principle of Noetherian induction (0, 2.2.2) to the set $M$ of closed subsets of $X$ contained in at least one of the $E_{α}$, we can thus suppose that every closed subset $Y \neq X$ of $X$ is contained in one of the $E_{α}$. The proposition is evident if $X$ is not irreducible, because each of the irreducible components $X_i$ of $X$ $(1 \leq i \leq m)$ is contained in an $E_{α_i}$, and there exists an $E_{α}$ containing all of the $E_{α_i}$. Now suppose that $X$ is irreducible. By hypothesis, there exists a $β$ such that $X = E_{β}$, so (9.2.2) $E_{β}$ contains a nonempty open subset $U$ of $X$. But then the closed set $X - U$ is contained in an $E_γ$, and it suffices to take an $E_{α}$ containing $E_{β}$ and $E_γ$. When every irreducible closed subset $Y$ of $X$ admits a generic point $y$, there exists $α$ such that $y \in E_{α}$, so $Y = \{y\} \subset E_{α}$, and condition (2nd) is therefore a consequence of (1st).

Proposition (9.2.5). — Let $X$ be a Noetherian space, $x$ a point of $X$, and $E$ a constructible subset of $X$. For $E$ to be a neighborhood of $x$, it is necessary and sufficient that for every irreducible closed subset $Y$ of $X$ containing $x$, $E \cap Y$ is dense in $Y$ (if there exists a generic point $y$ of $Y$, this also implies (9.2.2) that $y \in E$).

Proof. The condition is evidently necessary; we will prove that it is sufficient. Applying the principle of Noetherian induction to the set $M$ of closed subsets $Y$ of $X$ containing $x$ and such that $E \cap Y$ is a neighborhood of $x$ in $Y$, we can assert that every closed subset $Y \neq X$ of $X$ containing $x$ belongs to $M$. If $X$ is not irreducible, then each of the irreducible components $X_i$ of $X$ containing $x$ are distinct from $X$, hence $E \cap X_i$ is a neighborhood of $x$ with respect to $X_i$; as a result, $E$ is a neighborhood of $x$ in the union of the irreducible components of $X$ containing $x$, and as this union is a neighborhood of $x$ in $X$, so is $E$. If $X$ is irreducible, then $E$ is dense in $X$ by hypothesis, so it contains a nonempty open subset $U$ of $X$ (9.2.2); the proposition is then evident if $x \in U$; otherwise, $x$ is by hypothesis inside $E \cap (X - U)$ with respect to $X - U$, so the closure of $X - E$ in $X$ does not contain $x$, and the complement of this closure is a neighborhood of $x$ contained in $E$, which finishes the proof.

Corollary (9.2.6). — Let $X$ be a Noetherian space, $E$ a subset of $X$. For $E$ to be an open set in $X$, it is necessary and sufficient that for every irreducible closed subset $Y$ of $X$ intersecting $E$, $E \cap Y$ contains a nonempty open subset of $Y$.

Proof. The condition is evidently necessary; conversely, if it is satisfied, then it implies that $E$ is constructible by Proposition (9.2.3). In addition, Proposition (9.2.5) shows that $E$ is then a neighborhood of each of its points, hence the conclusion.

9.3. Constructible functions

Definition (9.3.1). — Let $h$ be a map from a topological space $X$ to a set $T$. We say that $h$ is constructible if $h^{-1}(t)$ is constructible for every $t \in T$, and empty except for finitely many values of $t$; then for every subset $S$ of $T$, $h^{-1}(S)$ is constructible. We say that $h$ is locally constructible if every $x \in X$ has an open neighborhood $V$ such that $h|V$ is constructible.

Every constructible function is locally constructible; the converse is true when $X$ is quasi-compact and admits a basis formed by the retrocompact open sets in $X$ (in particular, when $X$ is Noetherian).

Proposition (9.3.2). — Let $h$ be a map from a Noetherian space $X$ to a set $T$. For $h$ to be constructible, it is necessary and sufficient that for every irreducible closed subset $Y$ of $X$, there exists a nonempty subset $U$ of $Y$, open relative to $Y$, in which $h$ is constant.

Proof. The condition is necessary: indeed, by hypothesis, $h$ does not take finitely many values $t_i$ on $Y$, and each of the sets $h^{-1}(t_i) \cap Y$ is constructible in $Y$ (9.2.1); as they can not all be rare subsets of the space $Y$, at least one of them contains a nonempty open set (9.2.3).
Suppose that \( h \) is a constructible map from \( X \) to an ordered set. For \( h \) to be upper semi-continuous on \( X \), it is necessary and sufficient that for every \( x \in X \) and every specialization \((0,2.1.2)\) \( x' \) of \( x \), we have \( h(x') \leq h(x) \).

**Proposition (9.3.4).** — Let \( X \) be a Noetherian space in which every irreducible closed subset admits a generic point, \( h \) a constructible map from \( X \) to an ordered set. For \( h \) to be upper semi-continuous on \( X \), it is necessary and sufficient that for every \( x \in X \) and every specialization \((0,2.1.2)\) \( x' \) of \( x \), we have \( h(x') \leq h(x) \).

**Proof.** The function \( h \) does not take a finite number of values; therefore, to say that it is upper semi-continuous means that for every \( x \in X \), the set \( E \) of the \( y \in X \) such that \( h(y) \leq h(x) \) is a neighborhood of \( x \). By hypothesis, \( E \) is a constructible subset of \( X \); on the other hand, to say that an irreducible closed subset \( Y \) of \( X \) contains \( x \) means that its generic point \( y \) is a specialization of \( x \); the conclusion then follows from Proposition (9.2.5).

### §10. Supplement on Flat Modules

For any proofs missing in (10.1) and (10.2), we refer the reader to Bourbaki, *Alg. comm.*, chap. II and III.

**10.1. Relations between flat modules and free modules**

(10.1.1). Let \( A \) be a ring, \( \mathfrak{J} \) an ideal of \( A \), and \( M \) an \( A \)-module; for every integer \( p \geq 0 \), we have a canonical homomorphism of \((A/\mathfrak{J})\)-modules

\[
\phi_p : (M/\mathfrak{J} M) \otimes_{A/\mathfrak{J}} (\mathfrak{J}^p/\mathfrak{J}^{p+1}) \rightarrow \mathfrak{J}^p M/\mathfrak{J}^{p+1} M,
\]

which is evidently surjective. We denote by \( \text{gr}(A) = \bigoplus_{p \geq 0} \mathfrak{J}^p/\mathfrak{J}^{p+1} \) the graded ring associated to \( A \) filtered by the \( \mathfrak{J}^p \), and by \( \text{gr}(M) = \bigoplus_{p \geq 0} \mathfrak{J}^p M/\mathfrak{J}^{p+1} M \) the graded \( \text{gr}(A) \)-module associated to \( M \) filtered by the \( \mathfrak{J}^p M \); we then have \( \text{gr}_p(A) = \mathfrak{J}^p/\mathfrak{J}^{p+1} \), and \( \text{gr}_p(M) = \mathfrak{J}^p M/\mathfrak{J}^{p+1} M \); the \( \phi \) define a surjective homomorphism of graded \( \text{gr}(A) \)-modules

\[
\phi : \text{gr}_0(M) \otimes_{\text{gr}_0(A)} \text{gr}(A) \rightarrow \text{gr}(M).
\]
(d) \( \text{Tor}_1^B(M, A/\mathfrak{a}) = 0 \).

(e) The canonical homomorphism (10.1.1.2) is bijective.

This result can be applied, in particular, to the following two cases:

(i) \( M \) is an arbitrary module, over a local ring \( A \) whose maximal ideal \( \mathfrak{a} \) is nilpotent (for example, a local Artinian ring);

(ii) \( M \) is a module of finite type over a local Noetherian ring.

### 10.2. Local flatness criteria

(10.2.1). With the hypotheses and notation of (10.1.1), consider the following conditions.

(a) \( M \) is a flat \( A \)-module.

(b) \( \mathfrak{a} \mathfrak{a} M \) is a flat \( (A/\mathfrak{a}) \)-module, and \( \text{Tor}_1^B(M, A/\mathfrak{a}) = 0 \).

(c) \( \mathfrak{a} \mathfrak{a} M \) is a flat \( (A/\mathfrak{a}) \)-module, and the canonical homomorphism (10.1.1.2) is bijective.

(d) For all \( n \geq 0 \), \( \mathfrak{a} \mathfrak{a}^n M \) is a flat \( (A/\mathfrak{a}^n) \)-module.

Then we have the implications

\[(a) \implies (b) \implies (c) \implies (d),\]

and, if \( \mathfrak{a} \) is nilpotent, then the four conditions are equivalent. This is also the case if \( A \) is Noetherian and \( M \) is ideally separated, that is to say, for every ideal \( a \) of \( A \), the \( A \)-module \( a \otimes_A M \) is separated for the \( \mathfrak{a} \)-preadic topology.

(10.2.2). Let \( A \) be a Noetherian ring, \( B \) a commutative Noetherian \( A \)-algebra, \( \mathfrak{a} \) an ideal of \( A \) such that \( \mathfrak{a} B \) is contained in the radical of \( B \), and \( M \) a \( B \)-module of finite type. Then, when \( M \) is considered as an \( A \)-module, the four conditions of (10.2.1) are equivalent. This result applies first and foremost in the case where \( A \) and \( B \) are local Noetherian rings, with the homomorphism \( A \to B \) being a local homomorphism. More specifically, if \( \mathfrak{a} \) is then the maximal ideal of \( A \), we can, in conditions (b) and (c), remove the hypothesis that \( \mathfrak{a} \mathfrak{a} M \) is flat, since it is automatically satisfied, and condition (d) implies that the modules \( \mathfrak{a} \mathfrak{a}^n M \) are free over the \( A/\mathfrak{a}^n \).

(10.2.3). With the hypotheses on \( A, B, \mathfrak{a}, \) and \( M \) from the start of (10.2.2), let \( \hat{A} \) be the separated completion of \( A \) for the \( \mathfrak{a} \)-preadic topology, and \( \hat{M} \) the separated completion of \( M \) for the \( \mathfrak{a} B \)-preadic topology. Then, for \( M \) to be a flat \( A \)-module, it is necessary and sufficient for \( \hat{M} \) to be a flat \( \hat{A} \)-module.

(10.2.4). Let \( \rho : A \to B \) be a local homomorphism of local Noetherian rings, \( k \) the residue field of \( A \), and \( M \) and \( N \) both \( B \)-modules of finite type, with \( N \) assumed to be \( A \)-flat. Let \( u : M \to N \) be a \( B \)-homomorphism. Then the following conditions are equivalent.

(a) \( u \) is injective, and \( \text{Coker}(u) \) is a flat \( A \)-module.

(b) \( u \otimes 1 : M \otimes_A k \to N \otimes_A k \) is injective.

(10.2.5). Let \( \rho : A \to B \) and \( \sigma : B \to C \) be local homomorphisms of local Noetherian rings, \( k \) the residue field of \( A \), and \( M \) a \( C \)-module of finite type. Suppose that \( B \) is a flat \( A \)-module. Then the following conditions are equivalent.

(a) \( M \) is a flat \( B \)-module.

(b) \( M \) is a flat \( A \)-module, and \( M \otimes_A k \) is a flat \( (B \otimes_A k) \)-module.

**Proposition (10.2.6).** — Let \( A \) and \( B \) be local Noetherian rings, \( \rho : A \to B \) a local homomorphism, \( \mathfrak{a} \) an ideal of \( B \) contained in the maximal ideal, and \( M \) a \( B \)-module of finite type. Suppose that, for all \( n \geq 0 \), \( M_n = M/\mathfrak{a}^n M \) is a flat \( A \)-module. Then \( M \) is a flat \( A \)-module.

**Proof.** We have to prove that, for every injective \( B \)-module \( u : N' \to N \) of \( A \)-modules of finite type, \( v = 1 \otimes u : M \otimes_A N' \to M \otimes_A N \) is injective. But \( M \otimes_A N' \) and \( M \otimes_A N \) are \( B \)-modules of finite type, and thus separated for the \( \mathfrak{a} \)-preadic topology (0L, 7.3.5); it thus suffices to prove that the homomorphism \( \hat{\mathfrak{a}} : M \otimes_A N' \to M \otimes_A N \) of the separated completions is injective. But \( \hat{\mathfrak{a}} = \lim v_n \), where \( v_n \) is the homomorphism \( 1 \otimes u : M_n \otimes_A N' \to M_n \otimes_A N \); since, by hypothesis, \( M_n \) is \( A \)-flat, \( v_n \) is injective for all \( n \), and thus so too is \( v \), because the functor \( \lim \) is left exact.

**Corollary (10.2.7).** — Let \( A \) be a Noetherian ring, \( B \) a local Noetherian ring, \( \rho : A \to B \) a homomorphism, \( f \) an element of the maximal ideal of \( B \), and \( M \) a \( B \)-module of finite type. Suppose that the homothety \( f_M : x \to fx \) on \( M \) is injective, and that \( M/f M \) is a flat \( A \)-module. Then \( M \) is a flat \( A \)-module.
Proof. Let $M_i = f^i M$ for $i \geq 0$; since $f_M$ is injective, $M_i/M_{i+1}$ is isomorphic to $M/fM$, and thus $A$-flat for all $i \geq 0$; the exact sequence

$$0 \rightarrow M_i/M_{i+1} \rightarrow M/M_{i+1} \rightarrow M/M_i \rightarrow 0$$

gives us, by induction on $i$, that $M/M_i$ is $A$-flat for all $i \geq 0$ (0.1L.6.1.2); we can thus apply (10.2.6). We can also argue directly as follows: for every $A$-module $N$ of finite type, $M \otimes_A N$ is a $B$-module of finite type; since $f$ belongs to the radical $n$ of $B$, the $(f)$-adic topology on $M \otimes_A N$ is finer than the $u$-adic topology, and we know that the latter is separated (0.1L.0.7.3.5). Now, since $M/M_i$ is $A$-flat, we have that

$$f^i(M \otimes_A N) = \text{Im}(M_i \otimes_A N \rightarrow M \otimes_A N) = \text{Ker}(M \otimes_A N \rightarrow (M/M_i) \otimes_A N)$$

by (0.1L.6.1.2). So let $N$ be an $A$-module of finite type, and $N'$ a submodule of $N$, with canonical injection $j : N' \rightarrow N$; in the commutative diagram

$$\begin{array}{ccc}
M \otimes_A N' & \longrightarrow & (M/M_i) \otimes_A N' \\
1_{M \otimes j} & & 1_{M/M_i \otimes j} \\
M \otimes_A N & \longrightarrow & (M/M_i) \otimes_A N
\end{array}$$

$1_{M/M_i} \otimes j$ is injective, because $M/M_i$ is $A$-flat; we thus conclude that

$$\text{Ker}(M \otimes_A N' \rightarrow M \otimes_A N) \subset \text{Ker}(M \otimes_A N' \rightarrow (M/M_i) \otimes_A N')$$

for any $i$; since the intersection (over $i$) of the latter kernel is 0, as we saw above, so too is the intersection (over $i$) of the former, and so $M$ is $A$-flat.

**Proposition (10.2.8).** Let $A$ be a reduced Noetherian ring, and $M$ an $A$-module of finite type. Suppose that, for every $A$-algebra $B$ (which is then a discrete valuation ring), $M \otimes_A B$ is a flat $B$-module (and thus free (10.1.3)). Then $M$ is a flat $A$-module.

Proof. We know that, for $M$ to be flat, it is necessary and suffices for $M_m$ to be a flat $A_m$-module for every maximal ideal $m$ of $A$ (0.1L.6.3.3); we can thus restrict to the case where $A$ is local (0.1L.1.2.8). So let $m$ be the maximal ideal of $A$, $p_i$ ($1 \leq i \leq r$) the minimal prime ideals of $A$, and $k$ the residue field $A/m$. We know (II, 7.1.7) that there exists, for each $i$, a discrete valuation ring $B_i$ that has the same field of fractions $K_i$ as the integral ring $A/p_i$, and that, further, dominates $A/p_i$. Let $M_i = M \otimes_A B_i$. By hypothesis, $M_i$ is free over $B_i$, and so, denoting by $k_i$ the residue field of $B_i$, we have

(10.2.8.1) \[ \text{rg}_{k_i}(M_i \otimes_{B_i} k_i) = \text{rg}_{K_i}(M_i \otimes_{B_i} K_i). \]

But it is clear that the composite homomorphism $A \rightarrow A/p_i \rightarrow B_i$ is local, and so $k$ is an extension of $k_i$, and that we have $M_i \otimes_{B_i} k_i = M \otimes_A k_i = (M \otimes_A k) \otimes k_i$, and also that $M_i \otimes_{B_i} K_i = M \otimes_A K_i$. Equation (10.2.8.1) thus implies that

$$\text{rg}_{k_i}(M \otimes_A k) = \text{rg}_{K_i}(M \otimes_A K_i) \quad \text{for } 1 \leq i \leq r$$

and since $A$ is reduced, we know that this condition implies that $M$ is a free $A$-module (Bourbaki, *Alg. comm.*, chap. II, §3, n°2, prop. 7). \[ \square \]
10.3. Existence of flat extensions of local rings

**Proposition (10.3.1).** — Let $A$ be a local Noetherian ring, with maximal ideal $\mathfrak{a}$, and residue field $k = A/\mathfrak{a}$. Let $K$ be a field extension of $k$. Then there exists a local homomorphism from $A$ to a local Noetherian ring $B$, such that $B/\mathfrak{a}B$ is $k$-isomorphic to $K$, and such that $B$ is a flat $A$-module.

The rest of this section is devoted to proving this proposition, step-by-step.

(10.3.1). First suppose that $K = k(t)$, where $t$ is an indeterminate. In the ring of polynomials $A' = A[\mathfrak{a}]$, consider the prime ideal $\mathfrak{a}' = \mathfrak{a}A$, consisting of the polynomials that have coefficients in the ideal $\mathfrak{a}$; it is clear that $A'/\mathfrak{a}'$ is canonically isomorphic to $k[\mathfrak{a}]$. We will show that the ring of fractions $B = A'_{\mathfrak{a}'}$ is that for which we are searching (that is, a ring which satisfies the conditions of the conclusion of the proposition); it is clearly a local Noetherian ring, with maximal ideal $\mathfrak{a}$.

Now suppose that $K = k(t) = k[t]$, where $t$ is algebraic over $k$; let $f \in k[T]$ be the minimal polynomial of $t$; there exists a monic polynomial $F \in A[T]$ whose canonical image in $k[T]$ is $f$. So let $A' = A[T]$, and let $\mathfrak{a}'$ be the ideal $\mathfrak{a}A' + (F)$ in $A'$. We will see that the quotient ring $B = A'/F$ is that for which we are searching. First of all, it is clear that $B$ is a free $A$-module, and thus flat. The ring $A'/\mathfrak{a}'$ is isomorphic to $(A'/\mathfrak{a}A')/(\mathfrak{a} + (F)) = k[T]/(f) = K$.

the image $\mathfrak{a}'$ of $\mathfrak{a}'$ in $B$ is thus maximal, and we evidently have that $\mathfrak{a}' = \mathfrak{a}$. Finally, $B$ is a semi-local ring, because it is an $A$-module of finite type (Bourbaki, Alg. comm., chap. IV, §2, n° 5, cor. 3 of prop. 9), and its maximal ideals are in bijective correspondence with those of $B/\mathfrak{a}B$ ([SZ60, vol. I, p. 259]); the previous arguments then prove that $B$ is a local ring.

**Lemma (10.3.1.3).** — Let $(A_λ, f_λ, A_λ)$ be a filtered inductive system of local rings, such that the $f_λ$ are local homomorphisms; let $m_λ$ be the maximal ideal of $A_λ$, and let $K_λ = A_λ/m_λ$. Then $A' = \lim A_λ$ is a local ring, with maximal ideal $m = \lim m_λ$, and residue field $K = \lim K_λ$. Further, if $m_μ = m_λ A_μ$ with $λ < μ$, then we have $m' = m_λ A_μ'$ for all $λ$. If, further, for $λ < μ$, $A_μ$ is a flat $A_λ$-module, and if all the $A_λ$ are Noetherian, then $A'$ is a flat Noetherian $A_λ$-modules for all $λ$.

**Proof.** Since, by hypothesis, $(f_λ, A_λ)(m_λ) \subset m_μ$ for $λ < μ$, the $m_λ$ form an inductive system, and its limit $m'$ is evidently an ideal of $A'$. Further, if $x' \notin m'$, there exists a $λ$ such that $x' = f_λ(x_λ)$ for some $x_λ \in A_λ$ (where $f_λ : A_λ \to A'$ denotes the canonical homomorphism); because $x' \notin m'$, we necessarily have that $x_λ \notin m_λ$, and so $x_λ$ admits an inverse $y_λ$ in $A_λ$, and $y' = f_λ(y_λ)$ is the inverse of $x'$ in $A'$, which proves that $A'$ is a local ring with maximal ideal $m'$; the claim about $K$ follows immediately from the fact that $\lim μ$ is an exact functor. The hypothesis that $m_μ = m_λ A_μ$ implies that the canonical map $m_λ \otimes A_λ A_μ \to m_μ$ is surjective; the equality $m' = m_λ A_μ'$ then follows from, again, the fact that the functor $\lim μ$ is exact and commutes with the tensor product.

Now suppose that, for $λ < μ$, we have $m_μ = m_λ A_μ$, and that $A_μ$ is a flat $A_λ$-module. Then $A'$ is a flat $A_λ$-module for all $λ$, by (0, 6.2.3); since $A$ and $A_λ$ are local rings, and since $m' = m_λ A_μ'$, $A'$ is even a faithfully flat $A_λ$-module (0, 6.6.2). Finally, suppose further that the $A_λ$ are Noetherian; the $m_λ$-adic topologies are then separated (0, 7.3.5); we now show that, from this, it follows that the $m'$-adic topology on $A'$ is separated. Indeed, if $x' \in A'$ belongs to all the $m_μ^n (n \geq 0)$, then it is the image of some $x_μ \in A_μ$ for a specific index $μ$, and since the inverse image in $A_μ$ of $m_μ^n = m_μ^n A'$ is $m_μ^n (0, 6.6.1)$, $x_μ$ belongs to all the $m_μ^n$, so $x_μ = 0$, by hypothesis, and so $x' = 0$. Denote by $A'$ the completion of $A'$ for the $m'$-adic topology; the above shows that we have $A' \subset A'$. We now show that $A'$ is Noetherian and $A_λ$-flat for all $λ$; from this, it will follow that $A'$ is $A'$-flat (0, 6.2.3), and since $m' A' = A'$, that $A'$ is a faithfully flat $A'$-module (0, 6.6.2), whence the final conclusion that $A'$ is Noetherian (0, 6.5.2), which will finish the proof of the lemma.

We have $A' = \lim A'/m_μ^n$, by the fact that $A'$ is $A_λ$-flat, we have that $m_μ^n/m_μ^{n+1} = (m_λ^n/m_λ^{n+1}) A_λ A' = (m_λ^n/m_λ^{n+1}) K_λ (K_λ \otimes A_λ A') = (m_λ^n/m_λ^{n+1}) K_λ$. 


With this construction, it is clear that the ring $B$ is Noetherian. We further know that the maximal ideal of $\hat{A}'$ is $m'\hat{A}'$, and that $A'/m'\hat{A}'$ is isomorphic to $A'/m'^n$; since $A'/m'^n = (A_\lambda/m^n_\lambda) \otimes_{A_\lambda} A'$, we see that $A'/m'\hat{A}'$ is a flat $(A_\lambda/m^n_\lambda)$-module (0.1, 6.2.1); criterion (10.2.2) is thus applicable to the Noetherian $A_\lambda$-algebra $\hat{A}'$, and shows that $\hat{A}'$ is $A_\lambda$-flat. 

(10.3.1.4) We now treat the general case. There exists an ordinal $\gamma$ and, for every ordinal $\lambda \leq \gamma$, a subfield $k_\lambda$ of $K$ that contains $k$, such that (i) for all $\lambda < \gamma$, $k_{\lambda+1}$ is an extension of $k_\lambda$ generated by a single element; (ii) for every limit ordinal $\mu$, $k_\mu = \bigcup_{\lambda < \mu} k_\lambda$; and (iii) $K = k_\gamma$. In fact, it suffices to consider a bijection $\xi \mapsto t_\xi$ from the set of ordinals $\xi \leq \beta$ (for some suitable $\beta$) to $K$, and to define $k_\lambda$ by transfinite induction (for $\lambda \leq \beta$) as the union of the $k_\mu$ for $\mu < \lambda$ is a limit ordinal, and as $k_\lambda(t_\xi)$ if $\lambda = \nu + 1$, where $\xi$ is the smallest ordinal such that $t_\xi \notin k_\nu$; $\gamma$ is then, by definition, the smallest ordinal $\leq \beta$ such that $k_\gamma = K$.

With this in mind, we will define, by transfinite induction, a family of local Noetherian rings $A_\lambda$ for $\lambda \leq \gamma$, and local homomorphisms $f_{\mu\lambda} : A_\lambda \rightarrow A_\mu$ for $\lambda \leq \mu$, satisfying the following conditions:

(i) $(A_\lambda, f_{\mu\lambda})$ is an inductive system, and $A_0 = A$;
(ii) for all $\lambda$, we have a $k$-isomorphism $A_\lambda/\mathfrak{N}A_\lambda \simeq k_\lambda$;
(iii) for $\lambda \leq \mu$, $A_\mu$ is a flat $A_\lambda$-module.

So suppose that $A_\lambda$ and the $f_{\mu\lambda}$ are defined for $\lambda < \mu < \xi$, and suppose, first of all, that $\xi = \xi + 1$, so that $k_\xi = k_\xi(t_i)$. If $t$ is transcendental over $k_\xi$, we define $A_t$, following the procedure of (10.3.1.1), to be equal to $(A_\xi[t]_{\lambda \in \mathbb{N}})$; the canonical map is $f_{t\xi}$, and, for $\lambda < \xi$, we take $f_{t\lambda} = f_{t\xi} \cdot f_{\xi\lambda}$; the verification of conditions (i) to (iii) is then immediate, given that what we have shown in (10.3.1.1). So now suppose that $t$ is algebraic, and let $h$ be its minimal polynomial in $k_\xi[T]$, and $H$ a monic polynomial in $A_\xi[T]$ whose image in $k_\xi[T]$ is $h$; we then take $A_t$ to be equal to $A_\xi[T](h)$, with the $f_{t\lambda}$ being defined as before; the verification of conditions (i) to (iii) then follows from what we have shown in (10.3.1.2).

Now suppose that $\xi$ has no predecessor; we then take $A_\xi$ to be the inductive limit of the inductive system of local rings $(A_\lambda, f_{\mu\lambda})$ for $\lambda < \xi$; we define $f_{\xi\lambda}$ as the canonical map for $\lambda < \xi$. The fact that $A_\xi$ is local and Noetherian, that the $f_{\xi\lambda}$ are local homomorphism, and that conditions (i) to (iii) are satisfied for $\lambda \leq \xi$ then follows from the induction hypothesis, and from Lemma (10.3.1.3). With this construction, it is clear that the ring $B = A_\gamma$ satisfies the conditions of (10.3.1).

We note that, by (10.2.1, c), we have a canonical isomorphism

\[(10.3.1.5) \quad \text{gr}(A) \otimes_k K \xrightarrow{\sim} \text{gr}(B).\]

We can also replace $B$ by its $\mathfrak{N}B$-adic completion $\hat{B}$ without changing the conclusions of (10.3.1), because $\hat{B}$ is a flat $B$-module (0.1, 7.3.3), and thus a flat $A$-module (0.1, 6.2.1).

We have also shown the following:

**Corollary (10.3.2).** — If $K$ is an extension of finite degree, then we can assume that $B$ is a finite $A$-algebra.

### §11. Supplement on Homological Algebra

#### 11.1. Review of spectral sequences

(11.1.1) In the following, we use a more general notion of a spectral sequence than that defined in (T, 2.4); keeping the notations of (T, 2.4), we call a spectral sequence in an abelian category $\mathcal{C}$ a system $E$ consisting of the following:

(a) A family $(E_{pq}^r)$ of objects of $\mathcal{C}$ defined for $p, q \in \mathbb{Z}$ and $r \geq 2$.

(b) A family of morphisms $d_{pq}^r : E_{pq}^r \rightarrow E_{p+r,q-r+1}^{r+1}$ such that $d_{pq}^{r+1}d_{p+1,q-r+1}^{r+1} = 0$. We set $Z_{r+1}^r(E_{pq}^r) = \ker(d_{pq}^r)$ and $B_{r+1}^r(E_{pq}^r) = \text{im}(d_{p+1,q-r+1}^{r+1})$, so that

\[B_{r+1}^r(E_{pq}^r) \subset Z_{r+1}^r(E_{pq}^r) \subset E_{pq}^r.\]

(c) A family of isomorphisms $a_{r}^{pq} : Z_{r+1}^r(E_{pq}^r)/B_{r+1}^r(E_{pq}^r) \simeq E_{r+1}^p$. We then define for $k \geq r+1$, by induction on $k$, the subobjects $B_k(E_{pq}^r)$ and $Z_k(E_{pq}^r)$ as the inverse images, under the canonical morphism $E_{pq}^r \rightarrow E_{pq}^r/B_{r+1}^r(E_{pq}^r)$ of the subobjects $\ldots$
of this quotient identified via $a^{pq}_{r}$ with the subobjects $B_{k}(E^{pq}_{r+1})$ and $Z_{k}(E^{pq}_{r+1})$ respectively. It is clear that we then have, up to isomorphism,

$Z_{k}(E^{pq}_{r})/B_{k}(E^{pq}_{r}) = E^{pq}_{k}$ for $k \geq r + 1$,

and, if we set $B_{r}(E^{pq}_{r}) = 0$ and $Z_{r}(E^{pq}_{r}) = E^{pq}_{r}$, then we have the inclusion relations

$0 = B_{r}(E^{pq}_{r}) \subset B_{r+1}(E^{pq}_{r}) \subset B_{r+2}(E^{pq}_{r}) \subset \cdots \subset Z_{r+2}(E^{pq}_{r}) \subset Z_{r+1}(E^{pq}_{r}) \subset Z_{r}(E^{pq}_{r}) = E^{pq}_{r}$

The other parts of the data of $E$ are then:

(d) Two subobjects $B_{\infty}(E^{pq}_{2})$ and $Z_{\infty}(E^{pq}_{2})$ of $E^{pq}_{2}$ such that we have $B_{\infty}(E^{pq}_{2}) \subset Z_{\infty}(E^{pq}_{2})$ and, for every $k \geq 2$,

$B_{k}(E^{pq}_{2}) \subset B_{\infty}(E^{pq}_{2})$ and $Z_{\infty}(E^{pq}_{2}) \subset Z_{k}(E^{pq}_{2})$.

We set

$E^{pq}_{\infty} = Z_{\infty}(E^{pq}_{2})/B_{\infty}(E^{pq}_{2})$.

(e) A family $(E^{n})$ of objects of $C$, each equipped with a decreasing filtration $(F^{p}(E^{n}))_{p \in \mathbb{Z}}$. As usual, we denote by $gr(E^{n})$ the graded object associated to the filtered object $E^{n}$, the direct sum of the $gr_{p}(E^{n}) = F^{p}(E^{n})/F^{p+1}(E^{n})$.

(f) For every pair $(p, q) \in \mathbb{Z} \times \mathbb{Z}$, an isomorphism $B^{pq} : E^{pq}_{\infty} \simeq gr_{p}(E^{p+q})$.

The family $(E^{n})$, without the filtrations, is called the abutment (or limit) of the spectral sequence $E$.

Suppose that the category $C$ admits infinite direct sums, or that for every $r \geq 2$ and every $n \in \mathbb{Z}$, there are finitely many pairs $(p, q)$ such that $p + q = n$ and $E^{pq}_{r} \neq 0$ (it suffices for it to hold for $r = 2$). Then the $E^{n}_{r} = \sum_{p+q=n} E^{pq}_{r}$ are defined, and we if denote by $d^{(n)}_{r}$ the morphism $E^{n}_{r} \rightarrow E^{n}_{r+1}$, whose restriction to $E^{pq}_{r}$ is $d^{pq}_{r}$ for every pair $(p, q)$ such that $p + q = n$, then $d^{(n+1)}_{r} \circ d^{(n)}_{r} = 0$, in other words, $(E^{n}_{r})_{n \in \mathbb{Z}}$ is a complex $E^{*}_{r}$ in $C$, with differentials of degree $+1$, and it follows from (c) that $H^{n}(E^{*}_{r})$ is isomorphic to $E^{n}_{r+1}$ for every $r \geq 2$.

(11.1.2). A morphism $u : E \rightarrow E'$ from a spectral sequence $E$ to a spectral sequence $E' = (E^{pq}_{r}, E^{n})$ consists of systems of morphisms $u^{pq}_{r} : E^{pq}_{r} \rightarrow E'^{pq}_{r}$ and $u^{n} : E^{n} \rightarrow E'^{n}$, the $u^{n}$ compatible with the filtrations on $E^{n}$ and $E'^{n}$, and the diagrams

$$
\begin{array}{ccc}
E^{pq}_{r} & \xrightarrow{d^{pq}_{r}} & E^{pq}_{r+1} \\
\downarrow u^{pq}_{r} & & \downarrow u^{pq}_{r+1} \\
E'^{pq}_{r} & \xrightarrow{d'^{pq}_{r}} & E'^{pq}_{r+1}
\end{array}
$$

being commutative; in addition, by passing to quotients, $u^{pq}_{r}$ gives a morphism $\pi^{pq}_{r} : Z_{r+1}(E^{pq}_{r})/B_{r+1}(E^{pq}_{r}) \rightarrow Z_{r+1}(E'^{pq}_{r})/B_{r+1}(E'^{pq}_{r})$ and we must have $u^{pq}_{r+1} \circ \pi^{pq}_{r} = u^{pq}_{r} \circ \pi^{pq}_{r}$; finally, we must have $u^{pq}_{\infty}(B_{\infty}(E^{pq}_{2})) \subset B_{\infty}(E'^{pq}_{2})$ and $u^{pq}_{\infty}(Z_{\infty}(E^{pq}_{2})) \subset Z_{\infty}(E'^{pq}_{2})$; by passing to quotients, $u^{pq}_{2}$ then gives a morphism $u^{pq}_{\infty} : E^{pq}_{\infty} \rightarrow E'^{pq}_{\infty}$, and the diagram

$$
\begin{array}{ccc}
E^{pq}_{\infty} & \xrightarrow{u^{pq}_{\infty}} & E'^{pq}_{\infty} \\
\downarrow gr_{p}(E^{pq}) & & \downarrow gr_{p}(E'^{pq}) \\
gr_{p}(E^{p+q}) & \xrightarrow{gr_{p}(u^{p+q})} & gr_{p}(E'^{p+q})
\end{array}
$$

must be commutative.

The above definitions show, by induction on $r$, that if the $u^{pq}_{2}$ are isomorphisms, then so are the $u^{pq}_{r}$ for $r \geq 2$; if in addition we know that $u^{pq}_{2}(B_{\infty}(E^{pq}_{2})) = B_{\infty}(E'^{pq}_{2})$ and $u^{pq}_{2}(Z_{\infty}(E^{pq}_{2})) = Z_{\infty}(E'^{pq}_{2})$ and the $u^{n}$ are isomorphisms, then we can conclude that $u$ is an isomorphism.
(11.1.3). Recall that if \((F^p(X))_{p \in Z}\) is a \(\) (decreasing) filtration of an object \(X \in C\), then we say that this filtration is separated if \(\inf(F^p(X)) = 0\), discrete if there exists a \(p\) such that \(F^p(X) = 0\), exhaustive (or coseparated) if \(\sup(F^p(X)) = X\), codiscrete if there exists a \(p\) such that \(F^p(X) = X\).

We say that a spectral sequence \(E = (E^p, E^n)\) is weakly convergent if we have \(\text{B}_\infty(E^p) = \sup_{k}(B_k(E^p))\) and \(\text{Z}_\infty(E^p) = \inf_{k}(Z_k(E^p))\) (in other words, the objects of \(\text{B}_\infty(E^p)\) and \(\text{Z}_\infty(E^p)\) are determined from the data of (a) and (c) of the spectral sequence \(E\)). We say that the spectral sequence \(E\) is regular if it is weakly convergent and if in addition:

1st. For every \((p, q)\), the decreasing sequence \((Z_k(E^p_{pq}))_{k \geq 2}\) is stable; the hypothesis that \(E\) is weakly convergent then implies that \(\text{Z}_\infty(E^p_{pq}) = Z_k(E^p_{pq})\) for \(k\) large enough (depending on \(p\) and \(q\)).

2nd. For every \(n\), the filtration \((F^p(E^n))_{p \in Z}\) of \(E^n\) is discrete and exhaustive.

We say that the spectral sequence \(E\) is coregular if it is weakly convergent and if in addition:

3rd. For every \((p, q)\), the increasing sequence \((B_k(E^p_{pq}))_{k \geq 2}\) is stable, which implies that \(B_\infty(E^p_{pq}) = B_k(E^p_{pq})\), and as a result, \(E^p_\infty = \inf E^p_{pq}\).

4th. For every \(n\), the filtration of \(E^n\) is codiscrete.

Finally, we say that \(E\) is biregular if it is both regular and coregular, in other words if we have the following conditions:

(a) For every \((p, q)\), the sequences \((B_k(E^p_{pq}))_{k \geq 2}\) and \((Z_k(E^p_{pq}))_{k \geq 2}\) are stable and we have \(B_\infty(E^p_{pq}) = B_k(E^p_{pq})\) and \(Z_\infty(E^p_{pq}) = Z_k(E^p_{pq})\) for \(k\) large enough (which implies that \(E^p_\infty = E^p_k\)).

(b) For every \(n\), the filtration \((F^p(E^n))_{p \in Z}\) is discrete and codiscrete (which we also call finite).

The spectral sequences defined in (12.4) are thus biregular spectral sequences.

(11.1.4). Suppose that in the category \(C\), filtered inductive limits exist and the functor \(\text{lim}_-> X\) is exact (which is equivalent to saying that the axiom \((AB\) 5) of (T, 1.5) is satisfied (cf. T, 1.8)). The condition that the filtration \((F^p(X))_{p \in Z}\) of an object \(X \in C\) is exhaustive is then expressed as \(\text{lim}_-> F^p(X) = X\). If a spectral sequence \(E\) is weakly convergent, then we have \(\text{B}_\infty(E^p) = \lim_{k} B_k(E^p)\); if in addition \(u : E \to E'\) is a morphism from \(E\) to a weakly convergent spectral sequence \(E'\) in \(C\), then we have \(u^p_2(\text{B}_\infty(E^p_{pq})) = \text{B}_\infty(u^p_{pq})\), by the exactness of \(lim\). In addition:

**Proposition (11.1.5).** — Let \(C\) be an abelian category in which filtered inductive limits are exact, \(E\) and \(E'\) two regular spectral sequences in \(C\), \(u : E \to E'\) a morphism of spectral sequences. If the \(u^p_{pq}\) are isomorphisms, then so is \(u\).

**Proof.** We already know (11.1.2) that the \(u^p_{pq}\) are isomorphisms and that

\[
u^p_2(\text{B}_\infty(E^p_{pq})) = \text{B}_\infty(u^p_{pq});
\]

the hypothesis that \(E\) and \(E'\) are regular also implies that \(u^p_{pq}(\text{Z}_\infty(E^p_{pq})) = \text{Z}_\infty(u^p_{pq})\), and as \(u^p_{pq}\) is an isomorphism, so is \(u^p_{pq}\); we thus conclude that \(gr_p(E^{p+q})\) is also an isomorphism. But as the filtrations of the \(E^n\) and the \(E'^n\) are discrete and exhaustive, this implies that the \(u^n\) are also isomorphisms (Bourbaki, Alg. comm., chap. III, §2, n°8, th. 1). \(\square\)

(11.1.6). It follows from (11.1.2) and the definition (11.1.3) that if, for a spectral sequence \(E\), we have \(E^r_{pq} = 0\), then we have \(E^k_{pq} = 0\) for \(k \geq r\) and \(E^\infty_{pq} = 0\). We say that a spectral sequence degenerates if there exists an integer \(r \geq 2\) and, for every integer \(n \in Z\), an integer \(q(n)\) such that \(E^{r-1,q(n)}_r = 0\) for every \(q \neq q(n)\). We first deduce from the previous remark that we also have \(E^{r,q(q(n)}_r = 0\) for \(r \geq k\) (including \(k = \infty\) and \(q \neq q(n)\)). In addition, the definition of \(E^p_{r+1}\) shows that we have \(E^{r-1,q(n)}_r = E^{r,q(n)}_r;\) if \(E\) is weakly convergent, then we also have \(E^{r,q(n)}_r = E^{r-1,q(n)}_r\); in other words, for every \(n \in Z\), \(gr_p(E^n) = 0\) for \(p \neq q(n)\) and \(gr_q(E^n) = E^r_{q(n)}\). If in addition the filtration of \(E^n\) is discrete and exhaustive, then the spectral sequence \(E\) is regular, and we have \(E^n = E^r_{q(n)}\) up to unique isomorphism.
Suppose that filtered inductive limits exist and are exact in the category $\mathcal{C}$, and let $(E_{ij}, u_{ij})$ be an inductive system (over a filtered set of indices) of spectral sequences in $\mathcal{C}$. Then the inductive limit of this inductive system exists in the additive category of spectral sequences of objects of $\mathcal{C}$: to see this, it suffices to define $E_{pq}^{r+1}$, $d_{pq}^{r+1}$, $a_{pq}^{r+1}$, $B_{\infty}(E_{2}^{p+q})$, $Z_{\infty}(E_{2}^{p+q})$, $E_{n}^{p+q}(E_{n}^{0})$, and $\beta_{pq}^{r+1}$ as the respective inductive limits of the $E_{pq}^{r}$, $d_{pq}^{r}$, $a_{pq}^{r}$, $B_{\infty}(E_{2}^{p+q})$, $Z_{\infty}(E_{2}^{p+q})$, $E_{n}^{p+q}(E_{n}^{0})$, and $\beta_{pq}^{r}$; the verification of the conditions of (11.1.1) follows from the exactness of the functor $\lim_{\to}$ on $\mathcal{C}$.

Remark (11.1.8). — Suppose that the category $\mathcal{C}$ is the category of $A$-modules over a Noetherian ring $A$ (resp. a ring $A$). Then the definitions of (11.1.1) show that if, for a given $r$, the $E_{pq}^{r}$ are $A$-modules of finite type (resp. of finite length), then so are each of the modules $E_{pq}^{r}$ for $s \geq r$, hence so is $E_{pq}^{r}$. If in addition the filtration of the abutment/limit $(E^{n})$ is discrete or codiscrete for all $n$, then we conclude that each of the $E^{n}$ is also an $A$-module of finite type (resp. of finite length).

We will have to consider conditions which ensure that a spectral sequence $E$ is biregular is a “uniform” way in $p + q = n$. We will then use the following lemma:

Lemma (11.1.10). — Let $(E_{pq}^{r})$ be a family of objects of $\mathcal{C}$ related by the data of (a), (b), and (c) of (11.1.1). For a fixed integer $n$, the following properties are equivalent:

(a) There exists an integer $r(n)$ such that for $r \geq r(n)$, $p + q = n$ or $p + q = n - 1$, the morphisms $d_{pq}^{r}$ are zero.

(b) There exists an integer $r(n)$ such that for $p + q = n$ or $p + q = n + 1$, we have $B_{r}(E_{pq}^{r}) = B_{s}(E_{pq}^{s})$ for $s \geq r \geq r(n)$.

(c) There exists an integer $r(n)$ such that for $p + q = n$ or $p + q = n - 1$, we have $Z_{r}(E_{pq}^{r}) = Z_{s}(E_{pq}^{s})$ for $s \geq r \geq r(n)$.

(d) There exists an integer $r(n)$ such that for $p + q = n$, we have $B_{r}(E_{pq}^{r}) = B_{s}(E_{pq}^{s})$ and $Z_{r}(E_{pq}^{r}) = Z_{s}(E_{pq}^{s})$ for $s \geq r \geq r(n)$.

Proof. According to the conditions (a), (b), and (c) of (11.1.1), we have that saying $Z_{r+1}(E_{pq}^{r+1}) = Z_{r}(E_{pq}^{r})$ is equivalent to saying that $d_{pq}^{r+1} = 0$ and that saying $B_{r}(E_{pq}^{r+1}) = B_{r+1}(E_{pq}^{r+1})$ is equivalent to saying that $d_{pq}^{r} = 0$; the lemma immediately follows from this remark.

11.2. The spectral sequence of a filtered complex

Given an abelian category $\mathcal{C}$, we will agree to denote by notations such as $K^{\bullet}$ the complexes $(K_{i})_{i \in \mathbb{Z}}$ of objects of $\mathcal{C}$ whose differential is of degree $+1$, and by the notations such as $K_{\bullet}$ the complexes $(K_{i})_{i \in \mathbb{Z}}$ of objects of $\mathcal{C}$ whose differential is of degree $-1$. To each complex $K^{\bullet} = (K_{i})$ whose differential $d$ is of degree $+1$, we can associate a complex $K^{\bullet'} = (K_{i}')$ by setting $K_{i}' = K_{-i}$, the differential $K_{i}' \to K_{i'-1}'$ being the operator $d : K_{-i} \to K_{-(i+1)}$; and vice versa, which, depending on the circumstances, will allow one to consider either one of the types of complexes and translate any result from one type into results for the other. We similarly denote by notations such as $K^{\bullet \bullet} = (K^{i})$ (resp. $K_{\bullet \bullet} = (K_{ij})$) the bicomplexes (or double complexes) of objects of $\mathcal{C}$ in which the two differentials are of degree $+1$ (resp. $-1$); we can still pass from one type to the other by changing the signs of the indices, and we have similar notations and remarks for any multicomplexes. The notation $K^{\bullet}$ and $K_{\bullet}$ will also be used for $\mathbb{Z}$-graded objects of $\mathcal{C}$, which are not necessarily complexes (they can be considered as such for the zero differentials); for example, we write $H^{\bullet}(K^{\bullet}) = (H^{i}(K^{\bullet}))_{i \in \mathbb{Z}}$ for the cohomology of a complex $K^{\bullet}$ whose differential is of degree $+1$, and $H_{\bullet}(K_{\bullet}) = (H_{i}(K_{\bullet}))_{i \in \mathbb{Z}}$ for the homology of a complex $K_{\bullet}$ whose differential is of degree $-1$; when we pass from $K^{\bullet}$ to $K_{\bullet}$ by the method described above, we have $H_{i}(K_{i}) = H^{-i}(K^{i})$.

Recall in this case that for a complex $K^{\bullet}$ (resp. $K_{\bullet}$), we will write in general $Z^{i}(K^{\bullet}) = \text{Ker}(K^{i-1} \to K^{i})$ (“object of cocycles”) and $B^{i}(K^{\bullet}) = \text{Im}(K^{i-1} \to K^{i})$ (“object of coboundaries”) (resp. $Z_{i}(K_{\bullet}) = \text{Ker}(K_{i} \to K_{i-1})$ (“object of cycles”) and $B_{i}(K_{\bullet}) = \text{Im}(K_{i-1} \to K_{i})$ (“object of boundaries”) so that $H^{i}(K^{\bullet}) = Z^{i}(K^{\bullet}) / B^{i}(K^{\bullet})$ (resp. $H_{i}(K_{\bullet}) = Z_{i}(K_{\bullet}) / B_{i}(K_{\bullet})$).

If $K^{\bullet} = (K_{i})$ (resp. $K_{\bullet} = (K_{ij})$) is a complex in $\mathcal{C}$ and $T : \mathcal{C} \to \mathcal{C}'$ a functor from $\mathcal{C}$ to an abelian category $\mathcal{C}'$, then we denote by $T(K^{\bullet})$ (resp. $T(K_{\bullet})$) the complex $(T(K_{i}))$ (resp. $(T(K_{ij}))$) in $\mathcal{C}'$.

We will not review the definition of the $\partial^{\bullet}$-functor when the morphism $\partial$ decreases the degree of a unit, the context clarifying this point if there is cause for confusion.
Finally, we say that a graded object \((A_i)_{i \in \mathbb{Z}}\) of \(\mathcal{C}\) is bounded below (resp. above) if there exists an \(i_0\) such that \(A_i = 0\) for \(i < i_0\) (resp. \(i > i_0\)).

(11.2.2). Let \(K^*\) be a complex in \(\mathcal{C}\) whose differential \(d\) is of degree \(+1\), and suppose it is equipped with a filtration \(F(K^*) = (F^p(K^*))_{p \in \mathbb{Z}}\) consisting of graded subobjects of \(K^*\), in other words, \(F^p(K^*) = (K^i \cap F^p(K^*))_{i \in \mathbb{Z}}\); in addition, we assume that \(d(F^p(K^*)) \subset F^p(K^*)\) for every \(p \in \mathbb{Z}\). Let us quickly recall how one functorially defines a spectral sequence \(E(K^*)\) from \(K^*\) (M, XV, 4 and I, 4.3). For \(r \geq 2\), the canonical morphism \(F^p(K^*)/F^{p+r}(K^*) \to F^p(K^*)/F^{p+1}(K^*)\) defines a morphism in cohomology

\[
H^{p+q}(F^p(K^*)/F^{p+r}(K^*)) \to H^{p+q}(F^p(K^*)/F^{p+1}(K^*)).
\]

We denote by \(Z^p_1(K^*)\) the image of this morphism. Similarly, from the exact sequence

\[
0 \to F^p(K^*)/F^{p+1}(K^*) \to F^{p-r+1}(K^*)/F^{p+1}(K^*) \to F^{p-r+1}(K^*)/F^p(K^*) \to 0,
\]

we deduce from the exact sequence in cohomology a morphism

\[
H^{p+q-1}(F^{p-r+1}(K^*)/F^{p+1}(K^*)) \to H^{p+q}(F^p(K^*)/F^{p+1}(K^*)�)
\]

and we denote by \(B^p_1(K^*)\) the image of this morphism; we show that \(B^p_1(K^*) \subset Z^p_1(K^*)\) and we take \(E^p_1(K^*) = Z^p_1(K^*)/B^p_1(K^*)\); we will not specify the definition of the \(d^p_1\) or the \(\alpha^p_1\).

We note here that all the \(Z^p_1(K^*)\) and \(B^p_1(K^*)\), for \(p\) and \(q\) fixed, are subobjects of the same object \(H^{p+q}(F^p(K^*)/F^{p+1}(K^*))\), which we denote by \(Z^p_1(K^*)\); we set \(B^p_1(K^*) = 0\), so that the above definitions of \(Z^p_1(K^*)\) and \(B^p_1(K^*)\) also work for \(r = 1\); we still set \(E^p_1(K^*) = Z^p_1(K^*)\). We define \(d^p_1\) and \(\alpha^p_1\) such that the conditions of (11.1.1) are satisfied for \(r = 1\). On the other hand, we define the subobjects \(Z^p_0(K^*)\) as the image of the morphism

\[
H^{p+q}(F^p(K^*)) \to H^{p+q}(F^p(K^*)/F^{p+1}(K^*)) = E^p_0(K^*)�
\]

and \(B^p_0(K^*)\) as the image of the morphism

\[
H^{p+q-1}(K^*/F^{p+1}(K^*)) \to H^{p+q}(F^p(K^*)/F^{p+1}(K^*)) = E^p_1(K^*)�
\]

induced as above from the exact sequence in cohomology. We set \(Z_\infty(E^p_2(K^*))\) and \(B_\infty(E^p_2(K^*))\) to be the canonical images of \(E^p_2(K^*)\) in \(Z^p_2(K^*)\) and \(B^p_2(K^*)\).

Finally, we denote by \(F^p(H^p(K^*))\) the image in \(H^p(K^*)\) of the morphism \(H^p(F^p(K^*)) \to H^p(K^*)\); by the exact sequence in cohomology, this is also the kernel of the morphism \(H^p(K^*) \to H^p(K^*/F^{p+1}(K^*))\). This defines a filtration on \(E^p(K^*) = H^p(K^*)\); we will not give here the definition of the isomorphisms \(\beta^p_1\).

(11.2.3). The functorial nature of \(E(K^*)\) is understood in the following way: given two filtered complexes \(K^*\) and \(K'^*\) in \(\mathcal{C}\) and a morphism of complexes \(u : K^* \to K'^*\) that is compatible with the filtrations, we induce in an evident way the morphisms \(u^p_{pq}\) (for \(r \geq 1\)) and \(u^0\), and we show that these morphisms are compatible with the \(d^p_1\), \(\alpha^p_1\), and \(\beta^p_1\) in the sense of (11.1.2), and thus given a well-defined morphism \(E(u) : E(K^*) \to E(K'^*)\) of spectral sequences. In addition, we show that if \(u\) and \(v\) are morphisms \(K^* \to K'^*\) of the above type, homotopic in degree \(\leq k\), then \(u^p_{pq} = v^p_{pq}\) for \(r > k\) and \(u^0 = v^0\) for all \(n\) (M, XV, 3.1).

(11.2.4). Suppose that filtered inductive limits in \(\mathcal{C}\) are exact. Then if the filtration \((F^p(K^*))\) of \(K^*\) is exhaustive, then so is the filtration \((F^p(H^p(K^*)))\) for all \(n\), since by hypothesis we have \(K^* = \lim_{\to p \to -\infty} F^p(K^*)\) and since the hypothesis on \(\mathcal{C}\) implies that cohomology commutes with inductive limits. In addition, for the same reason, we have \(B_\infty(E^p_2(K^*)) = \sup_k B_k(E^p_2(K^*))\). We say that the filtration \((F^p(K^*))\) of \(K^*\) is regular if for every \(n\) there exists an integer \(u(n)\) such that \(H^n(F^p(K^*)) = 0\) for \(p > u(n)\). This is particularly the case when the filtration of \(K^*\) is discrete. When the filtration of \(K^*\) is regular and exhaustive, and filtered inductive limits are exact in \(\mathcal{C}\), we have (M, XV, 4) that the spectral sequence \(E(K^*)\) is regular.
11.3. The spectral sequences of a bicomplex

(11.3.1). With regard the conventions for bicomplexes, we follow those of (T, 2.4) rather than those of (M), the two differentials $d'$ and $d''$ (of degree $-1$) of such a bicomplex $K^{ullet, ullet} = (K^{ij})$ being thus assumed to be permutable. Suppose that one of the following two conditions is satisfied: 1st. Infinite direct sums exist in $C$; 2nd. For all $n \in \mathbb{Z}$, there is only a finite number of pairs $(p, q)$ such that $p + q = n$ and $K^{pq}_{\mathbb{Z}} \neq 0$. Then, the bicomplex $K^{ullet, ullet}$ defines a (simple) complex (11.3.2).

With regard the conventions for bicomplexes, we follow those of (T, 2.4) rather than those (F, 11.3.2.1). Let $K$ be a complex defined by $H_i(K) = \sum_{j \geq n} K^{ij}$, where $K^{ij}$ is the simple complex (11.3.1). For all $n \in \mathbb{Z}$, it will always be understood that of the above conditions is satisfied. We adopt the analogous conventions for multicomplexes.

We denote by $K^{i, ullet}$ (resp. $K^{\bullet, j}$) the simple complex $(K^{ij})_{j \in \mathbb{Z}}$ (resp. $(K^{ij})_{i \in \mathbb{Z}}$), by $Z^n_{II}(K^{ij})$, $B^n_{II}(K^{ij})$, $H^n_{II}(K^{ij})$ (resp. $Z^n_{II}(K^{ji})$, $B^n_{II}(K^{ji})$, $H^n_{II}(K^{ji})$) its $p$ objects of cocycles, of coboundaries, and of cohomology, respectively; the differential $d' : K^{i, ullet} \to K^{i+1, ullet}$ is a morphism of complexes, which thus gives an operator on the cocycles, coboundaries, and cohomology,

\[ d' : Z^n_{II}(K^{ij}) \to Z^n_{II}(K^{i+1, j}), \]
\[ d' : B^n_{II}(K^{ij}) \to B^n_{II}(K^{i+1, j}), \]
\[ d' : H^n_{II}(K^{ij}) \to H^n_{II}(K^{i+1, j}), \]

and it is clear that for these operators, $(Z^n_{II}(K^{ij}))_{i \in \mathbb{Z}}$, $(B^n_{II}(K^{ij}))_{i \in \mathbb{Z}}$, and $(H^n_{II}(K^{ij}))_{i \in \mathbb{Z}}$ are complexes; we denote the complex $(H^n_{II}(K^{ij}))_{i \in \mathbb{Z}}$ by $H^n_{II}(K^{ij})$, its $q$ objects of cocycles, coboundaries, and cohomology by $Z_q^i(H^n_{II}(K^{ij}))$, $B_q^i(H^n_{II}(K^{ij}))$, and $H_q^i(H^n_{II}(K^{ij}))$. We similarly define the complexes $H^n_p(K^{ij})$ and their cohomology objects $H^n_{II}(H^n_p(K^{ij}))$. Recall on the other hand that $H^n_p(K^{ij})$ denotes the $n$ object of the cohomology of the (simple) complex defined by $K^{ij}$.

(11.3.2). On the complex defined by a bicomplex $K^{ullet, ullet}$, we can consider two canonical filtrations $(F^n_{I}(K^{ij}))$ and $(F^n_{II}(K^{ij}))$ given by

\[ F^n_{I}(K^{ij}) = \left( \sum_{i+j=n, i \geq p} K^{ij} \right)_{n \in \mathbb{Z}}, \quad \text{and} \quad F^n_{II}(K^{ij}) = \left( \sum_{i+j=n, j \geq p} K^{ij} \right)_{n \in \mathbb{Z}}, \]

which, by definition, are graded subobjects of the (simple) complex define by $K^{ij}$, and thus make this complex a filtered complex; moreover, is is clear that these filtrations are exhaustive and separated.

There corresponds to each of these filtrations a spectral sequence (11.2.2); we denote by $E(K^{ij})$ and $E(K^{ij})$ the spectral sequences corresponding to $(F^n_{I}(K^{ij}))$ and $(F^n_{II}(K^{ij}))$ respectively, called the spectral sequence of the bicomplex $K^{ij}$, and both having as their abutment the cohomology $(H^n_p(K^{ij}))$. We show in addition (M, XV, 6) that we have

\[ E_2^{pq}(K^{ij}) = H_p^q(H^n_p(K^{ij})), \quad \text{and} \quad E_2^{pq}(K^{ij}) = H_p^q(H^n_p(K^{ij})). \]

Every morphism $u : K^{ij} \to K^{ij}$ of bicomplexes is ipso facto compatible with the filtrations of the same type of $K^{ij}$ and $K^{ij}$, thus define a morphism for each of the two spectral sequences; in addition, two homotopy morphisms define a homotopy of order $\leq 1$ of the corresponding filtered (simple) complexes, thus the same morphism for each of the two spectral sequences (M, XV, 6.1).

**Proposition (11.3.3).** — Let $K^{ij} = (K^{ij})$ be a bicomplex in an abelian category $C$.

(i) If there exist $i_0$ and $j_0$ such that $K^{ij} = 0$ for $i < i_0$ or $j < j_0$ (resp. $i > i_0$ or $j > j_0$), then the two spectral sequences $E(K^{ij})$ and $E(K^{ij})$ are biregular.

(ii) If there exist $i_0$ and $j_1$ such that $K^{ij} = 0$ for $i < i_0$ or $j > j_1$ (resp. if there exist $j_0$ and $j_1$ such that $K^{ij} = 0$ for $j > j_0$ or $i > j_1$), then the two spectral sequences $E(K^{ij})$ and $E(K^{ij})$ are biregular.

(iii) If there exists $i_0$ such that $K^{ij} = 0$ for $i > i_0$ (resp. if there exists $j_0$ such that $K^{ij} = 0$ for $j < j_0$), then the spectral sequence $E(K^{ij})$ is regular.

(iv) If there exists $i_0$ such that $K^{ij} = 0$ for $i < i_0$ (resp. if there exists $j_0$ such that $K^{ij} = 0$ for $j > j_0$), then the spectral sequence $E(K^{ij})$ is regular.

**Proof.** The proposition follows immediately from the definitions (11.1.3) and from (11.2.4).
§12. SUPPLEMENT ON SHEAF COHOMOLOGY

12.1. Cohomology of sheaves of modules on ringed spaces

§13. PROJECTIVE LIMITS IN HOMOLOGICAL ALGEBRA

§14. COMBINATORIAL DIMENSION OF A TOPOLOGICAL SPACE

Summary

§14. Combinatorial dimension of a topological space.
§15. $M$-regular sequences and $\mathcal{F}$-regular sequences.
§16. Dimension and depth of Noetherian local rings.
§17. Regular rings.
§20. Derivations and differentials.
§22. Differential criteria for smoothness and regularity.

Almost all of the preceding sections have been focused on the exposition of ideas of commutative algebra that will be used throughout Chapter IV. Even though a large amount of these ideas already appear in multiple works ([CC], [Sam53a], [SZ60], [Ser55b], [Nag62]), we thought that it would be more practical for the reader to have a coherent, vaguely independent exposition. Together with §§5, 6, and 7 of Chapter IV (where we use the language of schemes), these sections constitute, in the middle of our treatise, a miniature special treatise, somewhat independent of Chapters I to III, and one that aims to present, in a coherent manner, the properties of rings that “behave well” relative to operations such as completion, or integral closure, by systematically associating these properties to more general ideas.  

14.1. Combinatorial dimension of a topological space

(14.1.1). Let $I$ be an ordered set; a chain of elements of $I$ is, by definition, a strictly-increasing finite sequence $i_0 < i_1 < \cdots < i_n$ of elements of $I$ ($n \geq 0$); by definition, the length of this chain is $n$. If $X$ is a topological space, the set of its irreducible closed subsets is ordered by inclusion, and so we have the notion of a chain of irreducible closed subsets of $X$.

**Definition (14.1.2).** — Let $X$ be a topological space. We define the combinatorial dimension of $X$ (or simply the dimension of $X$), denoted by $\dimc(X)$, to be the upper bound of lengths of chains of irreducible closed subsets of $X$. For all $x \in X$, we define the combinatorial dimension of $X$ at $x$ (or simply the dimension of $X$ at $x$), denoted by $\dimc_x(X)$, to be the number $\inf_U (\dimc(U))$, where $U$ varies over the open neighbourhoods of $x$ in $X$.

It follows from this definition that we have

$$\dimc(\emptyset) = -\infty$$

(the upper bound in $\overline{R}$ of the empty set being $-\infty$). If $(X_a)$ is the family of irreducible components of $X$, then we have

$$\dimc(X) = \sup_a \dimc(X_a),$$

(14.1.2.1)

because every chain of irreducible closed subsets of $X$ is, by definition, contained in some irreducible component of $X$, and, conversely, the irreducible components are closed in $X$, so every irreducible closed subset of an $X_a$ is a irreducible closed subset of $X$.

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11The majority of properties which we discuss were discovered by Chevalley, Zariski, Nagata, and Serre. The method used here was first developed in the autumn of 1961, in a course taught at Harvard University by A. Grothendieck.
**Definition (14.1.3).** — We say that a topological space $X$ is equidimensional if all its irreducible components have the same dimension (which is thus equal to $\dim(X)$, by (14.1.2.1)).

**Proposition (14.1.4).** —

(i) For every closed subset $Y$ of a topological space $X$, we have $\dim(Y) \leq \dim(X)$.

(ii) If a topological space $X$ is a finite union of closed subsets $X_i$, then we have $\dim(X) = \sup_i \dim(X_i)$.

**Proof.** For every irreducible closed subset $Z$ of $Y$, the closure $\overline{Z}$ of $Z$ in $X$ is irreducible ($0_1$, 2.1.2), and $\overline{Z} \cap Y = Z$, whence (i). Now, if $X = \bigcup_{i=1}^n X_i$, where the $X_i$ are closed, then every irreducible closed subset of $X$ is contained in one of the $X_i$ ($0_1$, 2.1.1), and so every chain of irreducible closed subsets of $X$ is contained in one of the $X_i$, whence (ii). □

From (14.1.4, i), we see that, for all $x \in X$, we can also write

\[
\dim_x(X) = \lim_U \dim(U),
\]

where the limit is taken over the downward-directed set of open neighbourhoods of $x$ in $X$.

**Corollary (14.1.5).** — Let $X$ be a topological space, $x$ a point of $X$, $U$ a neighbourhood of $x$, and $Y_i$ ($1 \leq i \leq n$) closed subsets of $U$ such that, for all $i$, $x \in Y_i$, and such that $U$ is the union of the $Y_i$. Then we have

\[
\dim_x(X) = \sup_i \dim_x(Y_i).
\]

**Proof.** It follows from (14.1.4, ii) that we have $\dim_x(X) = \inf_V (\sup_i (\dim(Y_i \cap V)))$, where $V$ ranges over the set of open neighbourhoods of $x$ that are contained in $U$; similarly, we have $\dim_x(Y_i) = \inf_V (\dim(Y_i \cap V))$ for all $i$. The corollary is thus evident if

\[
\sup_i \dim_x(Y_i) = +\infty;
\]

if this were not the case, then there would be an open neighbourhood $V_0 \subset U$ of $x$ such that $\dim(Y_i \cap V) = \dim_x(Y_i)$ for $1 \leq i \leq n$ and for all $V \subset V_0$, whence the conclusion. □

**Proposition (14.1.6).** — For every topological space $X$, we have $\dim(X) = \sup_{x \in X} \dim_x(X)$.

**Proof.** It follows from Definition (14.1.2) and Proposition (14.1.4) that $\dim_x(X) \leq \dim(X)$ for all $x \in X$. Now, let $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ be a chain of irreducible closed subsets of $X$, and let $x \in Z_0$; for every open subset $U \subset X$ that contains $x$, $U \cap Z_i$ is irreducible ($0_1$, 2.1.6) and closed in $U$, and since we have $U \cap Z_i = Z_i$ in $X$, the $U \cap Z_i$ are pairwise distinct; thus $\dim(U) \geq n$, which finishes the proof. □

**Corollary (14.1.7).** — If $(X_a)$ is an open, or closed and locally finite, cover of $X$, then $\dim(X) = \sup_a (\dim(X_a))$.

**Proof.** If $X_a$ is a neighbourhood of $x \in X$, then $\dim_x(X) \leq \dim(X_a)$, whence the claim for open covers. On the other hand, if the $X_a$ are closed, and $U$ is a neighbourhood of $x \in X$ which meets only finitely many of the $X_a$, then

\[
\dim_x(X) \leq \dim(U) = \sup_a (\dim(U \cap X_a)) \leq \sup_a (\dim(X_a))
\]

by (14.1.4), whence the other claim. □

**Corollary (14.1.8).** — Let $X$ be a Noetherian Kolmogoroff space ($0_1$, 2.1.3), and $F$ the set of closed points of $X$. Then $\dim(X) = \sup_{x \in F} \dim_x(X)$.

**Proof.** With the notation from the proof of (14.1.6), it suffices to note that there exists a closed point in $Z_0$ ($0_1$, 2.1.3). □

**Proposition (14.1.9).** — Let $X$ be a nonempty Noetherian Kolmogoroff space. To have $\dim(X) = 0$, it is necessary and sufficient for $X$ to be finite and discrete.
Proof. If a space $X$ is separated (and a fortiori if $X$ is a discrete space), then all the irreducible closed subsets of $X$ are single points, and so $\dim(X) = 0$. Conversely, suppose that $X$ is a Noetherian Kolmogoroff space such that $\dim(X) = 0$; since every irreducible component of $X$ contains a closed point $(0, 2.1.3)$, it must be exactly this single point. Since $X$ has only a finite number of irreducible components, it is thus finite and discrete. □

Corollary (14.1.10). — Let $X$ be a Noetherian Kolmogoroff space. For a point $x \in X$ to be isolated, it is necessary and sufficient to have $\dim_x(X) = 0$.

Proof. The condition is clearly necessary (without any hypotheses on $X$). It is also sufficient, because it implies that $\dim(U) = 0$ for any open neighbourhood $U$ of $x$, and since $U$ is a Noetherian Kolmogoroff space, $U$ is finite and discrete. □

Proposition (14.1.11). — The function $x \mapsto \dim_x(X)$ is upper semi-continuous on $X$.

Proof. It is clear that this function is upper semi-continuous at every point where its value is $+\infty$. So suppose that $\dim_x(X) = n < +\infty$; then Equation (14.1.4.1) shows that there exists an open neighbourhood $U_0$ of $x$ such that $\dim(U) = n$ for every open neighbourhood $U \subset U_0$ of $x$. So, for all $y \in U_0$ and every open neighbourhood $V \subset U_0$ of $y$, we have $\dim(V) \leq \dim(U_0) = n$ (14.1.4); we thus deduce from (14.1.4.1) that $\dim_y(X) \leq n$. □

Remark (14.1.12). — If $X$ and $Y$ are topological spaces, and $f : X \to Y$ a continuous map, then it can be the case that $\dim(f(X)) > \dim(X)$; we obtain such an example by taking $X$ to be a discrete space with 2 points, $a$ and $b$, and $Y$ to be the set $\{a, b\}$ endowed with the topology for which the closed sets are $\emptyset$, $\{a\}$, and $\{a, b\}$; if $f : X \to Y$ is the identity map, then $\dim(Y) = 1$ and $\dim(X) = 0$. We note that $Y$ is the spectrum of a discrete valuation ring $A$, of which $a$ is the unique closed point, and $b$ the generic point; if $K$ and $k$ are the field of fractions and the residue field of $A$ (respectively), then $X$ is the spectrum of the ring $k \times K$, and $f$ is the continuous map corresponding to the homomorphism $(\phi, \psi) : A \to k \times K$, where $\phi : A \to k$ and $\psi : A \to K$ are the canonical homomorphism (cf. (IV, 5.4.3)).

14.2. Codimension of a closed subset

Definition (14.2.1). — Given an irreducible closed subset $Y$ of a topological space $X$, we define the combinatorial codimension (or simply codimension) of $Y$ in $X$, denoted by $\text{codim}(Y, X)$, to be the upper bound of the lengths of chains of irreducible closed subsets of $X$ of which $Y$ is the smallest element. If $Y$ is an arbitrary closed subset of $X$, then we define the codimension of $Y$ in $X$, again denoted by $\text{codim}(Y, X)$, to be the lower bound of the codimensions in $X$ of the irreducible components of $Y$. We say that $X$ is equicodimensional if all the minimal irreducible closed subsets of $X$ has the same codimension in $X$.

It follows from this definition that $\text{codim}(\emptyset, X) = +\infty$, since the lower bound of the empty set of $\mathbb{R}$ is $+\infty$. If $Y$ is closed in $X$, and if $(X_\alpha)$ (resp. $(Y_\alpha)$) is the family of irreducible components of $X$ (resp. $Y$), then every $Y_\beta$ is contained in some $X_\alpha$, and, more generally, every chain of irreducible closed subsets of $X$ of which $Y_\beta$ is the smallest element is formed of subsets of some $X_\alpha$; we thus have

$$\text{(14.2.1.1)} \quad \text{codim}(Y, X) = \inf(\sup_{\beta, \alpha}(\text{codim}(Y_\beta, X_\alpha))),$$

where, for every $\beta$, $\alpha$ ranges over the set of indices such that $Y_\beta \subset X_\alpha$.

Proposition (14.2.2). — Let $X$ be a topological space.

(i) If $\Phi$ is the set of irreducible closed subsets of $X$, then

$$\text{(14.2.2.1)} \quad \dim(X) = \sup_{Y \in \Phi}(\text{codim}(Y, X)).$$

(ii) For every nonempty closed subset $Y$ of $X$, we have

$$\text{(14.2.2.2)} \quad \dim(Y) + \text{codim}(Y, X) \leq \dim(X).$$

\[\text{Trans.} \] This is now often referred to as the Sierpiński space, or the connected two-point set.
(iii) If \( Y, Z, \) and \( T \) are closed subsets of \( X \) such that \( Y \subset Z \subset T \), then
\[
\text{codim}(Y, Z) + \text{codim}(Z, T) \leq \text{codim}(Y, T).
\]

(iv) For a closed subset \( Y \) of \( X \) to be such that \( \text{codim}(Y, X) = 0 \), it is necessary and sufficient for \( Y \) to contain an irreducible component of \( X \).

**Proof.** Claims (i) and (iv) are immediate consequences of Definition (14.2.1). To show (ii), we can restrict to the case where \( Y \) is irreducible, and then the equation follows from Definitions (14.1.1) and (14.2.1). Finally, to show (iii), we can, by Definition (14.2.1), first restrict to the case where \( Y \) is irreducible; then \( \text{codim}(Y, Z) = \sup_{\alpha} (\text{codim}(Y, Z_{\alpha})) \) for the irreducible components \( Z_{\alpha} \) of \( Z \) that contain \( Y \); it is clear that \( \text{codim}(Y, T) \geq \text{codim}(Y, Z) \), so the inequality is true if \( \text{codim}(Y, Z) = +\infty \); but if this were not the case, then there would exist some \( \alpha \) such that \( \text{codim}(Y, Z_{\alpha}) = \text{codim}(Y) \), and by (14.2.1), we can restrict to the case where \( Z \) itself is irreducible; but then the inequality in (14.2.3) is an evident consequence of Definition (14.2.1). \( \square \)

**Proposition (14.2.3).** Let \( X \) be a topological space, and \( Y \) a closed subset of \( X \). For every open subset \( U \) of \( X \), we have
\[
\text{codim}(Y \cap U, U) \geq \text{codim}(Y, X).
\]
Furthermore, for this inequality (14.2.3.1) to be an equality, it is necessary and sufficient to have \( \text{codim}(Y, X) = \inf_{i} (\text{codim}(Y_{i}, X)) \), where \( (Y_{i}) \) is the family of irreducible components of \( Y \) that meet \( U \).

**Proof.** We know (0, 2.1.6) that \( Z \mapsto \text{codim}(Z) \) is a bijection from the set of irreducible closed subsets of \( U \) to the set of irreducible closed subsets of \( X \) that meet \( U \), and, in particular, induces a correspondence between the irreducible components of \( Y \cap U \) and the irreducible components of \( Y \cap T \); if \( Y_{i} \) is one of the latter such components, then we have \( \text{codim}(Y_{i}, X) = \text{codim}(Y_{i} \cap U, U) \), and the proposition then follows from Definition (14.2.1). \( \square \)

**Definition (14.2.4).** Let \( X \) be a topological space, \( Y \) a closed subset of \( X \), and \( x \) a point of \( X \). We define the codimension of \( Y \) in \( X \) at the point \( x \), denoted by \( \text{codim}_{x}(Y, X) \), to be the number \( \sup_{U} (\text{codim}(Y \cap U, U)) \), where \( U \) ranges over the set of open neighbourhoods of \( x \) in \( X \).

By (14.2.3), we can also write
\[
\text{codim}_{x}(Y, X) = \lim_{U} (\text{codim}(Y \cap U, U)),
\]
where the limit is taken over the downward-directed set of open neighbourhoods of \( x \) in \( X \). We note that we have
\[
\text{codim}_{x}(Y, X) = +\infty \text{ if } x \in X - Y.
\]

**Proposition (14.2.5).** If \( (Y_{i})_{1 \leq i \leq n} \) is a finite family of closed subsets of a topological space \( X \), and \( Y \) is the union of this family, then
\[
\text{codim}(Y, X) = \inf_{i} (\text{codim}(Y_{i}, X)).
\]

**Proof.** Every irreducible component of one of the \( Y_{i} \) is contained in an irreducible component of \( Y \), and, conversely, every irreducible component of \( Y \) is also an irreducible component of one of the \( Y_{i} \) (0, 2.1.1); the conclusion then follows from Definition (14.2.1) and the inequality in (14.2.3). \( \square \)

**Corollary.** Let \( X \) be a topological space, and \( Y \) a locally-Noetherian closed subspace of \( X \).

(i) For all \( x \in X \), there exists only a finite number of irreducible components \( Y_{i} (1 \leq i \leq n) \) of \( Y \) that contain \( x \), and we have \( \text{codim}_{x}(Y_{i}, X) = \inf_{i} (\text{codim}(Y_{i}, X)) \).

(ii) The function \( x \mapsto \text{codim}_{x}(Y, X) \) is lower semi-continuous on \( X \).

**Proof.** By hypothesis, there exists an open neighbourhood \( U_{0} \) of \( x \) in \( X \) such that \( Y \cap U_{0} \) is Noetherian, and so \( U_{0} \) has only a finite number of irreducible components, which are the intersections of \( U_{0} \) with the irreducible components of \( Y \); \textit{a fortiori} there are only a finite number of irreducible components \( Y_{i} (1 \leq i \leq n) \) of \( Y \) that contain \( x \), and we can, by replacing \( U_{0} \) with an open neighbourhood \( U \subset U_{0} \) of \( x \) that doesn’t meet any of the \( Y_{i} \) that do not contain \( x \), assume that the \( Y_{i} \cap U \) are the irreducible components of \( Y \cap U \); for every open neighbourhood \( V \subset U \)
of $x$ in $X$, the $Y_i \cap V$ are thus the irreducible components of $Y \cap V$, and (14.2.3) then shows that \(\text{codim}(Y_i, X) = \text{codim}(Y_i \cap V, V)\), which proves (i). Further, for every $x' \in U$, the irreducible components of $Y$ that contain $x'$ are certain $Y_i$, and so $\text{codim}_{x'}(Y, X) \geq \text{codim}_x(Y, X)$, which proves (ii).

14.3. The chain condition

(14.3.1). In a topological space $X$, we say that a chain $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ of irreducible closed subsets if saturated if there does not exist an irreducible closed subset $Z'$, distinct from each of the $Z_i$, such that $Z_k \subset Z' \subset Z_{k+1}$ for any $k$.

Proposition (14.3.2). — Let $X$ be a topological space such that, for any two irreducible closed subsets $Y$ and $Z$ of $X$ with $Y \subset Z$, we have $\text{codim}(Y, Z) < +\infty$. The following two conditions are equivalent.

(a) Any two saturated chains of closed irreducible subsets of $X$ that have the same first and last elements as one another have the same length.

(b) If $Y$, $Z$, and $T$ are irreducible closed subsets of $X$ such that $Y \subset Z \subset T$, then

\[
\text{codim}(Y, T) = \text{codim}(Y, Z) + \text{codim}(Z, T).
\]

Proof. It is immediate that (a) implies (b). Conversely, suppose that (b) is satisfied, and we will show that if we have two saturated chains with the same first and last elements as one another, of lengths $m$ and $n \leq m$ (respectively), then $m = n$. We proceed by induction on $n$, with the proposition being clear for $n = 1$. So suppose that $1 < n < m$, and let $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ be a saturated chain such that there exists another saturated chain, with first element $Z_0$ and last element $Z_n$, of length $m$. Since $\text{codim}(Z_0, Z_n) \geq m > n$, and $\text{codim}(Z_0, Z_1) = 1$, it follows from (b) that $\text{codim}(Z_1, Z_n) = \text{codim}(Z_0, Z_n) - 1 > n - 1$, which contradicts our induction hypothesis.

When the conditions of (14.3.2) are satisfied, we say that $X$ satisfies the chain condition, or that it is a catenary space. It is clear that every closed subspace of a catenary space is catenary.

Proposition (14.3.3). — Let $X$ be a Noetherian Kolmogoroff space of finite dimension. The following conditions are equivalent.

(a) Any two maximal chains of irreducible closed subsets of $X$ have the same length.

(b) $X$ is equidimensional, equicodimensional, and catenary.

(c) $X$ is equidimensional, and, for any irreducible closed subsets $Y$ and $Z$ of $X$ with $Y \subset Z$, we have

\[
\dim(Z) = \dim(Y) + \text{codim}(Y, Z).
\]

(d) $X$ is equicodimensional, and, for any irreducible closed subsets $Y$ and $Z$ of $X$ with $Y \subset Z$, we have

\[
\text{codim}(Y, Z) = \text{codim}(Y, Z) + \text{codim}(Z, X).
\]

Proof. The hypotheses on $X$ imply that the first and last elements of a maximal chain of irreducible closed subsets of $X$ are necessarily a closed point and an irreducible component of $X$ (respectively) (0.1; 2.1.3); further, every saturated chain with first element $Y$ and last element $Z$ (thus $Y \subset Z$) is contained in a maximal chain whose elements differ from those of the given chain, or are contained in $Y$, or contain $Z$. These remarks immediately establish the equivalence between (a) and (b), and also show that if (a) is satisfied, then we have, for every irreducible closed subset $Y$ to $X$, $\dim(Y) + \text{codim}(Y, X) = \dim(X)$; from (14.3.2.1), we immediately deduce (14.3.3.1) and (14.3.3.2) from (14.3.3.3). Conversely, (14.3.3.1) implies (14.3.2.1), and so (14.3.3.1) implies the chain condition, by (14.3.2.2); further, by applying (14.3.3.1) to the case where $Y$ is a single closed point $x$ of $X$, and $Z$ is an irreducible component of $X$, we get that $\text{codim}(\{x\}, X) = \dim(Z)$; we thus conclude that (c) implies (b). Similarly, (14.3.3.2) implies (14.3.2.1), and thus the chain condition; further, with the same choice of $Y$ and $Z$ as above, (14.3.3.2) again implies that $\text{codim}(\{x\}, X) = \dim(Z)$, and so (since every irreducible component of $X$ contains a closed point, by (0.1; 2.1.3)), (d) implies (b).

We say that a Noetherian Kolmogoroff space is biequidimensional if it is of finite dimension and if it verifies any of the (equivalent) conditions of (14.3.3).
Corollary (14.3.4). — Let $X$ be a biequidimensional Noetherian Kolmogoroff space; then, for every closed point $x$ of $X$, and every irreducible component $Z$ of $X$, we have

\[(14.3.4.1) \dim(X) = \dim(Z) = \text{codim}(\{x\}, X) = \dim_x(X).\]

**Proof.** The last equality follows from the fact that, if $Y_0 = \{x\} \subset Y_1 \subset \cdots \subset Y_m$ is a maximal chain of irreducible closed subsets of $X$, and $U$ an open neighbourhood of $x$, then the $U \cap Y_i$ are pairwise disjoint irreducible closed subsets of $U$ (because $U \cap Y_i = Y_i$), whence $\dim(U) = \dim(X)$, by (14.1.4).

Corollary (14.3.5). — Let $X$ be a Noetherian Kolmogoroff space; if $X$ is biequidimensional, then so is every union of irreducible components of $X$, and every irreducible closed subset of $X$. In addition, for every closed subset $Y$ of $X$, we have

\[(14.3.5.1) \dim(Y) + \text{codim}(Y, X) = \dim(X).\]

**Proof.** Every chain of irreducible closed subsets of $X$ is contained in an irreducible component of $X$, and so the first claim follows immediately from (14.3.3). Further, if $X'$ is an irreducible closed subset of $X$, then $X'$ trivially satisfies the conditions of (14.3.3, c), whence the second claim.

Finally, to show (14.3.5.1), note that we have seen, in the proof of (14.3.3), that this equation is true whenever $Y$ is irreducible; if $Y_i (1 \leq i \leq m)$ are the irreducible components of $Y$, then the $Y_i$ for which $\dim(Y_i)$ is the largest are also those for which $\text{codim}(Y_i, X)$ is the smallest; so (14.3.5.1) follows from the definitions of $\dim(Y)$ and $\text{codim}(Y, X)$.

**Remark (14.3.6).** — The reader will note that the proof of (14.3.2) applies to any ordered set, and the fact that we are working with the example of a set of irreducible closed subsets of a topological space is not used at all in the proof. It is the same in the proof of (14.3.3), which holds, more generally, for any ordered set $E$ such that, for all $x \in E$, there exists some $z \leq x$ which is minimal in $E$, and such that the length of chains of elements of $E$ is bounded.
CHAPTER I

The language of schemes (EGA I)

SUMMARY

§1. Affine schemes.
§2. Preschemes and morphisms of preschemes.
§3. Products of preschemes.
§4. Subpreschemes and immersion morphisms.
§5. Reduced preschemes; the separation condition.
§6. Finiteness conditions.
§7. Rational maps.
§8. Chevalley schemes.
§10. Formal schemes.

In §§1–8 we do little more than develop a language to be used in what follows. It should be noted, however, that, in accordance with the general spirit of this treatise, §§7–8 will be used less than the others, and in a less essential way; we speak of Chevalley schemes only for the purpose of relating to the language of Chevalley [CC] and Nagata [Nag58a]. Then, in §9, we give definitions and results concerning quasi-coherent sheaves, some of which are no longer simply a translation of known notions of commutative algebra into a “geometric” language, but are instead already of a global nature; they will be indispensable, in the following chapters, when it comes to the global study of morphisms. Finally, in §10, we introduce a generalization of the notion of a scheme, which will be used as an intermediary in Chapter III to conveniently formulate and prove the fundamental results of the cohomological study of proper morphisms; moreover, it should be noted that the notion of formal schemes seems indispensable in expressing certain facts about the “theory of modules” (classification problems of algebraic varieties). The results of §10 will not be used before §3 of Chapter III, and it is recommended to skip their reading until then.

§1. AFFINE SCHEMES

1.1. The prime spectrum of a ring

(1.1.1). Notation. Let $A$ be a (commutative) ring, and $M$ an $A$-module. In this chapter and the following, we will constantly use the following notation:

- $\text{Spec}(A) =$ set of prime ideals of $A$, also called the prime spectrum of $A$; for $x \in X = \text{Spec}(A)$, it will often be convenient to write $1_x$ instead of $x$. For $\text{Spec}(A)$ to be empty, it is necessary and sufficient for the ring $A$ to be 0.
- $A_x = A_{1_x} =$ (local) ring of fractions $S^{-1}A$, where $S = A - 1_x$.
- $m_x = 1_xA_{1_x} =$ maximal ideal of $A_x$.
- $k(x) = A_x/m_x =$ residue field of $A_x$, canonically isomorphic to the field of fractions of the integral ring $A/1_x$, with which we identify it.
- $f(x) =$ class of $f$ mod. $1_x$ in $A/1_x \subset k(x)$, for $f \in A$ and $x \in X$. We also say that $f(x)$ is the value of $f$ at a point $x \in \text{Spec}(A)$; the equations $f(x) = 0$ and $f \in 1_x$ are equivalent.
- $M_x = M \otimes_A A_x =$ module of fractions with denominators in $A - 1_x$.
- $\mathfrak{r}(E) =$ radical of the ideal of $A$ generated by a subset $E$ of $A$. 

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\* \* V(E) = set of x \in X such that E \subset i_x (or the set of x \in X such that f(x) = 0 for all f \in E), for E \subset A. So we have \\
(1.1.1.1) \quad \tau(E) = \bigcap_{x \in V(E)} i_x. \\
\* \* V(f) = V(\{f\}) for f \in A. \\
\* \* D(f) = X - V(f) = set of x \in X where f(x) \neq 0.

**Proposition (1.1.2).** — We have the following properties:

(i) \( V(0) = X, V(1) = \emptyset. \)

(ii) The relation \( E \subset E' \) implies \( V(E) \supset V(E') \).

(iii) For each family \((E_j)\) of subsets of \( A \), \( V(\bigcup \lambda E_\lambda) = V(\sum \lambda E_\lambda) = \bigcap \lambda V(E_\lambda). \)

(iv) \( V(EE') = V(E) \cup V(E'). \)

(v) \( V(E) = V(\tau(E)). \)

**Proof.** The properties (i), (ii), (iii) are trivial, and (v) follows from (ii) and from equation (1.1.1.1). It is evident that \( V(EE') \supset V(E) \cap V(E'); \) conversely, if \( x \notin V(E) \) and \( x \notin V(E') \), then there exists \( f \in E \) and \( f' \in E' \) such that \( f(x) \neq 0 \) and \( f'(x) \neq 0 \) in \( k(x) \), hence \( f(x)f'(x) \neq 0 \), i.e., \( x \notin V(EE') \), which proves (iv). □

Proposition (1.1.2) shows, among other things, that sets of the form \( V(E) \) (where \( E \) varies over the subsets of \( A \)) are the closed sets of a topology on \( X \), which we will call the spectral topology; unless expressly stated otherwise, we always assume that \( X = \text{Spec}(A) \) is equipped with the spectral topology.

**Proposition (1.1.4).** —

(i) For each subset \( E \) of \( A \), we have \( j(V(E)) = \tau(E). \)

(ii) For each subset \( Y \) of \( X \), \( V(j(Y)) = \overline{Y}, \) the closure of \( Y \) in \( X. \)

**Proof.** (i) is an immediate consequence of the definitions and (1.1.1.1); on the other hand, \( V(j(Y)) \) is closed and contains \( Y \); conversely, if \( Y \subset V(E) \), we have \( f(y) = 0 \) for all \( y \in Y \), so \( E \subset j(Y) \), \( V(E) \supset V(j(Y)) \), which proves (ii). □

**Corollary (1.1.5).** — The closed subsets of \( X = \text{Spec}(A) \) and the ideals of \( A \) equal to their radicals (in other words, those that are the intersection of prime ideals) correspond bijectively by the inclusion-reversing maps \( Y \mapsto j(Y), a \mapsto V(a); \) the union \( Y_1 \cup Y_2 \) of two closed subsets corresponds to \( j(Y_1) \cap j(Y_2) \), and the intersection of any family \( \{Y_\lambda\} \) of closed subsets corresponds to the radical of the sum of the \( j(Y_\lambda) \).

**Corollary (1.1.6).** — If \( A \) is a Noetherian ring, \( X = \text{Spec}(A) \) is a Noetherian space.

Note that the converse of this corollary is false, as shown by any non-Noetherian integral ring with a single prime ideal \( \neq \{0\} \) (for example a nondiscrete valuation ring of rank 1).

As an example of ring \( A \) whose spectrum is not a Noetherian space, one can consider the ring \( C(Y) \) of continuous real functions on an infinite compact space \( Y \); we know that, as a set, \( Y \) corresponds to the set of maximal ideals of \( A \), and it is easy to see that the topology induced on \( Y \) by that of \( X = \text{Spec}(A) \) is the original topology of \( Y. \) Since \( Y \) is not a Noetherian space, the same is true for \( X. \)

\footnote{The introduction of this topology in algebraic geometry is due to Zariski. So this topology is usually called the “Zariski topology” on \( X. \).}
Corollary (1.1.7). — For each $x \in X$, the closure of $\{x\}$ is the set of $y \in X$ such that $\overline{j}_x \supset j_y$. For $\{x\}$ to be closed, it is necessary and sufficient that $\overline{j}_x$ is maximal.

Corollary (1.1.8). — The space $X = \text{Spec}(A)$ is a Kolmogoroff space.

Proof. If $x$ and $y$ are two distinct points of $X$, we have either $\overline{j}_x \not\supset j_y$ or $\overline{j}_y \not\supset j_x$, so one of the points $x, y$ does not belong to the closure of the other. \hfill \Box

(1.1.9). According to Proposition (1.1.12, iv), for two elements $f, g$ of $A$, we have

\[ D(fg) = D(f) \cap D(g). \tag{1.1.9.1} \]

Note also that the equality $D(f) = D(g)$ means, according to Proposition (1.1.4, i) and Proposition (1.1.2, v), that $\tau(f) = \tau(g)$, or that the minimal prime ideals containing $(f)$ and $(g)$ are the same; in particular, it is also the case when $f = ug$, where $u$ is invertible.

Proposition (1.1.10). —

(i) When $f$ ranges over $A$, the sets $D(f)$ forms a basis for the topology of $X$.

(ii) For every $f \in A$, $D(f)$ is quasi-compact. In particular, $X = D(1)$ is quasi-compact.

Proof.

\begin{enumerate}
  \item Let $U$ be an open set in $X$; by definition, we have $U = X - V(E)$ where $E$ is a subset of $A$, and $V(E) = \bigcap_{f \in E} V(f)$, hence $U = \bigcup_{f \in E} D(f)$.
  \item By (i), it suffices to prove that, if $(f_\lambda)_{\lambda \in L}$ is a family of elements of $A$ such that $D(f) \subset \bigcup_{\lambda \in L} D(f_\lambda)$, then there exists a finite subset $J$ of $L$ such that $D(f) \subset \bigcup_{\lambda \in J} D(f_\lambda)$. Let $a$ be the ideal of $A$ generated by the $f_\lambda$; we have, by hypothesis, that $V(f) \supset V(a)$, so $\tau(f) \subset \tau(a)$; since $f \in \tau(f)$, there exists an integer $n \geq 0$ such that $f^n \in a$. But then $f^n$ belongs to the ideal $b$ generated by the finite subfamily $(f_\lambda)_{\lambda \in J}$, and we have $V(f) = V(f^n) \supset V(b) = \bigcap_{\lambda \in J} V(f_\lambda)$, that is to say, $D(f) \supset \bigcup_{\lambda \in J} D(f_\lambda)$.
\end{enumerate}

\Box

Proposition (1.1.11). — For each ideal $a$ of $A$, $\text{Spec}(A/a)$ is canonically identified with the closed subspace $V(a)$ of $\text{Spec}(A)$.

Proof. We know there is a canonical bijective correspondence (respecting the inclusion order structure) between ideals (resp. prime ideals) of $A/a$ and ideals (resp. prime ideals) of $A$ containing $a$.

Recall that the set $\mathfrak{N}$ of nilpotent elements of $A$ (the nilradical of $A$) is an ideal equal to $\tau(0)$, the intersection of all the prime ideals of $A$ (0, 1.1.1).

Corollary (1.1.12). — The topological spaces $\text{Spec}(A)$ and $\text{Spec}(A/\mathfrak{N})$ are canonically homeomorphic.

Proposition (1.1.13). — For $X = \text{Spec}(A)$ to be irreducible (0, 2.1.1), it is necessary and sufficient that the ring $A/\mathfrak{N}$ is integral (or, equivalently, that the ideal $\mathfrak{N}$ is prime).

Proof. By virtue of Corollary (1.1.12), we can restrict to the case where $\mathfrak{N} = 0$. If $X$ is reducible, then there exist two distinct closed subsets $Y_1$ and $Y_2$ of $X$ such that $X = Y_1 \cup Y_2$, so $i(X) = i(Y_1) \cap i(Y_2) = 0$, since the ideals $i(Y_1)$ and $i(Y_2)$ are distinct from (0) (1.1.5); so $A$ is not integral. Conversely, if there are elements $f \neq 0, g \neq 0$ of $A$ such that $fg = 0$, we have $V(f) \neq X$, $V(g) \neq X$ (since the intersection of all the prime ideals of $A$ is (0)), and $X = V(fg) = V(f) \cup V(g)$. \hfill \Box

Corollary (1.1.14). —

(i) In the bijective correspondence between closed subsets of $X = \text{Spec}(A)$ and ideals of $A$ equal to their radicals, the irreducible closed subsets of $X$ correspond to the prime ideals of $A$. In particular, the irreducible components of $X$ correspond to the minimal prime ideals of $A$.

(ii) The map $x \mapsto \{x\}$ establishes a bijective correspondence between $X$ and the set of closed irreducible subsets of $X$ (in other words, all closed irreducible subsets of $X$ admit exactly one generic point).
Proof. (i) follows immediately from (1.1.13) and (1.1.11); and for proving (ii), we can, by (1.1.11), restrict to the case where \( X \) is irreducible; then, according to Proposition (1.1.13), there exists a smaller prime ideal \( \mathfrak{m} \) in \( A \), which corresponds to the generic point of \( X \); in addition, \( X \) admits at most one generic point since it is a Kolmogoroff space ((1.1.8) and (0, 2.1.3)).

Proposition (1.1.15). — If \( \mathfrak{a} \) is an ideal in \( A \) containing the radical \( \mathfrak{r}(A) \), the only neighborhood of \( V(\mathfrak{a}) \) in \( X = \text{Spec}(A) \) is the whole space \( X \).

Proof. Each maximal ideal of \( A \) belongs, by definition, to \( V(\mathfrak{a}) \). As each ideal \( \mathfrak{a} \neq A \) of \( A \) is contained in a maximal ideal, we have \( V(\mathfrak{a}) \cap V(\mathfrak{a}) \neq 0 \), whence the proposition.

1.2. Functorial properties of prime spectra of rings

(1.2.1). Let \( A, A' \) be two rings, and

\[ \phi : A' \rightarrow A \]

a homomorphism of rings. For each prime ideal \( x = \mathfrak{j}_x \in \text{Spec}(A) = X \), the ring \( A'/\phi^{-1}(\mathfrak{j}_x) \) is canonically isomorphic to a subring of \( A/\mathfrak{j}_x \), and so it is integral, or, in other words, \( \phi^{-1}(\mathfrak{j}_x) \) is a prime ideal of \( A' \); we denote it by \( \phi(x) \), and we have thus defined a map

\[ \phi^x : X = \text{Spec}(A) \rightarrow X' = \text{Spec}(A') \]

(also denoted \( \text{Spec}(\phi) \)), that we call the map associated to the homomorphism \( \phi \). We denote by \( \phi^x \) the injective homomorphism from \( A'/\phi^{-1}(\mathfrak{j}_x) \) to \( A/\mathfrak{j}_x \) induced by \( \phi \) by passing to quotients, as well as its canonical extension to a monomorphism of fields

\[ \phi^x : k(\phi(x)) \rightarrow k(x) \]

for each \( f' \in A' \), we therefore have, by definition,

(1.2.1.1) \[ \phi^x(f'(\phi(x))) = (\phi(f'))(x) \quad (x \in X). \]

Proposition (1.2.2). —

(i) For each subset \( E' \) of \( A' \), we have

(1.2.2.1) \[ \phi^{-1}(V(E')) = V(\phi(E')), \]

and in particular, for each \( f' \in A' \),

(1.2.2.2) \[ \phi^{-1}(D(f')) = D(\phi(f')). \]

(ii) For each ideal \( \mathfrak{a} \) of \( A \), we have

(1.2.2.3) \[ \phi(V(\mathfrak{a})) = V(\phi^{-1}(\mathfrak{a})). \]

Proof. The relation \( \phi(x) \in V(E') \) is, by definition, equivalent to \( E' \subset \phi^{-1}(\mathfrak{j}_x) \), so \( \phi(E') \subset \mathfrak{j}_x \), and finally \( x \in V(\phi(E')) \), hence (i). To prove (ii), we can suppose that \( \mathfrak{a} \) is equal to its radical, since \( V(\mathfrak{r}(\mathfrak{a})) = V(\mathfrak{a}) \) (1.1.2, v) and \( \phi^{-1}(\mathfrak{r}(\mathfrak{a})) = \mathfrak{r}(\phi^{-1}(\mathfrak{a})) \); the relation \( f' \in \mathfrak{a}' \) is, by definition, equivalent to \( f'(x') = 0 \) for each \( x \in \phi(Y) \), so, by Equation (1.2.1.1), it is also equivalent to \( \phi(f')(x) = 0 \) for each \( x \in Y \), or to \( \phi(f') \in j(Y) = \mathfrak{a} \), since \( \mathfrak{a} \) is equal to its radical; hence (ii).

Corollary (1.2.3). — The map \( \phi^x \) is continuous.

We remark that, if \( A'' \) is a third ring, and \( \phi \) a homomorphism \( A'' \rightarrow A' \), then we have \( \phi(\phi' \circ \phi) = \phi \circ \phi^x \); this result, with Corollary (1.2.3), says that \( \text{Spec}(A) \) is a contravariant functor in \( A \), from the category of rings to that of topological spaces.

Corollary (1.2.4). — Suppose that \( \phi \) is such that every \( f \in A \) can be written as \( f = h(\phi(f')) \), where \( h \) is invertible in \( A \) (which, in particular, is the case when \( \phi \) is surjective). Then \( \phi^x \) is a homeomorphism from \( X \) to \( \phi(X) \).

Proof. We show that for each subset \( E \subset A \), there exists a subset \( E' \) of \( A' \) such that \( V(E) = V(\phi(E')) \); according to the \((T_0)\) axiom (1.1.8) and the formula (1.2.2.1), this implies first of all that \( \phi^x \) is injective, and then, by (1.2.2.1), that \( \phi^x \) is a homeomorphism. But it suffices, for each \( f \in E \), to take \( f' \in A' \) such that \( h(\phi(f')) = f \) with \( h \) invertible in \( A \); the set \( E' \) of these elements \( f' \) is exactly what we are searching for.
(1.2.5). In particular, when \( \phi \) is the canonical homomorphism from \( A \) to a ring quotient \( A/a \), we again get (1.1.12), and \( \phi^a \) is the canonical injection of \( V(a) \), identified with \( \text{Spec}(A/a) \), into \( X = \text{Spec}(A) \).

Another particular case of (1.2.4):

**Corollary (1.2.6).** — If \( S \) is a multiplicative subset of \( A \), the spectrum \( \text{Spec}(S^{-1}A) \) is canonically identified (with its topology) with the subspace of \( X = \text{Spec}(A) \) consisting of the \( x \) such that \( j_x \cap S = \emptyset \).

**Proof.** We know by (0, 1.2.6) that the prime ideals of \( S^{-1}A \) are the ideals \( S^{-1}j_x \) such that \( j_x \cap S = \emptyset \), and that we have \( j_x = (i^S_A)^{-1}(S^{-1}j_x) \). It then suffices to apply Corollary (1.2.4) to the \( i^S_A \).

**Corollary (1.2.7).** — For \( \phi(X) \) to be dense in \( X' \), it is necessary and sufficient for each element of the kernel \( \text{Ker} \phi \) to be nilpotent.

**Proof.** Applying Equation (1.2.2.3) to the ideal \( a = (0) \), we have \( \phi(a) = V(\text{Ker} \phi) \), and for \( V(\text{Ker} \phi) = X' \) to hold, it is necessary and sufficient for \( \text{Ker} \phi \) to be contained in all the prime ideals of \( A' \), or, equivalently, in the nilradical \( t' \) of \( A' \).

### 1.3. Sheaf associated to a module

**Lemma (1.3.2).** — The following conditions are equivalent:

(a) \( g \in S'_f \);
(b) \( S'_g \subset S'_f \);
(c) \( f \in \tau(g) \);
(d) \( \tau(f) \subset \tau(g) \);
(e) \( V(g) \subset V(f) \);
(f) \( D(f) \subset D(g) \).

**Proof.** This follows immediately from the definitions and (1.1.5).

**Lemma (1.3.3).** If \( D(f) = D(g) \), then Lemma (1.3.2, b) shows that \( M_f = M_g \). More generally, if \( D(f) \supset D(g) \), then \( S'_f \subset S'_g \), and we know (0, 1.4.1) that there exists a canonical functorial homomorphism

\[
\rho_{g,f} : M_f \rightarrow M_g,
\]

and if \( D(f) \supset D(g) \supset D(h) \), we have (0, 1.4.4)

\[
\rho_{h,g} \circ \rho_{g,f} = \rho_{h,f}.
\]

When \( f \) ranges over the elements of \( A - i_x \) (for a given \( x \) in \( X = \text{Spec}(A) \)), the sets \( S'_f \) constitute an increasing filtered set of subsets of \( A - i_x \), since for elements \( f \) and \( g \) of \( A - i_x \), \( S'_f \) and \( S'_g \) are contained in \( S'_{fg} \); since the union of the \( S'_f \) over \( f \in A - i_x \) is \( A - i_x \), we conclude (0, 1.4.5) that the \( A_x \)-module \( M_x \) is canonically identified with the inductive limit \( \lim_{\rightarrow} M_f \), relative to the family of homomorphisms \( (\rho_{g,f}) \). We denote by

\[
\rho^f_x : M_f \rightarrow M_x
\]

the canonical homomorphism for \( f \in A - i_x \) (or, equivalently, \( x \in D(f) \)).

**Definition (1.3.4).** — We define the structure sheaf of the prime spectrum \( X = \text{Spec}(A) \) (resp. the sheaf associated to the \( A \)-module \( M \)), denoted by \( \mathcal{A} \) or \( \mathcal{O}_X \) (resp. \( M \)) as the sheaf of rings (resp. the \( \mathcal{A} \)-module) associated to the presheaf \( D(f) \mapsto A_f \) (resp. \( D(f) \mapsto M_f \)), defined on the basis \( \mathcal{B} \) of \( X \) consisting of the \( D(f) \) for \( f \in A \) ((1.1.10), (0, 3.2.1), and (0, 3.5.6)).
We saw (0, 3.2.4) that the stalk $\tilde{A}_x$ (resp. $\tilde{M}_x$) can be identified with the ring $A_x$ (resp. the $A_x$-module $M_x$); we denote by

$$\theta_f : A_f \longrightarrow \Gamma(D(f), \tilde{A})$$

(resp. $\theta_f : M_f \longrightarrow \Gamma(D(f), \tilde{M})$),

the canonical map, so that, for all $x \in D(f)$ and all $\xi \in M_f$, we have

$$\theta_f(\xi)_x = \rho^f_x(\xi).$$

**Proposition (1.3.5).** — $\tilde{M}$ is an exact functor, covariant in $M$, from the category of $A$-modules to the category of $\tilde{A}$-modules.

**Proof.** Indeed, let $M, N$ be two $A$-modules, and $u$ a homomorphism $M \rightarrow N$; for each $f \in A$, $u$ corresponds canonically to a homomorphism $u_f$ from the $A_f$-module $M_f$ to the $A_f$-module $N_f$, and the diagram (for $D(g) \subset D(f)$)

$$\begin{array}{ccc}
M_f & \xrightarrow{u_f} & N_f \\
\downarrow{\rho_{g,f}} & & \downarrow{\rho_{g,f}} \\
M_g & \xrightarrow{u_g} & N_g
\end{array}$$

is commutative (1.4.1); these homomorphisms then define a homomorphism of $\tilde{A}$-modules $\tilde{u} : \tilde{M} \rightarrow \tilde{N}$ (0, 3.2.3). In addition, for each $x \in X$, $\tilde{u}_x$ is the inductive limit of the $u_f$ for $x \in D(f)$ ($f \in A$), and as a result (0, 1.4.5), if we can identify $\tilde{M}_x$ and $\tilde{N}_x$ with $M_x$ and $N_x$ respectively, then $\tilde{u}_x$ is identified with the homomorphism $u_x$ canonically induced by $u$. If $P$ is a third $A$-module, $v$ a homomorphism $N \rightarrow P$, and $w = v \circ u$, it is immediate that $w_x = v_x \circ u_x$, so $\tilde{w} = \tilde{v} \circ \tilde{u}$. We have therefore clearly defined a covariant (in $M$) functor $\tilde{M}$, from the category of $A$-modules to that of $\tilde{A}$-modules. This functor is exact, since, for each $x \in X$, $M_x$ is an exact functor in $M$ (0, 1.3.2); in addition, we have $\text{Supp}(M) = \text{Supp}(\tilde{M})$, by definition ((0, 1.7.1) and (0, 3.1.6)).

**Proposition (1.3.6).** — For each $f \in A$, the open subset $D(f) \subset X$ is canonically identified with the prime spectrum $\text{Spec}(A_f)$, and the sheaf $\tilde{M}_f$ associated to the $A_f$-module $M_f$ is canonically identified with the restriction $\tilde{M}|D(f)$.

**Proof.** The first assertion is a particular case of (1.2.6). In addition, if $g \in A$ is such that $D(g) \subset D(f)$, then $M_g$ is canonically identified with the module of fractions of $M_f$ whose denominators are the powers of the canonical image of $g$ in $A_f$ (0, 1.4.6). The canonical identification of $\tilde{M}_f$ with $\tilde{M}|D(f)$ then follows from the definitions.

**Theorem (1.3.7).** — For each $A$-module $M$ and each $f \in A$, the homomorphism

$$\theta_f : M_f \longrightarrow \Gamma(D(f), \tilde{M})$$

is bijective (in other words, the presheaf $D(f) \mapsto M_f$ is a sheaf). In particular, $M$ can be identified with $\Gamma(X, \tilde{M})$ via $\theta_1$.

**Proof.** We note that, if $M = A$, then $\theta_f$ is a homomorphism of structure rings; Theorem (1.3.7) then implies that, if we identify the rings $A_f$ and $\Gamma(D(f), \tilde{A})$ via $\theta_f$, the homomorphism $\theta_f : M_f \rightarrow \Gamma(D(f), \tilde{M})$ is an isomorphism of modules.

We show first that $\theta_f$ is injective. Indeed, if $\xi \in M_f$ is such that $\theta_f(\xi) = 0$, then, for each prime ideal $p$ of $A_f$, there exists $h \not\in p$ such that $h\xi = 0$; as the annihilator of $\xi$ is not contained in any prime ideal of $A_f$, each $A_f$ integral, and so $h = 0$.

It remains to show that $\theta_f$ is surjective; we can restrict to the case where $f = 1$ (the general case then following by “localizing”, using (1.3.6)). Now let $s$ be a section of $\tilde{M}$ over $X$; according to (1.3.4) and (1.1.10, ii), there exists a finite cover $(D(f_i))_{i \in I}$ of $X$ ($f_i \in A$) such that, for each $i \in I$, "$f_i$
the restriction \( s_i = s|D(f_i) \) is of the form \( \theta f_i(\xi_i) \), where \( \xi_i \in M_{f_i} \). If \( i, j \) are indices of \( I \), and if the restrictions of \( s_i \) and \( s_j \) to \( D(f_i) \cap D(f_j) = D(f_if_j) \) are equal, then it follows, by definition of \( M \), that
\[
\rho f_if_j f_if_j(\xi_i) = \rho f_if_j f_if_j(\xi_j).
\]
By definition, we can write, for each \( i \in I \), \( \xi_i = z_i/f_i^{n_i} \), where \( z_i \in M \), and. since \( I \) is finite, by multiplying each \( z_i \) by a power of \( f_i \), we can assume that all the \( n_i \) are equal to one single \( n \). Then, by definition, \((1.3.7.1)\) implies that there exists an integer \( m_{ij} \geq 0 \) such that \( (f_if_j)^{m_{ij}}(f_i^n z_i - f_i^n z_j) = 0 \), and we can moreover suppose that the \( m_{ij} \) are equal to the one single \( m \); then replacing the \( z_i \) by \( f_i^m z_i \), it remains to prove the case where \( m = 0 \), in other words, the case where we have
\[
f_i^m z_i = f_i^m z_j
\]
for any \( i, j \). We have \( D(f_i^n) = D(f_i) \), and since the \( D(f_i) \) form a cover of \( X \), the ideal generated by the \( f_i^n \) is \( A \); in other words, there exist elements \( g_i \in A \) such that \( \sum_i g_i f_i^n = 1 \). Then consider the element \( z = \sum_i g_i z_i \) of \( M \); in \((1.3.7.2)\), we have \( f_i^n z = \sum_i g_i f_i^m z_i = (\sum_i g_i f_i^m) z_i = z_i \), where, by definition, \( \theta_i = z_i^1/1 \in M_{f_i} \). We conclude that the \( s_i \) are the restrictions to \( D(f_i) \) of \( \theta_i(z) \), which proves that \( s = \theta_i(z) \) and finishes the proof.

Corollary (1.3.8). — Let \( M \) and \( N \) be \( A \)-modules; the canonical homomorphism \( u \mapsto \tilde{u} \) from \( \text{Hom}_A(M, N) \) to \( \text{Hom}_{\tilde{A}}(\tilde{M}, \tilde{N}) \) is bijective. In particular, the equations \( M = 0 \) and \( \tilde{M} = 0 \) are equivalent.

Proof. Consider the canonical homomorphism \( v \mapsto \Gamma(v) \) from \( \text{Hom}_{\tilde{A}}(\tilde{M}, \tilde{N}) \) to \( \text{Hom}_{\Gamma(\tilde{A})}(\Gamma(\tilde{M}), \Gamma(\tilde{N})) \); the latter module is canonically identified with \( \text{Hom}_{\tilde{A}}(M, N) \), by Theorem (1.3.7). It remains to show that \( u \mapsto \tilde{u} \) and \( v \mapsto \Gamma(v) \) are inverses of each other; it is evident that \( \Gamma(\tilde{u}) = u \) by definition of \( \tilde{u} \); on the other hand, if we let \( u = \Gamma(v) \) for \( v \in \text{Hom}_{\tilde{A}}(\tilde{M}, \tilde{N}) \), then the map \( w : \Gamma(D(f), \tilde{M}) \to \Gamma(D(f), \tilde{N}) \) canonically induced from \( v \) is such that the diagram
\[
\begin{array}{ccc}
M & \xrightarrow{u} & N \\
\downarrow{\rho_{f_1}} & & \downarrow{\rho_{f_1}} \\
M_f & \xrightarrow{w} & N_f
\end{array}
\]
is commutative; so we necessarily have that \( w = u_f \) for all \( f \in A(0, 1.2.4) \), which shows that \( \Gamma(v) = v \). □

Corollary (1.3.9). —

(i) Let \( u \) be a homomorphism from an \( A \)-module \( M \) to an \( A \)-module \( N \); then the sheaves associated to \( \ker u \), \( \text{im} u \), and \( \text{coker} u \) are \( \ker \tilde{u} \), \( \text{im} \tilde{u} \), and \( \text{coker} \tilde{u} \) (respectively). In particular, for \( \tilde{u} \) to be injective (resp. surjective, bijective), it is necessary and sufficient for \( u \) to be so too.

(ii) If \( M \) is an inductive limit (resp. direct sum) of a family of \( A \)-modules \( (M_\lambda) \), then \( \tilde{M} \) is the inductive limit (resp. direct sum) of the family \( (M_\lambda) \), via a canonical isomorphism.

Proof.

(i) It suffices to apply the fact that \( \tilde{M} \) is an exact functor in \( M \) (1.3.5) to the two exact sequences of \( A \)-modules
\[
0 \to \ker u \to M \to \text{im} u \to 0,
0 \to \text{im} u \to N \to \text{coker} u \to 0.
\]
The second claim then follows from Theorem (1.3.7).

(ii) Let \( (M_\lambda, g_{\lambda\mu}) \) be an inductive system of \( A \)-modules, with inductive limit \( M \), and let \( g_\lambda \) be the canonical homomorphism \( M_\lambda \to M \). Since we have \( \tilde{g}_{\lambda\mu} \circ \tilde{g}_{\mu\lambda} = \tilde{g}_{\lambda\mu} \tilde{g}_{\mu\lambda} \) and \( \tilde{g}_{\lambda} = \tilde{g}_{\mu} \circ \tilde{g}_{\mu\lambda} \) for \( \lambda \leq \mu \leq \nu \), it follows that \( (M_\lambda, \tilde{g}_{\lambda\mu}) \) is an inductive system of sheaves on \( X \), and if we denote by \( h_\lambda \) the canonical homomorphism \( M_\lambda \to \text{lim} M_\lambda \), then there is a unique homomorphism \( v : \text{lim} M_\lambda \to \tilde{M} \) such that \( v \circ h_\lambda = \tilde{g}_{\lambda} \). To see that \( v \) is bijective, it suffices to check, for each \( x \) in \( X \), that \( v_x \) is a bijection from \( \text{lim} M_\lambda \) to \( \tilde{M}_x \); but \( \tilde{M}_x = M_x \), and
\[
(\text{lim} M_\lambda)_x = \text{lim} (M_\lambda)_x = \text{lim} M_x = M_x \quad (0, 1.3.3).
\]
Conversely, it follows from the definitions that \((\bar{g}_\lambda)_x\) and \((h_\lambda)\) are both equal to the canonical map from \((M_\lambda)_x\) to \(M_x\); since \((\bar{g}_\lambda)_x = v_x \circ (h_\lambda)_x\), \(v_x\) is the identity. Finally, if \(M\) is the direct sum of two \(A\)-modules \(N\) and \(P\), it is immediate that \(\tilde{M} = \tilde{N} \oplus \tilde{P}\); each direct sum being the inductive limit of finite direct sums, the claims of (ii) are thus proved. □

We note that Corollary (1.3.9) proves that the sheaves isomorphic to the associated sheaves of \(A\)-modules form an abelian category (T, I, 1.4).

We also note that Corollary (1.3.9) implies that, if \(M\) is an \(A\)-module of finite type (that is to say, there exists a surjective homomorphism \(A^n \rightarrow M\) then there exists a surjective homomorphism \(\tilde{A^n} \rightarrow \tilde{M}\), or, in other words, the \(\tilde{A}\)-module \(\tilde{M}\) is generated by a finite family of sections over \(X(0, 5.1.1)\), and vice versa.

**Corollary (1.3.11).** — On the category of sheaves isomorphic to the associated sheaves of \(A\)-modules, the functor \(\Gamma\) is exact.

**Proof.** Let \(\tilde{M} \xrightarrow{\bar{u}} \tilde{N} \xrightarrow{\bar{v}} \tilde{P}\) be an exact sequence corresponding to two homomorphisms \(u : M \rightarrow N\) and \(v : N \rightarrow P\) of \(A\)-modules. If \(Q = \text{Im} u\) and \(R = \text{Ker} v\), we have \(\bar{Q} = \text{Im} \bar{u} = \text{Ker} \bar{v} = \bar{R}\) (Corollary (1.3.9)), hence \(Q = R\). □

**Corollary (1.3.12).** — Let \(M\) and \(N\) be \(A\)-modules.

(i) The sheaf associated to \(M \otimes_A N\) is canonically identified with \(\tilde{M} \otimes_{\tilde{A}} \tilde{N}\).

(ii) If, in addition, \(M\) admits a finite presentation, then the sheaf associated to \(\text{Hom}_A(M, N)\) is canonically identified with \(\text{Hom}_A(\tilde{M}, \tilde{N})\).

**Proof.**

(i) The sheaf \(\mathcal{F} = \tilde{M} \otimes_{\tilde{A}} \tilde{N}\) is associated to the presheaf

\[
U \mapsto \mathcal{F}(U) = \Gamma(U, \tilde{M}) \otimes_{\Gamma(U, \tilde{A})} \Gamma(U, \tilde{N}),
\]

with \(U\) varying over the basis (1.1.10, i) of \(X\) consisting of the \(D(f)\), where \(f \in A\). We know that \(\mathcal{F}(D(f))\) is canonically identified with \(M_f \otimes_{A_f} N_f\), by (1.3.7) and (1.3.6). Moreover, we know that the \(A_f\)-module \(M_f \otimes_{A_f} N_f\) is canonically isomorphic to \((M \otimes_A N)_f\) (0, 1.3.4), which is itself canonically isomorphic to \(\Gamma(D(f), (M \otimes_A N)\)) (Theorem (1.3.7) and Proposition (1.3.6)). In addition, we see immediately that the canonical isomorphisms

\[
\mathcal{F}(D(f)) \simeq \Gamma(D(f), (M \otimes_A N)\)
\]

thus obtained satisfy the compatibility conditions with respect to the restriction operations (0, 1.4.2), so they define a canonical functorial isomorphism

\[
\tilde{M} \otimes_{\tilde{A}} \tilde{N} \simeq (M \otimes_A N)\)
\]

(ii) The sheaf \(\mathcal{G} = \text{Hom}_A(\tilde{M}, \tilde{N})\) is associated to the presheaf

\[
U \mapsto \mathcal{G}(U) = \text{Hom}_A(\tilde{M}|U, \tilde{N}|U),
\]

with \(U\) varying over the basis of \(X\) consisting of the \(D(f)\). We know that \(\mathcal{G}(D(f))\) is canonically identified with \(\text{Hom}_A(M_f, N_f)\) (Proposition (1.3.6) and Corollary (1.3.8)),
which, according to the hypotheses on \( M \), is canonically identified with \((\text{Hom}_A(M,N))_f(0, 1.3.5)\). Finally, \((\text{Hom}_A(M,N))_f\) is canonically identified with \(\Gamma(D(f), (\text{Hom}_A(M,N)))\) (Proposition (1.3.6) and Theorem (1.3.7)), and the canonical isomorphisms \(\mathcal{H}(D(f)) \simeq \Gamma(D(f), (\text{Hom}_A(M,N)))\) thus obtained are compatible with the restriction operations \((0, 1.4.2)\); they thus define a canonical isomorphism \(\mathcal{H}\text{om}_A(M,\tilde{N}) \simeq (\text{Hom}_A(M,N))_\tau\).

\(\Box\)

(1.3.13). Now let \( B \) be a (commutative) \( A \)-algebra; this can be understood by saying that \( B \) is an \( A \)-module such that we have some given element \( e \in B \) and an \( A \)-homomorphism \( \phi : B \otimes_A B \to B \), so that \( (a) \) the diagrams

\[
\begin{array}{ccc}
B \otimes_A B & \xrightarrow{\phi} & B \otimes_A B \\
\downarrow 1 \otimes \phi & & \downarrow \phi \\
B \otimes_A B & \xrightarrow{\phi} & B
\end{array}
\]

\( (\sigma \) being the canonical symmetry map) are commutative; and \( (b) \) \( \phi(e \otimes x) = \phi(x \otimes e) = x \). By Corollary (1.3.12), the homomorphism \( \tilde{\phi} : \tilde{B} \otimes_{\tilde{A}} \tilde{B} \to \tilde{B} \) of \( \tilde{A} \)-modules satisfies the analogous conditions, and so it defines an \( \tilde{A} \)-algebra structure on \( \tilde{B} \). In a similar way, the data of a \( B \)-module \( N \) is the same as the data of an \( A \)-module and an \( A \)-homomorphism \( \psi : B \otimes_A N \to N \) such that the diagram

\[
\begin{array}{ccc}
B \otimes_A B \otimes_A N & \xrightarrow{\phi \otimes 1} & B \otimes_A N \\
\downarrow 1 \otimes \phi & & \downarrow \phi \\
B \otimes_A N & \xrightarrow{\psi} & N
\end{array}
\]

is commutative and \( \psi(e \otimes n) = n \); the homomorphism \( \tilde{\psi} : \tilde{B} \otimes_{\tilde{A}} \tilde{N} \to \tilde{N} \) satisfies the analogous condition, and so defines a \( \tilde{B} \)-module structure on \( \tilde{N} \).

In a similar way, we see that if \( u : B \to B' \) (resp. \( v : N \to N' \)) is a homomorphism of \( A \)-algebras (resp. of \( B \)-modules), then \( \tilde{u} \) (resp. \( \tilde{v} \)) is a homomorphism of \( \tilde{A} \)-algebras (resp. of \( \tilde{B} \)-modules), and \( \text{Ker} \tilde{u} \) is a \( \tilde{B} \)-ideal (resp. \( \text{Ker} \tilde{v} \), \( \text{Coker} \tilde{v} \), and \( \text{Im} \tilde{v} \) are \( \tilde{B} \)-modules). If \( N \) is a \( B \)-module, then \( \tilde{N} \) is a \( \tilde{B} \)-module of finite type if and only if \( N \) is a \( B \)-module of finite type \((0, 5.2.3)\).

If \( M, N \) are \( B \)-modules, then the \( \tilde{B} \)-module \( \tilde{M} \otimes_{\tilde{B}} \tilde{N} \) is canonically identified with \( (M \otimes_B N)_\tau \) similarly \( \mathcal{H}\text{om}_{\tilde{B}}(\tilde{M}, \tilde{N}) \) is canonically identified with \( (\text{Hom}_B(M,N))_\tau \) whenever \( M \) admits a finite presentation; the proofs are similar to those for Corollary (1.3.12).

If \( \mathfrak{a} \) is an ideal of \( B \), and \( N \) is a \( B \)-module, then we have \( (\mathfrak{a} N)_\tau = \mathfrak{a} \cdot \tilde{N} \).

Finally, if \( B \) is an \( A \)-algebra graded by the \( A \)-submodules \( B_n \) \((n \in \mathbb{Z})\), then the \( \tilde{A} \)-algebra \( \tilde{B} \), the direct sum of the \( \tilde{A} \)-modules \( \tilde{B}_n \) \((1.3.9)\), is graded by these \( \tilde{A} \)-submodules, the axiom of graded algebras saying that the image of the homomorphism \( B_m \otimes B_n \to B \) is contained in \( B_{m+n} \). Similarly, if \( M \) is a \( B \)-module graded by the submodules \( M_n \), then \( \tilde{M} \) is a \( \tilde{B} \)-module graded by the \( \tilde{M}_n \).

(1.3.14). If \( B \) is an \( A \)-algebra, and \( M \) a submodule of \( B \), then the \( \tilde{A} \)-subalgebra of \( \tilde{B} \) generated by \( \tilde{M} \) \((0, 4.1.3)\) is the \( \tilde{A} \)-subalgebra \( \tilde{C} \), where we denote by \( C \) the subalgebra of \( B \) generated by \( M \). Indeed, \( C \) is the direct sum of the submodules of \( B \) which are the images of the homomorphisms \( \otimes^m M \to B \) \((n \geq 0)\), so it suffices to apply \((1.3.9)\) and \((1.3.12)\).
1.4. Quasi-coherent sheaves over a prime spectrum

**Theorem (1.4.1).** — Let $X$ be the prime spectrum of a ring $A$, $V$ a quasi-compact open subset of $X$, and $\mathcal{F}$ an $(\mathcal{O}_X|V)$-module. The following four conditions are equivalent.

(a) There exists an $A$-module $M$ such that $\mathcal{F}$ is isomorphic to $\tilde{M}|V$.

(b) There exists a finite open cover $(V_i)$ of $V$ by sets of the form $D(f_i)$ ($f_i \in A$) contained in $V$, such that, for each $i$, $\mathcal{F}|V_i$ is isomorphic to a sheaf of the form $M_i$, where $M_i$ is an $A_{f_i}$-module.

(c) The sheaf $\mathcal{F}$ is quasi-coherent (0.5.1.3).

(d) The two following properties are satisfied:

(d1) For each $f \in A$ such that $D(f) \subset V$, and for each section $s \in \Gamma(D(f), \mathcal{F})$, there exists an integer $n \geq 0$ such that $f^n s$ extends to a section of $\mathcal{F}$ over $V$.

(d2) For each $f \in A$ such that $D(f) \subset V$ and for each section $t \in \Gamma(V, \mathcal{F})$ such that the restriction of $t$ to $D(f)$ is zero, there exists an integer $n \geq 0$ such that $f^n t = 0$.

(In the statement of the conditions (d1) and (d2), we have tacitly identified $A$ and $\Gamma(\tilde{A})$ using Theorem (1.3.7)).

**Proof.** The fact that (a) implies (b) is an immediate consequence of Proposition (1.3.6) and the fact that the $D(f_i)$ form a basis for the topology of $X$ (1.1.10). As each $A$-module is isomorphic to the cokernel of a homomorphism of the form $A^{(l)} \to A^{(l)}$, (1.3.6) implies that each sheaf associated to an $A$-module is quasi-coherent; so (b) implies (c). Conversely, if $\mathcal{F}$ is quasi-coherent, each $x \in V$ has a neighborhood of the form $D(f) \subset V$ such that $\mathcal{F}|D(f)$ is isomorphic to the cokernel of a homomorphism $\tilde{A}_f^{(l)} \to \tilde{A}_f^{(l)}$, so also to the sheaf $\tilde{N}$ associated to the module $N$, the cokernel of the corresponding homomorphism $\tilde{A}_f^{(l)} \to \tilde{A}_f^{(l)}$ (Corollaries (1.3.8) and (1.3.9)); since $V$ is quasi-compact, it is clear that (c) implies (b).

To prove that (b) implies (d1) and (d2), we first assume that $V = D(g)$ for some $g \in A$, and that $\mathcal{F}$ is isomorphic to the sheaf $\tilde{N}$ associated to an $A_g$-module $N$; by replacing $X$ with $V$ and $A$ with $A_g$ (1.3.6), we can reduce to the case where $g = 1$. Then $\Gamma(D(f), \tilde{N})$ and $N_f$ are canonically identified with one another (Proposition (1.3.6) and Theorem (1.3.7)), so a section $s \in \Gamma(D(f), \tilde{N})$ is identified with an element of the form $z/f^n$, where $z \in N$; the section $f^n s$ is identified with the element $z/1$ of $N_f$ and, as a result, is the restriction to $D(f)$ of the section of $\tilde{N}$ over $X$ that is identified with the element $z \in N$; hence (d1) in this case. Similarly, $t \in \Gamma(V, \tilde{N})$ is identified with an element $z' \in N_f$, the restriction of $t$ to $D(f)$ is identified with the image $z'/1$ of $z'$ in $N_f$, and to say that this image is zero means that there exists some $n \geq 0$ such that $f^n z' = 0$ in $N$, or, equivalently, $f^n t = 0$.

To finish the proof, that (b) implies (d1) and (d2), it suffices to establish the following lemma.

**Lemma (1.4.1.1).** — Suppose that $V$ is the finite union of sets of the form $D(g_i)$, and that all of the sheaves $\mathcal{F}|D(g_i)$ and $\mathcal{F}|(D(g_i) \cap D(g_j)) = \mathcal{F}|D(g_i g_j)$ satisfy (d1) and (d2); then $\mathcal{F}$ has the following two properties:

(d1') For each $f \in A$ and for each section $s \in \Gamma(D(f) \cap V, \mathcal{F})$, there exists an integer $n \geq 0$ such that $f^n s$ extends to a section of $\mathcal{F}$ over $V$.

(d2') For each $f \in A$ and for each section $t \in \Gamma(V, \mathcal{F})$ such that the restriction of $t$ to $D(f) \cap V$ is zero, there exists an integer $n \geq 0$ such that $f^n t = 0$.

We first prove (d2'); since $D(f) \cap D(g_i) = D(f g_i)$, there exists, for each $i$, an integer $n_i$ such that the restriction of $(f g_i)^{n_i} t$ to $D(g_i)$ is zero; since the image of $g_i$ in $A_{g_i}$ is invertible, the restriction of $f^{n_i} t$ to $D(g_i)$ is also zero; taking $n$ to be the largest of the $n_i$, we have proved (d2').

To show (d1'), we apply (d1) to the sheaf $\mathcal{F}|D(g_i)$: there exists an integer $n_i \geq 0$ and a section $s_i$ of $\mathcal{F}$ over $D(g_i)$ extending the restriction of $(f g_i)^{n_i} s_i$ to $D(f g_i)$; since the image of $g_i$ in $A_{g_i}$ is invertible, there is a section $s_i$ of $\mathcal{F}$ over $D(g_i)$ such that $s_i = g_i^{n_i} s_i$ and $s_i$ extends the restriction of $f^{n_i} s$ to $D(g_i)$; in addition we can suppose that all the $n_i$ are equal to a single integer $n$. By construction, the restriction of $s_i - s_i$ to $D(f) \cap D(g_i) \cap D(g_j)$ is zero; by (d2) applied to the sheaf $\mathcal{F}|D(g_i g_j)$, there exists an integer $m_{ij} \geq 0$ such that the restriction to $D(g_i g_j)$ of $(f g_i g_j)^{m_{ij}}(s_i - s_j)$ is zero; since the image of $g_i g_j$ in $A_{g_i g_j}$ is invertible, the restriction of $f^{m_{ij}}(s_i - s_j)$ to $D(g_i g_j)$ is zero. We can then assume that all the $m_{ij}$ are equal to a single integer $m$, and so there
exists a section \( s' \in \Gamma(V, \mathcal{F}) \) extending the \( f^m s' \); as a result, this section extends \( f^{n+m} s \), hence we have proved (d’1).

It remains to show that (d1) and (d2) imply (a). We first show that (d1) and (d2) imply that these conditions are satisfied for each sheaf \( \mathcal{F}/D(g) \), where \( g \in A \) is such that \( D(g) \subset V \). It is evident for (d1); on the other hand, if \( t \in \Gamma(D(g), \mathcal{F}) \) is such that its restriction to \( D(f) \subset D(g) \) is zero, there exists, by (d1), an integer \( m \geq 0 \) such that \( g^m t \) extends to a section \( s \) of \( \mathcal{F} \) over \( V \); applying (d2), we see that there exists an integer \( n \geq 0 \) such that \( f^n g^m t = 0 \), and as the image of \( g \) in \( A_\mathfrak{g} \) is invertible, \( f^n t = 0 \).

That being so, since \( V \) is quasi-compact, Lemma (1.4.1.1) proves that the conditions (d’1) and (d’2) are satisfied. Consider then the \( A \)-module \( M = \Gamma(V, \mathcal{F}) \), and define a homomorphism of \( \tilde{A} \)-modules \( u : \tilde{M} \to j_! \mathcal{F} \), where \( j \) is the canonical injection \( V \to X \). Since the \( D(f) \) form a basis for the topology of \( X \), it suffices, for each \( f \in A \), to define a homomorphism \( u_f : M_f \to \Gamma(D(f), j_! \mathcal{F}) = \Gamma(D(f) \cap V, \mathcal{F}) \), with the usual compatibility conditions (0, 3.2.5). Since the canonical image of \( f \) in \( A_f \) is invertible, the restriction homomorphism \( M = \Gamma(V, \mathcal{F}) \to \Gamma(D(f) \cap V, \mathcal{F}) \) factors as \( M \to M_f \xrightarrow{u_f} \Gamma(D(f) \cap V, \mathcal{F}) \) (0, 1.2.4), and the verification of these compatibility conditions for \( D(g) \subset D(f) \) is immediate. This being so, we show that the condition (d’1) (resp. (d’2)) implies that each of the \( u_f \) are surjective (resp. injective), which proves that \( u \) is bijective, and as a result that \( \mathcal{F} \) is the restriction to \( V \) of an \( \tilde{A} \)-module isomorphic to \( \tilde{M} \). If \( s \in \Gamma(D(f) \cap V, \mathcal{F}) \), there exists, by (d’1), an integer \( n \geq 0 \) such that \( f^n s \) extends to a section \( z \in M \); we then have \( u_f(z/f^n) = s \), so \( u_f \) is surjective. Similarly, if \( z \in M \) is such that \( u_f(z/1) = 0 \), this means that the restriction to \( D(f) \cap V \) of the section \( z \) is zero; according to (d’2), there exists an integer \( n \geq 0 \) such that \( f^n z = 0 \), hence \( z/1 = 0 \in M_f \), and so \( u_f \) is injective.

\[ \square \]

Corollary (1.4.2). — Each quasi-coherent sheaf over a quasi-compact open subset of \( X \) is induced by a quasi-coherent sheaf on \( X \).

Corollary (1.4.3). — Every quasi-coherent \( \mathcal{O}_X \)-algebra over \( X = \text{Spec}(A) \) is isomorphic to an \( \mathcal{O}_X \)-algebra of the form \( \mathcal{B} \), where \( B \) is an algebra over \( A \); every quasi-coherent \( B \)-module is isomorphic to a \( \mathcal{B} \)-module of the form \( \mathcal{N} \), where \( N \) is a \( B \)-module.

Proof. Indeed, a quasi-coherent \( \mathcal{O}_X \)-algebra is a quasi-coherent \( \mathcal{O}_X \)-module, and therefore of the form \( \mathcal{B} \), where \( B \) is an \( A \)-module; the fact that \( B \) is an \( A \)-algebra follows from the characterization of the structure of an \( \mathcal{O}_X \)-algebra using the homomorphism \( \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \to \mathcal{B} \) of \( \mathcal{A} \)-modules, as well as Corollary (1.3.12). If \( \mathcal{G} \) is a quasi-coherent \( \mathcal{B} \)-module, it suffices to show, in a similar way, that it is also a quasi-coherent \( \mathcal{A} \)-module to conclude the proof; since the question is local, we can, by restricting to an open subset of \( X \) of the form \( D(f) \), assume that \( \mathcal{G} \) is the cokernel of a homomorphism \( \mathcal{B}(f) \to \mathcal{B}(f) \) of \( \mathcal{B} \)-modules (and \( \text{a fortiori} \) of \( \mathcal{A} \)-modules); the proposition then follows from Corollaries (1.3.8) and (1.3.9).

\[ \square \]

### 1.5. Coherent sheaves over a prime spectrum

Theorem (1.5.1). — Let \( A \) be a Noetherian ring, \( X = \text{Spec}(A) \) its prime spectrum, \( V \) an open subset of \( X \), and \( \mathcal{F} \) an \( (\mathcal{O}_X|V) \)-module. The following conditions are equivalent.

(a) \( \mathcal{F} \) is coherent.

(b) \( \mathcal{F} \) is of finite type and quasi-coherent.

(c) There exists an \( A \)-module \( M \) of finite type such that \( \mathcal{F} \) is isomorphic to the sheaf \( \widetilde{M}|V \).

Proof. (a) trivially implies (b). To see that (b) implies (c), note that, since \( V \) is quasi-compact (0, 2.2.3), we have previously seen that \( \mathcal{F} \) is isomorphic to a sheaf \( \mathcal{N}|V \), where \( N \) is an \( A \)-module (1.4.1). We have \( \mathcal{N} = \varinjlim M_\lambda \), where \( M_\lambda \) run over the set of \( A \)-submodules of \( N \) of finite type, hence (1.3.9) \( \mathcal{F} = \mathcal{N}|V = \varinjlim M_\lambda|V \); but since \( \mathcal{F} \) is of finite type, and \( V \) is quasi-compact, there exists an index \( \lambda \) such that \( \mathcal{F} = M_\lambda|V \) (0, 5.2.3).

Finally, we show that (c) implies (a). It is clear that \( \mathcal{F} \) is then of finite type ((1.3.6) and (1.3.9)); in addition, since the questions is local, we can restrict to the case where \( V = D(f) \), \( f \in A \). Since \( A_f \) is Noetherian, we see that it suffices to prove that the kernel of a homomorphism \( \mathcal{A}^n \to \mathcal{M} \),
where $M$ is an $A$-module, is of finite type. But such a homomorphism is of the form $\tilde{u}$, where $u$ is a homomorphism $A^u \to M$ (1.3.8), and if $P = \text{Ker } u$ then we have $\tilde{P} = \text{Ker } \tilde{u}$ (1.3.9). Since $A$ is Noetherian, $P$ is of finite type, which finishes the proof. 

\[ \square \]

**Corollary (1.5.2).** — Under the hypotheses of (1.5.1), the sheaf $\mathcal{O}_X$ is a quasi-coherent sheaf of rings.

**Corollary (1.5.3).** — Under the hypotheses of (1.5.1), every coherent sheaf over an open subset of $X$ is induced by a coherent sheaf on $X$.

**Corollary (1.5.4).** — Under the hypotheses of (1.5.1), every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is the inductive limit of the coherent $\mathcal{O}_X$-submodules of $\mathcal{F}$.

**Proof.** Indeed, $\mathcal{F} = \tilde{M}$, where $M$ is an $A$-module, and $M$ is the inductive limit of its submodules of finite type; we conclude the proof by appealing to (1.3.9) and (1.5.1). 

\[ \square \]

### 1.6. Functorial properties of quasi-coherent sheaves over a prime spectrum

**Example (1.6.2).** — Let $S$ be a multiplicative subset of $A$, and $\phi$ the canonical homomorphism $A \to S^{-1}A$; we have already seen (1.2.6) that $\phi$ is a homeomorphism from $Y = \text{Spec}(S^{-1}A)$ to the subspace of $X = \text{Spec}(A)$ consisting of the $x$ such that $i_x \cap S = \emptyset$. In addition, for each $x$ in this subspace, which is thus of the form $\phi(x')$ with $x' \in Y$, the homomorphism $\phi_x : \mathcal{O}_Y \to \mathcal{O}_X$ is bijective (0, 1.2.6); in other words, $\mathcal{O}_Y$ is identified with the sheaf on $Y$ induced by $\mathcal{O}_X$.

**Proposition (1.6.3).** — For every $A$-module $M$, there exists a canonical functorial isomorphism from the $\mathcal{O}_X$-module $(M|_x)^\sim$ to the direct image $\Phi_x(M)$. 

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PROOF. For purposes of abbreviation, we write $M' = M_{[\phi]}$, and for each $f' \in A'$, we put $f = \phi(f')$. The modules of sections $\Gamma(D(f'), \tilde{M})$ and $\Gamma(D(f), \tilde{M})$ are identified, respectively, with the modules $M'_f$ and $M_f$ (over $A'_f$ and $A_f$, respectively); in addition, the $A'_f$-module $(M_f)_{[\phi]}$ is canonically isomorphic to $M'_f$ (0, 1.5.2). We thus have a functorial isomorphism of $\Gamma(D(f'), A')$-modules: $\Gamma(D(f'), \tilde{M}) \cong \Gamma(\phi^{-1}(D(f')), \tilde{M})_{[\phi]}$ and these isomorphisms satisfy the usual compatibility conditions with the restrictions (0, 1.5.6), thus defining the desired functorial isomorphism. We note that, in a precise way, if $u : M_1 \to M_2$ is a homomorphism of $A$-modules, it can be considered as a homomorphism $(M_1)_{[\phi]} \to (M_2)_{[\phi]}$ of $A'$-modules; if we denote this homomorphism by $u_{[\phi]}$, then $\Phi_s(\tilde{u})$ is identified with $(u_{[\phi]})^\vee$.

This proof also shows that, for each $A$-algebra $B$, the canonical functorial isomorphism $(B_{[\phi]})^\sim \Phi_{s}(\tilde{B})$ is an isomorphism of $\mathcal{O}_X$-algebras; if $M$ is a $B$-module, the canonical functorial isomorphism $(M_{[\phi]})^\sim \Phi_{s}(\tilde{M})$ is an isomorphism of $\Phi_s(\tilde{B})$-modules.

Corollary (1.6.4). — The direct image functor $\Phi_s$ is exact on the category of quasi-coherent $\mathcal{O}_X$-modules.

PROOF. Indeed, it is clear that $M_{[\phi]}$ is an exact functor in $M$ and $\tilde{M}'$ is an exact functor in $M'$ (1.3.5).

Proposition (1.6.5). — Let $N'$ be an $A'$-module, and $N$ the $A$-module $N' \otimes_{A'} A_{[\phi]}$; then there exists a canonical functorial isomorphism from $\mathcal{O}_X$-modules $\Phi^*(\tilde{N}')$ to $\tilde{N}$.

PROOF. We first remark that $j : z' \to z' \otimes 1$ is an $A'$-homomorphism from $N'$ to $N_{[\phi]}$: indeed, by definition, for $f' \in A'$, we have $(f'z') \otimes 1 = z' \otimes \phi(f') = \phi(f')(z' \otimes 1)$. We have (1.3.8) a homomorphism $\tilde{j} : \tilde{N'} \to (N_{[\phi]})^\sim$ of $\mathcal{O}_X$-modules, and, thanks to (1.6.3), we can consider $\tilde{j}$ as mapping $\tilde{N'}$ to $\Phi_{s}(\tilde{N})$. There canonically corresponds to this homomorphism $j$ a homomorphism $h = j^\prime \tilde{f}$ from $\Phi^*(\tilde{N'})$ to $\tilde{N}$ (0, 4.4.3); we will see that, for each stalk, $h_x$ is bijective. Put $z' = \phi(x)$ and let $f' \in A'$ be such that $x' \in D(f')$; let $f = \phi(f')$. The ring $\Gamma(D(f), \tilde{A})$ is identified with $A_f$, the modules $\Gamma(D(f'), \tilde{N'})$ and $\Gamma(D(f), \tilde{M}')$ with $N_f$ and $N'_f$, respectively; let $s \in \Gamma(D(f'), \tilde{N'})$, identified with $n' / f'^P (n' \in N')$, and $s$ be its image under $\tilde{j}$ in $\Gamma(D(f), \tilde{N})$; $s'$ is identified with $(n' \otimes 1) / f'^P$. On the other hand, let $t \in \Gamma(D(f), \tilde{A})$, identified with $g / f'^q (g \in A)$; then, by definition, we have $h_s(s'_x \otimes t_x) = t_x \cdot s_x (0, 4.4.3)$. But we can canonically identify $N_f$ with $N'_f \otimes A_{[\phi]} (A_f)_{[\phi]}$ (0, 1.5.4); $s$ then corresponds to the element $(n' / f'^P) \otimes 1$, and the section $y \to t_x \cdot s_y$ with $(n' / f'^P) \otimes (g / f'^q)$.

The compatibility diagram of (0, 1.5.6) show that $h_x$ is exactly the canonical isomorphism (1.6.5.1)

$$N'_x \otimes_{A_{[\phi]}} (A_x)_{[\phi]} \cong N_x = (N' \otimes_{A'} A_{[\phi]})_x.$$ 

In addition, let $v : N'_x \to N'_y$ be a homomorphism of $A'$-modules; since $v_x' = v_x$ for each $x' \in X'$, it follows immediately from the above that $\Phi^*(\tilde{v})$ is canonically identified with $(v \otimes 1)^\sim$, which finishes the proof of (1.6.5).

If $B'$ is an $A'$-algebra, the canonical isomorphism from $\Phi^*(\tilde{B'})$ to $(B' \otimes_{A'} A_{[\phi]})^\sim$ is an isomorphism of $\mathcal{O}_X$-algebras; if, in addition, $N'$ is a $B'$-module, then the canonical isomorphism from $\Phi^*(\tilde{N'})$ to $(N' \otimes_{A'} A_{[\phi]})^\sim$ is an isomorphism of $\Phi^*(\tilde{B'})$-modules.

Corollary (1.6.6). — The sections of $\Phi^*(\tilde{N'})$, the canonical images of the sections $s'$, where $s'$ varies over the $A'$-module $\Gamma(\tilde{N}')$, generate the $A$-module $\Gamma(\Phi^*(\tilde{N'}))$.

PROOF. Indeed, these images are identified with the elements $z' \otimes 1$ of $N$, when we identify $N'$ and $N$ with $\Gamma(\tilde{N}')$ and $\Gamma(\tilde{N})$ (respectively) (1.3.7), and $z'$ varies over $N'$.

(1.6.7). In the proof of (1.6.5), we had proved in passing that the canonical map (0, 4.4.3.2) $\rho : \tilde{N'} \to \Phi_s(\Phi^*(\tilde{N'}))$ is exactly the homomorphism $\tilde{j}$, where $j : N' \to N' \otimes_{A'} A_{[\phi]}$ is the homomorphism $z' \to z' \otimes 1$. Similarly, the canonical map (0, 4.4.3.3) $\sigma : \Phi^*(\Phi_s(\tilde{M})) \to \tilde{M}$ is exactly $\tilde{p}$, where $p : M_{[\phi]} \otimes A_{[\phi]} \to \tilde{M}$ is the canonical homomorphism, which sends each tensor product $z \otimes a (z \in M, a \in A)$ to $a \cdot z$; this follows immediately from the definitions ((0, 3.7.1), (0, 4.4.3), and (1.3.7)).
We conclude ((0, 4.4.3) and (0, 3.5.4.4)) that if \( \psi : N' \to M[\phi] \) is an \( A' \)-homomorphism, then 
\[ \tilde{\psi} = (\psi \otimes 1)^{-}. \]

**Theorem (1.6.8).** — Let \( N'_1 \) and \( N'_2 \) be \( A' \)-modules, and assume \( N'_1 \) admits a finite presentation; it then follows from (1.6.7) and (1.3.12, ii) that the canonical homomorphism (0, 4.4.6)
\[
\Phi^* \left( \text{Hom}_A \left( \tilde{N}'_1, \tilde{N}'_2 \right) \right) \to \text{Hom}_A \left( \Phi^*(\tilde{N}'_1), \Phi^*(\tilde{N}'_2) \right)
\]
is exactly \( \gamma \), where \( \gamma \) denotes the canonical homomorphism of \( A \)-modules \( \text{Hom}_A (N'_1, N'_2) \otimes_A A \to \text{Hom}_A (N'_1 \otimes_A A, N'_2 \otimes_A A) \).

**Theorem (1.6.9).** Let \( \mathfrak{A} \) be an ideal of \( A' \), and \( M \) an \( A \)-module; since, by definition, \( \mathfrak{A} \mathcal{M} \) is the image of the canonical homomorphism \( \Phi^*(\mathfrak{A}) \otimes_A \tilde{\mathcal{M}} \to \tilde{\mathcal{M}} \), it follows from Proposition (1.6.5) and Corollary (1.3.12, i) that \( \mathfrak{A} \mathcal{M} \) canonically identifies with \( (\mathfrak{A} \mathcal{M})^{-} \); in particular, \( \Phi^*(\mathfrak{A})^{-} \) is identified with \( (\mathfrak{A}^{-} A)^{-} \), and, taking the right exactness of the functor \( \Phi^* \) into account, the \( A^{-} \)-algebra \( \Phi^*(\mathfrak{A}' / \mathfrak{A}^{-} A)^{-} \) is identified with \( (A / \mathfrak{A}^{-} A)^{-} \).

**Theorem (1.6.10).** Let \( A'' \) be a third ring, \( \phi' \) a homomorphism \( A'' \to A' \), and write \( \phi'' = \phi \circ \phi' \). It follows immediately from the definitions that \( \phi'' = (\phi' \circ \phi) \circ \phi' = \bar{\phi} \circ \phi' \). We conclude that \( \Phi'' = \Phi' \circ \Phi; \) in other words, \( (\text{Spec}(A), \mathcal{A}) \) is a functor from the category of rings to that of ringed spaces.

### 1.7. Characterization of morphisms of affine schemes

**Definition (1.7.1).** — We say that a ringed space \( (X, \mathcal{O}_X) \) is an **affine scheme** if it is isomorphic to a ringed space of the form \( (\text{Spec}(A), \mathcal{A}) \), where \( A \) is a ring; we then say that \( (X, \mathcal{O}_X) \), which is canonically identified with the ring \( A \) (1.3.7), is the ring of the affine scheme \( (X, \mathcal{O}_X) \), and we denote it by \( A(X) \) when there is no chance of confusion.

By abuse of language, when we speak of an affine scheme \( \text{Spec}(A) \), it will always be the ringed space \( (\text{Spec}(A), \mathcal{A}) \).

**Theorem (1.7.2).** Let \( A \) and \( B \) be rings, and \( (X, \mathcal{O}_X) \) and \( (Y, \mathcal{O}_Y) \) the affine schemes corresponding to the prime spectra \( X = \text{Spec}(A), Y = \text{Spec}(B) \). We have seen (1.6.1) that each ring homomorphism \( \phi : B \to A \) corresponds to a morphism \( \Phi = (\phi, \bar{\phi}) = \text{Spec}(\phi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \). We denote by \( \phi \) the entire homomorphism of rings \( A(Y) \to A(X) \) corresponding to \( \Phi \).

**Theorem (1.7.3).** — Let \( (X, \mathcal{O}_X), (Y, \mathcal{O}_Y) \) be affine schemes. For a morphism of ringed spaces \( (\psi, \theta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \) of the form \( (\phi, \bar{\phi}) \), where \( \phi \) is a homomorphism of rings \( A(Y) \to A(X) \), it is necessary and sufficient that, for each \( x \in X \), \( \mathfrak{p}_x \) is a local homomorphism: \( \psi_*(\mathcal{O}_x) \to \mathcal{O}_x \).

**Proof.** Let \( A = A(X), B = A(Y) \). The condition is necessary, since we saw (1.6.1) that \( \bar{\phi}_x \) is the homomorphism from \( B \mathfrak{p}_x \) to \( A \mathfrak{p}_x \), canonically induced by \( \phi \), and, by definition, of \( \phi(x) = \phi^{-1}(i_x) \), this homomorphism is local.

We now prove that the condition is sufficient. By definition, \( \theta \) is a homomorphism \( \mathcal{O}_Y \to \psi_*(\mathcal{O}_X) \), and we canonically obtain a ring homomorphism
\[
\phi = \Gamma(\theta) : B = \Gamma(Y, \mathcal{O}_Y) \to \Gamma(Y, \psi_*(\mathcal{O}_X)) = \Gamma(X, \mathcal{O}_X) = A.
\]

The homomorphisms on \( \mathfrak{p}_x \) mean that this homomorphism induces, by passing to quotients, a monomorphism \( \mathfrak{p}_x \) from the residue field \( k(\psi(x)) \) to the residue field \( k(x) \), such that, for each section \( f \in \Gamma(Y, \mathcal{O}_Y) = B \), we have \( \theta^*(f(\psi(x))) = \phi(f)(x) \). The relation \( f(\psi(x)) = 0 \) is therefore equivalent to \( \phi(f)(x) = 0 \), which means that \( i_{\psi(x)} = i_{\phi(x)} \), and we now write \( \psi(x) = \phi(x) \) for each \( x \in X \), or \( \psi = \phi \). We also know that the diagram
\[
\begin{array}{ccc}
B = \Gamma(Y, \mathcal{O}_Y) & \xrightarrow{\phi} & \Gamma(X, \mathcal{O}_X) = A \\
\downarrow & & \downarrow \\
B_{\psi(x)} & \xrightarrow{\bar{\phi}_x} & A_x
\end{array}
\]

\[\text{[Trans.] See (1.8) and the footnote there.}\]
is commutative (0, 3.7.2), which means that $\theta_x^2$ is equal to the homomorphism $\phi_x : B_{\phi(x)} \to A_x$ canonically induced by $\phi (0, 1.5.1)$. As the data of the $\theta_x^2$ completely characterize $\theta^2$, and as a result $\theta (0, 3.7.1)$, we conclude that we have $\theta = \bar{\phi}$, by the definition of $\bar{\phi} (1.6.1)$. □

We say that a morphism $(\psi, \theta)$ of ringed spaces satisfying the condition of (1.7.3) is a morphism of affine schemes.

**Corollary (1.7.4).** — If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are affine schemes, there exists a canonical isomorphism from the set of morphisms of affine schemes $\text{Hom}((X, \mathcal{O}_X), (Y, \mathcal{O}_Y))$ to the set of ring homomorphisms from $B$ to $A$, where $A = \Gamma(\mathcal{O}_X)$ and $B = \Gamma(\mathcal{O}_Y)$.

Furthermore, we can say that the functors $(\text{Spec}(A), A)$ in $A$ and $\Gamma(X, \mathcal{O}_X)$ in $(X, \mathcal{O}_X)$ define an equivalence between the category of commutative rings and the opposite category of affine schemes (I, 1.2).

**Corollary (1.7.5).** — If $\phi : B \to A$ is surjective, then the corresponding morphism $(^a\phi, \nu)$ is a monomorphism of ringed spaces (cf. (4.1.7)).

**Proof.** Indeed, we know that $^a\phi$ is injective (1.2.5), and, since $\phi$ is surjective, for each $x \in X$, $\phi_x^2 : B_{\phi(x)} \to A_x$, which is induced by $\phi$ by passing to rings of fractions, is also surjective (0, 1.5.1); hence the conclusion (0, 4.1.1). □

### 1.8. Morphisms from locally ringed spaces to affine schemes

Due to a remark by J. Tate, the statements of Theorem (1.7.3) and Proposition (2.2.4) can be generalized as follows:

**Proposition (1.8.1).** — Let $(S, \mathcal{O}_S)$ be an affine scheme, and $(X, \mathcal{O}_X)$ a locally ringed space. Then there is a canonical bijection from the set of ring homomorphisms $\Gamma(S, \mathcal{O}_S) \to \Gamma(X, \mathcal{O}_X)$ to the set of morphisms of ringed spaces $(\psi, \theta) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ such that, for each $x \in X$, $\theta^2_x$ is a local homomorphism $\mathcal{O}_{\phi(x)} \to \mathcal{O}_x$.

**Proof.** We note first that if $(X, \mathcal{O}_X)$ and $(S, \mathcal{O}_S)$ are any two ringed spaces, then a morphism $(\psi, \theta)$ from $(X, \mathcal{O}_X)$ to $(S, \mathcal{O}_S)$ canonically defines a ring homomorphism $\Gamma(\theta) : \Gamma(S, \mathcal{O}_S) \to \Gamma(X, \mathcal{O}_X)$, hence a first map

$$(1.8.1.1) \quad \rho : \text{Hom}((X, \mathcal{O}_X), (S, \mathcal{O}_S)) \to \text{Hom}(\Gamma(S, \mathcal{O}_S), \Gamma(X, \mathcal{O}_X)).$$

Conversely, under the stated hypotheses, we set $A = \Gamma(S, \mathcal{O}_S)$, and consider a ring homomorphism $\phi : A \to \Gamma(X, \mathcal{O}_X)$. For each $x \in X$, it is clear that the set of the $f \in A$ such that $\phi(f)(x) = 0$ is a prime ideal of $A$, since $\mathcal{O}_x/m_x = k(x)$ is a field; it is therefore an element of $S = \text{Spec}(A)$, which we denote by $^a\phi(x)$. In addition, for each $f \in A$, we have, by definition (0, 5.5.2), that $^a\phi^{-1}(D(f)) = X_f$, which proves that $^a\phi$ is a continuous map $X \to S$. We then define a homomorphism

$$\bar{\phi} : \mathcal{O}_S \to ^a\phi_*(\mathcal{O}_X)$$

of $\mathcal{O}_S$-modules; for each $f \in A$, we have $\Gamma(D(f), \mathcal{O}_S) = A_f$ (1.3.6); for each $s \in A$, we associate to $s/f \in A_f$ the element $(\phi(s)|_{X_f})(\phi(f)|_{X_f})^{-1}$ of $\Gamma(X_f, \mathcal{O}_X) = \Gamma(D(f), ^a\phi_*(\mathcal{O}_X))$, and we immediately see (by passing from $D(f)$ to $D(fg)$) that this is a well-defined homomorphism of $\mathcal{O}_S$-modules, hence a morphism $(^a\phi, \nu)$ of ringed spaces. In addition, with the same notation, and setting $y = ^a\phi(x)$ for brevity, we immediately see (0, 3.7.1) that we have $\bar{\phi}^{-1}_y(s_y/f_y) = (\phi(s)_x)(\phi(f)_x)^{-1}$; since the relation $s_y \in m_y$ is, by definition, equivalent to $\phi(s)_x \in m_x$, we see that $\bar{\phi}^2_x$ is a local homomorphism $\mathcal{O}_y \to \mathcal{O}_x$, and we have thus defined a second map $\sigma : \text{Hom}(\Gamma(S, \mathcal{O}_S), \Gamma(X, \mathcal{O}_X)) \to \mathcal{L}$, where $\mathcal{L}$ is the set of the morphisms $(\psi, \theta) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ such that $\theta^2_x$ is local for each $x \in X$. It remains to prove that $\sigma$ and $\rho$ (restricted to $\mathcal{L}$) are inverses of each other; the definition of $\bar{\phi}$ immediately shows that $\Gamma(\bar{\phi}) = \phi$, and, as a result, that $\rho \circ \sigma$ is the identity. To see that $\sigma \circ \rho$ is the identity, start with a morphism $(\psi, \theta) \in \mathcal{L}$ and let $\phi = \Gamma(\theta)$; the hypotheses on $\theta^2_x$ mean that this morphism induces, by passing to quotients, a monomorphism $\theta^3 : k(\psi(x)) \to k(x)$ such that for each section

\[\text{Trans.} \quad \text{The following section (1.1.8) was added in the errata of EGA II, hence the temporary change in page numbers, which refer to EGA II.}\]
with this definition of morphisms, it is clear that the locally ringed spaces form a category II (1.8.4).

Let \( \phi : \Gamma(X, \mathcal{O}_X) \to \Gamma(Y, \mathcal{O}_Y) \), \( \psi : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(Z, \mathcal{O}_Z) \) be morphisms of locally ringed spaces such that, for each \( x \in X \), \( \theta^x_x \) is a local homomorphism \( \mathcal{O}_{\phi(x)} \to \mathcal{O}_x \). Henceforth we will denote it by \( \theta^x_x \). Hence

This definition of morphisms makes the category of locally ringed spaces a category. For any two objects \( X \) and \( Y \) of this category, \( \text{Hom}(X, Y) \) denotes the set of morphisms of locally ringed spaces from \( X \) to \( Y \) (the set denoted \( \mathcal{C} \) in (1.8.1)); when we consider the set of morphisms of ringed spaces from \( X \) to \( Y \), we will denote it by \( \text{Hom}_{\text{rs}}(X, Y) \) to avoid any confusion. The map (1.8.1.1) is then written as

\[
(1.8.2.1) \quad \rho : \text{Hom}_{\text{rs}}(X, Y) \longrightarrow \text{Hom}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X))
\]

and its restriction

\[
(1.8.2.2) \quad \rho' : \text{Hom}(X, Y) \longrightarrow \text{Hom}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X))
\]

is a functorial map in \( X \) and \( Y \) on the category of locally ringed spaces.

\[ \Box \]

Corollary (1.8.3). — Let \( (X, \mathcal{O}_X) \) be a locally ringed space. For \( Y \) to be an affine scheme, it is necessary and sufficient that, for each locally ringed space \( (X, \mathcal{O}_X) \), the map (1.8.2.2) be bijective.

Proof. Proposition (1.8.1) shows that the condition is necessary. Conversely, if we suppose that the condition is satisfied, and if we put \( A = \Gamma(Y, \mathcal{O}_Y) \), then it follows from the hypotheses and from (1.8.1) that the functors \( X \mapsto \text{Hom}(X, Y) \) and \( X \mapsto \text{Hom}(X, \text{Spec}(A)) \), from the category of locally ringed spaces to that of sets, are isomorphic. We know that this implies the existence of a canonical isomorphism \( X \to \text{Spec}(A) \) (cf. 0, 8).

(1.8.4). Let \( S = \text{Spec}(A) \) be an affine scheme; denote by \( (S', A') \) the ringed space whose underlying space is a point and the structure sheaf \( A' \) is the (necessarily simple) sheaf on \( S' \) defined by the ring \( A \). Let \( \pi : S \to S' \) be the unique map from \( S \) to \( S' \); on the other hand, we note that, for each open subset \( U \) of \( S \), we have a canonical map \( \Gamma(S', \mathcal{O}_{S'}) = \Gamma(S, \mathcal{O}_S) \to \Gamma(U, \mathcal{O}_S) \) which thus defines a \( \pi \)-morphism \( i : \mathcal{O}_S \to \mathcal{O}_{S'} \) of sheaves of rings. We have thus canonically defined a morphism of ringed spaces \( i = (\pi, i) : (S, \mathcal{O}_S) \to (S', A') \). For each \( A \)-module \( M \), we denote by \( M' \) the simple sheaf on \( S' \) defined by \( M \), which is evidently an \( A' \)-module. It is clear that \( i_*(\tilde{M}) = M' \) (1.3.7).

Lemma (1.8.5). — With the notation of (1.8.4), for each \( A \)-module \( M \), the canonical functorial \( \mathcal{O}_{S'} \)-homomorphism (0, 4.3.3)

\[
(1.8.5.1) \quad i^*(i_*(\tilde{M})) \longrightarrow \tilde{M}
\]

is an isomorphism.

Proof. Indeed, the two parts of (1.8.5.1) are right exact (the functor \( M \mapsto i_*(\tilde{M}) \) evidently being exact) and commute with direct sums; by considering \( M \) as the cokernel of a homomorphism \( A^{(1)} \to A^{(l)} \), we can reduce to proving the lemma for the case where \( M = A \), and it is evident in this case.

\[ \Box \]

Corollary (1.8.6). — Let \( (X, \mathcal{O}_X) \) be a ringed space, and \( u : X \to S \) a morphism of ringed spaces. For each \( A \)-module \( M \), we have (with the notation of (1.8.4)) a canonical functorial isomorphism of \( \mathcal{O}_X \)-modules

\[
(1.8.6.1) \quad u^*(\tilde{M}) \simeq u^*(i^*(M')).
\]
Corollary (1.8.7). — Under the hypotheses of (1.8.6), we have, for each $A$-module $M$ and each $\mathcal{O}_X$-module $\mathcal{F}$, a canonical isomorphism, functorial in $M$ and $\mathcal{F}$,

\[(1.8.7.1) \quad \text{Hom}_{\mathcal{O}_X}(\overline{M}, u_*(\mathcal{F})) \cong \text{Hom}_A(M, \Gamma(X, \mathcal{F})).\]

**Proof.** We have, according to (0, 4.4.3) and Lemma (1.8.5), a canonical isomorphism of bifunctors

\[\text{Hom}_{\mathcal{O}_X}(\overline{M}, u_*(\mathcal{F})) \cong \text{Hom}_{A'}(\overline{M}', i_*u_*(\mathcal{F})).\]

and it is clear that the right-hand side is exactly $\text{Hom}_A(M, \Gamma(X, \mathcal{F}))$. We note that the canonical homomorphism (1.8.7.1) sends each $\mathcal{O}_S$-homomorphism $h : \overline{M} \to u_*(\mathcal{F})$ (in other words, each $u$-morphism $\overline{M} \to \mathcal{F}$) to the $A$-homomorphism $\Gamma(h) : M \to \Gamma(S, u_*(\mathcal{F})) = \Gamma(X, \mathcal{F})$. □

(1.8.8). With the notation of (1.8.4), it is clear (0, 4.1.1) that each morphism of ringed spaces $(\psi, \theta) : X \to S'$ is equivalent to the data of a ring homomorphism $A \to \Gamma(X, \mathcal{O}_X)$. We can thus interpret Proposition (1.8.1) as defining a canonical bijection $\text{Hom}(X, S) \cong \text{Hom}(X, S')$ (where we understand that the right-hand side is the collection of morphisms of ringed spaces, since in general $A$ is not a local ring). More generally, if $X$ and $Y$ are locally ringed spaces, and if $(Y', A')$ is the ringed space whose underlying space is a point and whose sheaf of rings $A'$ is the simple sheaf defined by the ring $\Gamma(Y, \mathcal{O}_Y)$, we can interpret (1.8.2.1) as a map

\[(1.8.8.1) \quad \rho : \text{Hom}_{\mathcal{O}_S}(X, Y) \to \text{Hom}(X, Y').\]

The result of Corollary (1.8.3) is interpreted by saying that affine schemes are characterized among locally ringed spaces as those for which the restriction of $\rho$ to $\text{Hom}(X, Y)$:

\[(1.8.8.2) \quad \rho' : \text{Hom}(X, Y) \to \text{Hom}(X, Y')\]

is bijective for every locally ringed space $X$. In the following chapter, we generalize this definition, which allows us to associate to any ringed space $Z$ (and not only to a ringed space whose underlying space is a point) a locally ringed space which we will call $\text{Spec}(Z)$; this will be the starting point for a “relative” theory of preschemes over any ringed space, extending the results of Chapter I.

(1.8.9). We can consider the pairs $(X, \mathcal{F})$ consisting of a locally ringed space $X$ and an $\mathcal{O}_X$-module $\mathcal{F}$ as forming a category, a morphism in this category being a pair $(u, h)$ consisting of a morphism of locally ringed spaces $u : X \to Y$ and a $u$-morphism $h : \mathcal{F} \to \mathcal{F}$ of modules; these morphisms (for $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ fixed) form a set which we denote by $\text{Hom}((X, \mathcal{F}), (Y, \mathcal{G}))$; the map $(u, h) \mapsto (\rho'(u), \Gamma(h))$ is a canonical map

\[(1.8.9.1) \quad \text{Hom}((X, \mathcal{F}), (Y, \mathcal{G})) \to \text{Hom}((\Gamma(Y, \mathcal{O}_Y), \Gamma(Y, \mathcal{G})), (\Gamma(X, \mathcal{O}_X), \Gamma(X, \mathcal{F})))\]

functorial in $(X, \mathcal{F})$ and $(Y, \mathcal{G})$, the right-hand side being the set of di-homomorphisms corresponding to the rings and modules considered (0, 1.0.2).

Corollary (1.8.10). — Let $Y$ be a locally ringed space, and $\mathcal{G}$ an $\mathcal{O}_Y$-module. For $Y$ to be an affine scheme and $\mathcal{G}$ to be a quasi-coherent $\mathcal{O}_Y$-module, it is necessary and sufficient that, for each pair $(X, \mathcal{F})$ consisting of a locally ringed space $X$ and an $\mathcal{O}_X$-module $\mathcal{F}$, the canonical map (1.8.9.1) be bijective.

We leave the proof, which is modelled on that of (1.8.3), using (1.8.1) and (1.8.7), to the reader.

Remark (1.8.11). — The statements (1.7.3), (1.7.4), and (2.2.4) are particular cases of (1.8.1), as well as the definition in (1.6.1); similarly, (2.2.5) follows from (1.8.7). Corollary (1.8.7) also implies (1.6.3) (and, as a result, (1.6.4)) as a particular case, since if $X$ is an affine scheme and $\Gamma(X, \mathcal{F}) = \mathbb{N}$, then the functors $M \mapsto \text{Hom}_{\mathcal{O}_X}(\overline{M}, u_*(\mathbb{N}))$ and $M \mapsto \text{Hom}_{\mathcal{O}_X}(\overline{M}, (N|_{\phi})^\sim)$ (where $\phi : A \to \Gamma(X, \mathcal{O}_X)$ corresponds to $\varphi$) are isomorphic, by Corollaries (1.8.7) and (1.3.8). Finally, (1.6.5) (and, as a result, (1.6.6)) follow from (1.8.6), and the fact that, for each $f \in A$, the $A_f$-modules $N' \otimes_{A'} A_f$ and $(N' \otimes_{A'} A)_f$ (with the notation of (1.6.5)) are canonically isomorphic.
§2. PRESCHEMES AND MORPHISMS OF PRESCHEMES

2.1. Definition of preschemes

(2.1.1). Given a ringed space \((X, \mathcal{O}_X)\), we say that an open subset \(V\) of \(X\) is an affine open subset if the ringed space \((V, \mathcal{O}_X|_V)\) is an affine scheme (1.7.1).

**Definition (2.1.2).** We define a prescheme to be a ringed space \((X, \mathcal{O}_X)\) such that every point of \(X\) admits an affine open neighborhood.

**Proposition (2.1.3).** If \((X, \mathcal{O}_X)\) is a prescheme, then its affine open subsets form a basis for the topology of \(X\).

**Proof.** If \(V\) is an arbitrary open neighborhood of \(x \in X\), then there exists by hypothesis an open neighborhood \(W\) of \(x\) such that \((W, \mathcal{O}_X|_W)\) is an affine scheme; we write \(A\) to mean its ring. In the space \(W\), \(V \cap W\) is an open neighborhood of \(x\); so there exists some \(f \in A\) such that \(D(f)\) is an open neighborhood of \(x\) contained inside \(V \cap W\) (1.10, i). The ringed space \((D(f), \mathcal{O}_X|_{D(f)})\) is thus an affine scheme, isomorphic to \(A_f\) (1.3.6), whence the proposition. \(\square\)

**Proposition (2.1.4).** The underlying space of a prescheme is a Kolmogoroff space.

**Proof.** If \(x\) and \(y\) are two distinct points of a prescheme \(X\), then it is clear that there exists an open neighborhood of one of these points that does not contain the other if \(x\) and \(y\) are not in the same affine open subset; and if they are in the same affine open subset, this is a result of \((1.1.8)\). \(\square\)

**Proposition (2.1.5).** If \((X, \mathcal{O}_X)\) is a prescheme, then every closed irreducible subset of \(X\) admits exactly one generic point, and the map \(x \mapsto \{\overline{x}\}\) is thus a bijection of \(X\) onto its set of closed irreducible subsets.

**Proof.** If \(Y\) is a closed irreducible subset of \(X\) and \(y \in Y\), and if \(U\) is an affine open neighborhood of \(y\) in \(X\), then \(U \cap Y\) is dense in \(Y\), and also irreducible ((0, 2.1.1) and (0, 2.1.4)); thus, by Corollary (1.14), \(U \cap Y\) is the closure in \(U\) of a point \(x\), and so \(Y \subset U\) is the closure of \(x\) in \(X\). The uniqueness of the generic point of \(X\) is a result of Proposition (2.1.4) and of (0, 2.1.3). \(\square\)

(2.1.6). If \(Y\) is a closed irreducible subset of \(X\), and \(y\) its generic point, then the local ring \(\mathcal{O}_Y\) (also written \(\mathcal{O}_{X/Y}\)) is called the local ring of \(X\) along \(Y\), or the local ring of \(Y\) in \(X\).

If \(X\) itself is irreducible and \(x\) its generic point then we say that \(\mathcal{O}_x\) is the ring of rational functions on \(X\) (cf. §7).

**Proposition (2.1.7).** If \((X, \mathcal{O}_X)\) is a prescheme, then the ringed space \((U, \mathcal{O}_X|_U)\) is a prescheme for every open subset \(U\).

**Proof.** This follows directly from Definition (2.1.2) and Proposition (2.1.3). \(\square\)

We say that \((U, \mathcal{O}_X|_U)\) is the prescheme induced on \(U\) by \((X, \mathcal{O}_X)\), or the restriction of \((X, \mathcal{O}_X)\) to \(U\).

(2.1.8). We say that a prescheme \((X, \mathcal{O}_X)\) is irreducible (resp. connected) if the underlying space \(X\) is irreducible (resp. connected). We say that a prescheme is integral if it is irreducible and reduced (cf. (5.1.4)). We say that a prescheme \((X, \mathcal{O}_X)\) is locally integral if every \(x \in X\) admits an open neighborhood \(U\) such that the prescheme induced on \(U\) by \((X, \mathcal{O}_X)\) is integral.
2.2. Morphisms of preschemes

Definition (2.2.1). — Given two preschemes, \((X, \mathcal{O}_X)\) and \((Y, \mathcal{O}_Y)\), we define a morphism (of preschemes) from \((X, \mathcal{O}_X)\) to \((Y, \mathcal{O}_Y)\) to be a morphism of ringed spaces \((\psi, \theta)\) such that, for all \(x \in X\), \(\theta_x^\psi\) is a local homomorphism \(\mathcal{O}_y(\psi(x)) \to \mathcal{O}_x\).

By passing to quotients, the map \(\theta_{\psi(x)} \to \theta_x\) gives us a monomorphism \(\theta^X : k(\psi(x)) \to k(x)\), which lets us consider \(k(x)\) as an extension of the field \(k(\psi(x))\).

(2.2.2). The composition \((\psi'', \theta'')\) of two morphisms \((\psi, \theta), (\psi', \theta')\) of preschemes is also a morphism of preschemes, since it is given by the formula \(\theta'' = \theta' \circ \psi^*(\theta'')\) \((0,3.5.5)\). From this we conclude that preschemes form a category; using the usual notation, we will write \(\text{Hom}(X,Y)\) to mean the set of morphisms from a prescheme \(X\) to a prescheme \(Y\).

Example (2.2.3). — If \(U\) is an open subset of \(X\), then the canonical injection \((0,4.1.2)\) of the induced prescheme \((U, \mathcal{O}_X|_U)\) into \((X, \mathcal{O}_X)\) is a morphism of preschemes; it is further a monomorphism of ringed spaces and, a fortiori, a monomorphism of preschemes, which follows rapidly from \((0,4.1.1)\).

Proposition (2.2.4). — \(^4\) Let \((X, \mathcal{O}_X)\) be a prescheme, and \((S, \mathcal{O}_S)\) an affine scheme associated to a ring \(A\). Then there exists a canonical bijective correspondence between morphisms of preschemes from \((X, \mathcal{O}_X)\) to \((S, \mathcal{O}_S)\) and ring homomorphisms \((A \to \Gamma(X, \mathcal{O}_X))\) to \(\Gamma(X, \mathcal{O}_X)\).

Proof. First note that, if \((X, \mathcal{O}_X)\) and \((Y, \mathcal{O}_Y)\) are two arbitrary ringed spaces, a morphism \((\psi, \theta)\) from \((X, \mathcal{O}_X)\) to \((Y, \mathcal{O}_Y)\) canonically defines a ring homomorphism \(\Gamma(\theta) : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(Y, \psi_*(\mathcal{O}_X)) = \Gamma(X, \mathcal{O}_X)\). In the case that we consider, everything boils down to showing that any homomorphism \(\phi : A \to \Gamma(X, \mathcal{O}_X)\) is of the form \(\Gamma(\theta)\) for exactly one \(\theta\). Now, by hypothesis, there is a covering \((V_a)\) of \(X\) by affine open subsets; by composing \(\phi\) with the restriction homomorphism \(\Gamma(X, \mathcal{O}_X) \to \Gamma(V_a, \mathcal{O}_X|_V)\), we obtain a homomorphism \(\phi_a : A \to \Gamma(V_a, \mathcal{O}_X|_V)\) that corresponds to a unique morphism \((\psi_a, \theta_a)\) from the prescheme \((V_a, \mathcal{O}_X|_V)\) to \((S, \mathcal{O}_S)\), by Theorem (1.7.3). Furthermore, for each pair of indices \((a, \beta)\), each point of \(V_a \cap V_\beta\) admits an affine open neighborhood \(W\) contained inside \(V_a \cap V_\beta\) \((1.3.1)\); it is clear that, by composing \(\phi_a\) and \(\phi_\beta\) with the restriction homomorphisms to \(W\), we obtain the same homomorphism \(\Gamma(S, \mathcal{O}_S) \to \Gamma(W, \mathcal{O}_X|_W)\), so, with the notation \((\theta_a^\beta)\) for all \(x \in V_a\) and all \(a\) \((1.6.1)\), the restriction to \(W\) of the morphisms \((\psi_a, \theta_a)\) and \((\psi_\beta, \theta_\beta)\) coincide. From this we conclude that there is a morphism \((\psi, \theta) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)\) of ringed spaces, and only one such that its restriction to each \(V_a\) is \((\psi_a, \theta_a)\), and it is clear that this morphism is a morphism of preschemes and such that \(\Gamma(\theta) = \phi\).

Let \(u : A \to \Gamma(X, \mathcal{O}_X)\) be a ring homomorphism, and \(v = (\psi, \theta)\) the corresponding morphism \((X, \mathcal{O}_X) \to (S, \mathcal{O}_S)\). For each \(f \in A\), we have that

\[\psi^{-1}(D(f)) = X_u(f)\]

with the notation of \((0, 5.5.2)\) relative to the locally free sheaf \(\mathcal{O}_X\). In fact, it suffices to verify this formula when \(X\) itself is affine, and then this is nothing but \((1.2.2)\).

Proposition (2.2.5). — Under the hypotheses of Proposition (2.2.4), let \(\phi : A \to \Gamma(X, \mathcal{O}_X)\) be a ring homomorphism, \(f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)\) the corresponding morphism of preschemes, \(\mathcal{F}\) (resp. \(\mathcal{F}\)) an \(\mathcal{O}_X\)-module (resp. \(\mathcal{O}_S\)-module), and \(M = \Gamma(S, \mathcal{F})\). Then there exists a canonical bijective correspondence between \(f\)-morphisms \(\mathcal{F} \to \mathcal{G}\) \((0, 4.4.1)\) and \(A\)-homomorphisms \(M \to (\Gamma(X, \mathcal{G}))\).

Proof. Reasoning as in Proposition (2.2.4), we reduce to the case where \(X\) is affine, and the proposition then follows from Proposition \((1.6.3)\) and from Corollary \((1.3.8)\).

(2.2.6). We say that a morphism of preschemes \((\psi, \theta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\) is open (resp. closed) if, for all open subsets \(U\) of \(X\) (resp. all closed subsets \(F\) of \(X\)), \(\psi(U)\) is open (resp. \(\psi(F)\) is closed) in \(Y\), we say that \((\psi, \theta)\) is dominant if \(\psi(X)\) is dense in \(Y\), and surjective if \(\psi\) is surjective. We note that these conditions rely only on the continuous map \(\psi\).

\(^4\) Trans. See (1.8) and the footnote there.
Proposition (2.2.7). — Let
\[ f = (\psi, \theta) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y) \]
and
\[ g = (\psi', \theta') : (Y, \mathcal{O}_Y) \rightarrow (Z, \mathcal{O}_Z) \]
be morphisms of preschemes.

(i) If \( f \) and \( g \) are both open (resp. closed, dominant, surjective), then so is \( g \circ f \).
(ii) If \( f \) is surjective and \( g \circ f \) closed, then \( g \) is closed.
(iii) If \( g \circ f \) is surjective, then \( g \) is surjective.

**Proof.** Claims (i) and (iii) are evident. Write \( g \circ f = (\psi'', \theta'') \). If \( F \) is closed in \( Y \) then \( \psi^{-1}(F) \) is closed in \( X \), so \( \psi''(\psi^{-1}(F)) \) is closed in \( Z \); but since \( \psi \) is surjective, \( \psi(\psi^{-1}(F)) = F \), so \( \psi''(\psi^{-1}(F)) = \psi'(F) \), which proves (ii). \( \square \)

Proposition (2.2.8). — Let \( f = (\psi, \theta) \) be a morphism \( (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y) \), and \( (U_a) \) an open cover of \( Y \). For \( f \) to be open (resp. closed, surjective, dominant), it is necessary and sufficient for its restriction to each induced prescheme \( (\psi^{-1}(U_a), \mathcal{O}_X|_{\psi^{-1}(U_a)}) \), considered as a morphism of preschemes from this induced prescheme to the induced prescheme \( (U_a, \mathcal{O}_Y|_{U_a}) \) to be open (resp. closed, surjective, dominant).

**Proof.** The proposition follows immediately from the definitions, taking into account the fact that a subset \( F \) of \( Y \) is closed (resp. open, dense) in \( Y \) if and only if each of the \( F \cap U_a \) are closed (resp. open, dense) in \( U_a \).

(2.2.9). Let \( (X, \mathcal{O}_X) \) and \( (Y, \mathcal{O}_Y) \) be two preschemes; suppose that \( X \) (resp. \( Y \)) has a finite number of irreducible components \( X_i \) (resp. \( Y_j \)) \((1 \leq i \leq n)\); let \( \xi_i \) (resp. \( \eta_j \)) be the generic point of \( X_i \) (resp. \( Y_j \)) \((2.1.5)\). We say that a morphism
\[ f = (\psi, \theta) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y) \]
is birational if, for all \( i \), \( \psi^{-1}(\eta_j) = \{ \xi_i \} \) and \( \theta^{\sharp}_{\xi_i} : \mathcal{O}_{\eta_j} \rightarrow \mathcal{O}_{\xi_i} \) is an isomorphism. It is clear that a birational morphism is dominant \((0, 2.1.8)\), and thus it is surjective if it is also closed.

Notation (2.2.10). — In all that follows, when there is no risk of confusion, we suppress the structure sheaf (resp. the morphism of structure sheaves) from the notation of a prescheme (resp. morphism of preschemes). If \( U \) is an open subset of the underlying space \( X \) of a prescheme, then whenever we speak of \( U \) as a prescheme we always mean the induced prescheme on \( U \).

2.3. Gluing preschemes

(2.3.1). It follows from Definition (2.1.2) that every ringed space obtained by gluing preschemes \((0, 4.1.7)\) is again a prescheme. In particular, although every prescheme admits, by definition, a cover by affine open sets, we see that every prescheme can actually be obtained by gluing affine schemes.

Example (2.3.2). — Let \( K \) be a field, \( B = K[s] \) and \( C = K[t] \) polynomial rings in one indeterminate over \( K \), and define \( X_1 = \text{Spec}(B) \) and \( X_2 = \text{Spec}(C) \), which are isomorphic affine schemes. In \( X_1 \) (resp. \( X_2 \)), let \( U_{12} \) (resp. \( U_{21} \)) be the affine open \( D(s) \) (resp. \( D(t) \)) where the ring \( B_s \) (resp. \( C_t \)) is formed of rational fractions of the form \( f(s)/s^n \) (resp. \( g(t)/t^n \)) with \( f \in B \) (resp. \( g \in C \)). Let \( u_{12} \) be the isomorphism of preschemes \( U_{21} \rightarrow U_{12} \) corresponding \((2.2.4)\) to the isomorphism from \( B_s \) to \( C_t \) that, to \( f(s)/s^n \), associates the rational fraction \( f(1/t)/(1/t^n) \). We can glue \( X_1 \) and \( X_2 \) along \( U_{12} \) and \( U_{21} \) by using \( u_{12} \), because there is clearly no gluing condition. We later show that the prescheme \( X \) obtained in this manner is a particular case of a general method of construction \((11, 2.4.3)\). Here we show only that \( X \) is not an affine scheme; this will follow from the fact that the ring \( \Gamma(X, \mathcal{O}_X) \) is isomorphic to \( K \), and so its spectrum reduces to a point. Indeed, a section of \( \mathcal{O}_X \) over \( X \) has a restriction over \( X_1 \) (resp. \( X_2 \)), identified with an affine open of \( X \), that is a polynomial \( f(s) \) (resp. \( g(t) \)), and it follows from the definitions that we should have \( g(t) = f(1/t) \), which is only possible if \( f = g \in K \).
2.4. Local schemes

(2.4.1). We say that an affine scheme is a local scheme if it is the affine scheme associated to a local ring \( A \); there then exists, in \( X = \text{Spec}(A) \), a single closed point \( a \in X \), and for all other \( b \in X \) we have that \( a \in \overline{\{ b \}} \) (1.1.7).

For all preschemes \( Y \) and points \( y \in Y \), the local scheme \( \text{Spec}(\mathcal{O}_y) \) is called the local scheme of \( Y \) at the point \( y \). Let \( V \) be an affine open subset of \( Y \) containing \( y \), and \( B \) the ring of the affine scheme \( V \); then \( \mathcal{O}_y \) is canonically identified with \( B_y \) (1.3.4), and the canonical homomorphism \( B \rightarrow B_y \) thus corresponds (1.6.1) to a morphism of preschemes \( \text{Spec}(\mathcal{O}_y) \rightarrow V \). If we compose this morphism with the canonical injection \( V \rightarrow Y \), then we obtain a morphism \( \text{Spec}(\mathcal{O}_y) \rightarrow Y \) which is independent of the affine open subset \( V \) (containing \( y \)) that we chose: indeed, if \( V' \) is some other affine open subset containing \( y \), then there exists a third affine open subset \( W \) that contains \( y \) and is such that \( W \subset V \cap V' \) (2.1.3); we can thus assume that \( V \subset V' \), and then if \( B' \) is the ring of \( V' \), so everything relies on remarking that the diagram

\[
\begin{array}{ccc}
B' & \rightarrow & B \\
\downarrow & & \downarrow \\
\mathcal{O}_y & \rightarrow & \mathcal{O}_y
\end{array}
\]

is commutative (0, 1.5.1). The morphism

\[ \text{Spec}(\mathcal{O}_y) \rightarrow Y \]

thus defined is said to be canonical.

Proposition (2.4.2). — Let \( (Y, \mathcal{O}_Y) \) be a prescheme; for all \( y \in Y \), let \( (\psi, \theta) \) be the canonical morphism \( (\text{Spec}(\mathcal{O}_y), \mathcal{O}_y) \rightarrow (Y, \mathcal{O}_Y) \). Then \( \psi \) is a homeomorphism from \( \text{Spec}(\mathcal{O}_y) \) to the subspace \( S_y \) of \( Y \) given by the \( z \) such that \( y \in \overline{\{ z \}} \) (or, equivalently, the generalizations of \( y \) (0, 2.1.2)); furthermore, if \( z = \psi(p) \), then \( \mathcal{O}_z : \mathcal{O}_z \rightarrow (\mathcal{O}_y)_p \) is an isomorphism; \( (\psi, \theta) \) is thus a monomorphism of ringed spaces.

Proof. Since the unique closed point \( a \) of \( \text{Spec}(\mathcal{O}_y) \) is contained in the closure of any point of this space, and since \( \psi(a) = y \), the image of \( \text{Spec}(\mathcal{O}_y) \) under the continuous map \( \psi \) is contained in \( S_y \). Since \( S_y \) is contained in every affine open containing \( y \), one can consider just the case where \( Y \) is an affine scheme; but then this proposition follows from (1.6.2). \( \square \)

We see (2.1.5) that there is a bijective correspondence between \( \text{Spec}(\mathcal{O}_y) \) and the set of closed irreducible subsets of \( Y \) containing \( y \).

Corollary (2.4.3). — For \( y \in Y \) to be the generic point of an irreducible component of \( Y \), it is necessary and sufficient for the only prime ideal of the local ring \( \mathcal{O}_y \) to be its maximal ideal (in other words, for \( \mathcal{O}_y \) to be of dimension zero).

Proposition (2.4.4). — Let \( (X, \mathcal{O}_X) \) be a local scheme of some ring \( A \), its unique closed point, and \( (Y, \mathcal{O}_Y) \) a prescheme. Every morphism \( u = (\psi, \theta) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y) \) then factors uniquely as \( X \rightarrow \text{Spec}(\mathcal{O}_{\psi(a)}) \rightarrow Y \), where the second arrow denotes the canonical morphism, and the first corresponds to a local homomorphism \( \mathcal{O}_a \rightarrow A \). This establishes a canonical bijective correspondence between the set of morphisms \( (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y) \) and the set of local homomorphisms \( \mathcal{O}_a \rightarrow A \) for \( (y \in Y) \).

Indeed, for all \( x \in X \), we have that \( a \in \overline{\{ x \}} \), so \( \psi(a) \in \overline{\{ \psi(x) \}} \), which shows that \( \psi(X) \) is contained in every affine open subset that contains \( \psi(a) \). So it suffices to consider the case where \( (Y, \mathcal{O}_Y) \) is an affine scheme of ring \( B \), and then we have that \( u = (\phi, \phi) \), where \( \phi \in \text{Hom}(B, A) \) (1.7.3). Further, we have that \( \phi^{-1}(1_b) = 1_{\mathcal{O}(a)} \), and hence that the image under \( \phi \) of any element of \( B - 1_{\mathcal{O}(a)} \) is invertible in the local ring \( A \); the factorization in the result follows from the universal property of the ring of fractions (0, 1.2.4). Conversely, to each local homomorphism \( \mathcal{O}_a \rightarrow A \) there is a unique corresponding morphism \( (\psi, \theta) : X \rightarrow \text{Spec}(\mathcal{O}_y) \) such that \( \psi(a) = y \) (1.7.3), and, by composing with the canonical morphism \( \text{Spec}(\mathcal{O}_y) \rightarrow Y \), we obtain a morphism \( X \rightarrow Y \), which proves the proposition.
The affine schemes whose ring is a field $K$ have an underlying space that is just a point. If $A$ is a local ring with maximal ideal $m$, then each local homomorphism $A \to K$ has kernel equal to $m$, and so factors as $A \to A/m \to K$, where the second arrow is a monomorphism. The morphisms $\text{Spec}(K) \to \text{Spec}(A)$ thus correspond bijectively to monomorphisms of fields $A/m \to K$.

Let $(Y, O_Y)$ be a prescheme; for each $y \in Y$ and each ideal $a_y$ of $O_y$, the canonical homomorphism $O_y \to O_y/a_y$ defines a morphism $\text{Spec}(O_y/a_y) \to \text{Spec}(O_y)$; if we compose this with the canonical morphism $\text{Spec}(O_y) \to Y$, then we obtain a morphism $\text{Spec}(O_y/a_y) \to Y$, again said to be canonical. For $a_y = m_y$, this says that $O_y/a_y = k(y)$, and so Proposition (2.4.4) says that:

Corollary (2.4.6). — Let $(X, O_X)$ be a local scheme whose ring $K$ is a field, $\xi$ the unique point of $X$, and $(Y, O_Y)$ a prescheme. Then each morphism $u : (X, O_X) \to (Y, O_Y)$ factors uniquely as $X \to \text{Spec}(k(\psi(\xi))) \to Y$, where the second arrow denotes the canonical morphism, and the first corresponds to a monomorphism $k(\psi(\xi)) \to K$. This establishes a canonical bijective correspondence between the set of morphisms $(X, O_X) \to (Y, O_Y)$ and the set of monomorphisms $k(y) \to K$ (for $y \in Y$).

Corollary (2.4.7). — For all $y \in Y$, every canonical morphism $\text{Spec}(O_y/a_y) \to Y$ is a monomorphism of ringed spaces.

Proof. We have already seen this when $a_y = 0$ (2.4.2), and it suffices to apply Corollary (1.75). □

Remark. — 2.4.8 Let $X$ be a local scheme, and $a$ its unique closed point. Since every affine open subset containing $a$ is necessarily equal to the whole of $X$, every invertible $O_X$-module (0, 5.4.1) is necessarily isomorphic to $O_X$ (or, as we say, again, trivial). This property does not hold in general for an arbitrary affine scheme $\text{Spec}(A)$; we will see in Chapter V that if $A$ is a normal ring then this is true when $A$ is a unique factorisation domain.

2.5. Preschemes over a prescheme

Definition (2.5.1). — Given a prescheme $S$, we say that the data of a prescheme $X$ and a morphism of preschemes $\phi : X \to S$ defines a prescheme $X$ over the prescheme $S$, or an $S$-prescheme; we say that $S$ is the base prescheme of the $S$-prescheme $X$. The morphism $\phi$ is called the structure morphism of the $S$-prescheme $X$. When $S$ is an affine scheme of ring $A$, we also say that $X$ endowed with $\phi$ is a prescheme over the ring $A$ (or an $A$-prescheme).

It follows from (2.2.4) that the data of a prescheme over a ring $A$ is equivalent to the data of a prescheme $(X, O_X)$ whose structure sheaf $O_X$ is a sheaf of $A$-algebras. An arbitrary prescheme can always be considered as a $\mathbf{Z}$-scheme in a unique way.

If $\phi : X \to S$ is the structure morphism of an $S$-prescheme $X$, we say that a point $x \in X$ is over a point $s \in S$ if $\phi(x) = s$. We say that $X$ dominates $S$ if $\phi$ is a dominant morphism (2.2.6).

(2.5.2). Let $X$ and $Y$ be $S$-preschemes; we say that a morphism of preschemes $u : X \to Y$ is a morphism of preschemes over $S$ (or an $S$-morphism) if the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow & & \downarrow \\
S & & \end{array}
\]

(where the diagonal arrows are the structure morphisms) is commutative: this ensures that, for all $s \in S$ and $x \in X$ over $s$, $u(x)$ also lies over $s$.

It follows immediately from this definition that the composition of any two $S$-morphisms is an $S$-morphism; $S$-preschemes thus form a category.

We denote by $\text{Hom}_S(X, Y)$ the set of $S$-morphisms from an $S$-prescheme $X$ to an $S$-prescheme $Y$; the identity morphism of an $S$-prescheme $X$ is denoted by $1_X$.

When $S$ is an affine scheme of ring $A$, we will also say $A$-morphism instead of $S$-morphism.

(2.5.3). If $X$ is an $S$-prescheme, and $v : X' \to X$ a morphism of preschemes, then the composition $X' \to X \to S$ endows $X'$ with the structure of an $S$-prescheme; in particular, every prescheme induced by an open set $U$ of $X$ can be considered as an $S$-prescheme by the canonical injection.
If \( u : X \to Y \) is an \( S \)-morphism of \( S \)-preschemes, then the restriction of \( u \) to any prescheme induced by an open subset \( U \) of \( X \) is also an \( S \)-morphism \( U \to Y \). Conversely, let \( (U_{\alpha}) \) be an open cover of \( X \), and for each \( \alpha \) let \( u_{\alpha} : U_{\alpha} \to Y \) be an \( S \)-morphism; if, for all pairs of indices \( (\alpha, \beta) \), the restrictions of \( u_{\alpha} \) and \( u_{\beta} \) to \( U_{\alpha} \cap U_{\beta} \) agree, then there exists an \( S \)-morphism \( X \to Y \), and exactly one such that the restriction to each \( U_{\alpha} \) is \( u_{\alpha} \).

If \( u : X \to Y \) is an \( S \)-morphism such that \( u(X) \subseteq V \), where \( V \) is an open subset of \( Y \), then \( u \), considered as a morphism from \( X \) to \( V \), is also an \( S \)-morphism.

\[ (2.5.4) \text{ Let } S' \to S \text{ be a morphism of preschemes; for all } S'-\text{preschemes, the composition } X \to S' \to S \text{ endows } X \text{ with the structure of an } S \text{-prescheme. Conversely, suppose that } S' \text{ is the induced prescheme of an open subset of } S; \text{ let } X \text{ be an } S \text{-prescheme and suppose that the structure morphism } f : X \to S \text{ is such that } f(X) \subseteq S'; \text{ then we can consider } X \text{ as an } S' \text{-prescheme. In this latter case, if } Y \text{ is another } S \text{-prescheme whose structure morphism sends the underlying space to } S', \text{ then every } S\text{-morphism from } X \text{ to } Y \text{ is also an } S' \text{-morphism.} \]

\[ (2.5.5) \text{ If } X \text{ is an } S \text{-prescheme, with structure morphism } \phi : X \to S, \text{ we define an } S \text{-section of } X \text{ to be an } S \text{-morphism from } S \text{ to } X, \text{ that is to say a morphism of preschemes } \psi : S \to X \text{ such that } \phi \circ \psi \text{ is the identity on } S. \text{ We denote by } \Gamma(X/S) \text{ the set of } S \text{-sections of } X. \]

### §3. PRODUCTS OF PRESCHEMES

#### 3.1. Sums of preschemes

Let \( (X_\alpha) \) be any family of preschemes; let \( X \) be a topological space which is the sum of the underlying spaces \( X_\alpha \); \( X \) is then the union of pairwise disjoint open subspaces \( X'_\alpha \), and for each \( \alpha \) there is a homomorphism \( \phi_\alpha \) from \( X_\alpha \) to \( X'_\alpha \). If we equip each of the \( X'_\alpha \) with the sheaf \( (\phi_\alpha)_*(\mathcal{O}_{X_\alpha}) \), it is clear that \( X \) becomes a prescheme, which we call the sum of the family of preschemes \( (X_\alpha) \) and which we denote \( \coprod X_\alpha \). If \( Y \) is a prescheme, then the map \( f \mapsto (f \circ \phi_\alpha) \) is a bijection from the set \( \text{Hom}(X, Y) \) to the product set \( \prod \text{Hom}(X_\alpha, Y) \). In particular, if the \( X_\alpha \) are \( S \)-preschemes, with structure morphisms \( \psi_\alpha \), then \( X \) is an \( S \)-prescheme by the unique morphism \( \psi : X \to S \) such that \( \psi \circ \phi_\alpha = \psi_\alpha \) for each \( \alpha \). The sum of two preschemes \( X \) and \( Y \) is denoted by \( X \sqcup Y \). It is immediate that, if \( X = \text{Spec}(A) \) and \( Y = \text{Spec}(B) \), then \( X \sqcup Y \) is canonically identified with \( \text{Spec}(A \times B) \).

#### 3.2. Products of preschemes

**Definition (3.2.1).** — Given \( S \)-preschemes \( X \) and \( Y \), we say that a triple \( (Z, p_1, p_2) \), consisting of an \( S \)-prescheme \( Z \), and \( S \)-morphisms \( p_1 : Z \to X \) and \( p_2 : Z \to Y \), is a product of the \( S \)-preschemes \( X \) and \( Y \), if, for each \( S \)-prescheme \( T \), the map \( f \mapsto (p_1 \circ f, p_2 \circ f) \) is a bijection from the set of \( S \)-morphisms from \( T \) to \( Z \), to the set of pairs consisting of an \( S \)-morphism \( T \to X \) and an \( S \)-morphism \( T \to Y \) (in other words, a bijection

\[ \text{Hom}_S(T, Z) \simeq \text{Hom}_S(T, X) \times \text{Hom}_S(T, Y). \]

This is the general notion of a product of two objects in a category, applied to the category of \( S \)-preschemes (\( T, I, 1.1 \)); in particular, a product of two \( S \)-preschemes is unique up to a unique \( S \)-isomorphism. Because of this uniqueness, most of the time we will denote a product of two \( S \)-preschemes \( X \) and \( Y \) by \( X \times_S Y \) (or simply \( X \times Y \), when there is no chance of confusion), with the morphisms \( p_1 \) and \( p_2 \) (the canonical projections of \( X \times_S Y \) to \( X \) and to \( Y \), respectively) being suppressed in the notation. If \( g : T \to X \) and \( h : T \to Y \) are \( S \)-morphisms, we denote by \( (g, h)_S \), or simply \( (g, h) \), the \( S \)-morphism \( f : T \to X \times_S Y \) such that \( p_1 \circ f = g \) and \( p_2 \circ f = h \). If \( X' \) and \( Y' \) are two \( S \)-preschemes, \( p'_1 \) and \( p'_2 \) the canonical projections of \( X' \times_S Y' \) (assumed to exist), and \( u : X' \to X \) and \( v : Y' \to Y \) \( S \)-morphisms, then we write \( u \times_S v \) (or simply \( u \times v \)) for the \( S \)-morphism \( (u \circ p'_1, v \circ p'_2)_S \) from \( X' \times_S Y' \) to \( X \times_S Y \).

When \( S \) is an affine scheme given by some ring \( A \), we often replace \( S \) by \( A \) in the above notation.

**Proposition (3.2.2).** — Let \( X, Y, \) and \( S \) be affine schemes, given by rings \( B, C, \) and \( A \) (respectively). Let \( Z = \text{Spec}(B \otimes_A C) \), and let \( p_1 \) and \( p_2 \) be the \( S \)-morphisms corresponding to \( (2.2.4) \) to the canonical \( A \)-homomorphisms \( u : b \mapsto b \otimes 1 \) and \( v : c \mapsto 1 \otimes c \) (respectively) from \( B \) and \( C \) to \( B \otimes_A C \); then \( (Z, p_1, p_2) \) is a product of \( X \) and \( Y \).
Proof. According to (2.2.4), it suffices to check that, if, to each \(A\)-homomorphism \(f : B \otimes_A C \to L\) (where \(L\) is an \(A\)-algebra), we associate the pair \((f \circ u, f \circ v)\), then this defines a bijection \(\text{Hom}_A(B \otimes_A C, L) \cong \text{Hom}_A(B, L) \times \text{Hom}_A(C, L)\), which follows immediately from the definitions and the fact that \(b \otimes c = (b \otimes 1)(1 \otimes c)\).

\[\Box\]

Corollary (3.2.3). — Let \(T\) be an affine scheme given by some ring \(D\), and \(a = (\xi, \bar{a})\) (resp. \(b = (\eta, \bar{b})\)) an \(S\)-morphism \(T \to X\) (resp. \(T \to Y\)), where \(\xi\) (resp. \(\eta\)) is an \(S\)-homomorphism from \(B\) (resp. \(C\)) to \(D\); then \((a, b)_{S} = (\xi, \eta, \bar{a}, \bar{b})\), where \(\bar{a}\) is the \(S\)-homomorphism \(B \otimes_A C \to D\) such that \(\tau(b \otimes c) = \rho(b) \circ \sigma(c)\).

Proposition (3.2.4). — Let \(f : S' \to S\) be a monomorphism of preschemes \((T, I, 1.1)\), and let \(X\) and \(Y\) be \(S'\)-preschemes, also considered as \(S\)-preschemes via \(f\). Every product of the \(S\)-preschemes \(X\) and \(Y\) is then a product of the \(S'\)-preschemes \(X\) and \(Y\), and vice versa.

Proof. Let \(\phi : X \to S'\) and \(\psi : Y \to S'\) be the structure morphisms. If \(T\) is an \(S\)-scheme, and \(u : T \to X\) and \(v : T \to Y\) are \(S\)-morphisms, then we have, by definition, that \(f \circ \phi \circ u = f \circ \psi \circ v = \theta\), the structure morphism of \(T\); the hypotheses on \(f\) imply that \(\phi \circ u = \psi \circ v = \theta'\), so we can consider \(T\) as an \(S'\)-prescheme with structure morphism \(\theta'\), and \(u\) and \(v\) as \(S'\)-morphisms. The conclusion of the proposition follows immediately, taking (3.2.1) into account.

\[\Box\]

Corollary (3.2.5). — Let \(X\) and \(Y\) be \(S\)-preschemes, with structure morphisms \(\phi : X \to S\) and \(\psi : Y \to S\), and let \(S'\) be an open subset of \(S\) such that \(\phi(X) \subset S'\) and \(\psi(Y) \subset S'\). Every product of the \(S\)-preschemes \(X\) and \(Y\) is then also a product of the \(S'\)-preschemes \(X\) and \(Y\), and conversely.

It suffices to apply (3.2.4) to the canonical injection \(S' \to S\).

Theorem (3.2.6). — Given \(S\)-preschemes \(X\) and \(Y\), there exists a product \(X \times_S Y\).

The proof proceeds in several steps.

Lemma (3.2.6.1). — Let \((Z, p, q)\) be a product of \(X\) and \(Y\), and \(U\) and \(V\) open subsets of \(X\) and \(Y\), respectively. If we let \(W = p^{-1}(U) \cap q^{-1}(V)\), then the triple consisting of \(W\) and the restrictions of \(p\) and \(q\) to \(W\) (considered as the morphisms \(W \to U\) and \(W \to V\), respectively) is a product of \(U\) and \(V\).

Indeed, if \(T\) is an \(S\)-scheme, then we can identify the \(S\)-morphisms \(T \to W\) with the \(S\)-morphisms \(T \to Z\) mapping \(T\) to \(W\). Then, if \(g : T \to U\) and \(h : T \to V\) are any two \(S\)-morphisms, we can consider them as \(S\)-morphisms from \(T\) to \(X\) and to \(Y\), respectively, and, by hypothesis, there is then a unique \(S\)-morphism \(f : T \to Z\) such that \(g = p \circ f\) and \(h = q \circ f\). Since \(p(f(Y)) \subset U\), \(q(f(T)) \subset V\), we have

\[f(T) \subset p^{-1}(U) \cap q^{-1}(V) = W,
\]
hence our claim.

Lemma (3.2.6.2). — Let \(Z\) be an \(S\)-prescheme, \(p : Z \to X\) and \(q : Z \to Y\) both \(S\)-morphisms, \((U_{\alpha})\) an open cover of \(X\), and \((V_{\lambda})\) an open cover of \(Y\). Suppose that, for each pair \((\alpha, \lambda)\), the \(S\)-prescheme \(W_{\alpha\lambda} = p^{-1}(U_{\alpha}) \cap q^{-1}(V_{\lambda})\) and the restrictions of \(p\) and \(q\) to \(W_{\alpha\lambda}\) form a product of \(U_{\alpha}\) and \(V_{\lambda}\). Then \((Z, p, q)\) is a product of \(X\) and \(Y\).

We first show that, if \(f_{1}\) and \(f_{2}\) are \(S\)-morphisms \(T \to Z\), then the equations \(p \circ f_{1} = p \circ f_{2}\) and \(q \circ f_{1} = q \circ f_{2}\) imply that \(f_{1} = f_{2}\). Indeed, \(Z\) is the union of the \(W_{\alpha\lambda}\), so the \(f_{1\alpha\lambda}^{-1}(W_{\alpha\lambda})\) form an open cover of \(T\), and similarly for \(f_{2\alpha\lambda}^{-1}(W_{\alpha\lambda})\). In addition, we have

\[f_{1\alpha\lambda}^{-1}(W_{\alpha\lambda}) = f_{2\alpha\lambda}^{-1}(p^{-1}(U_{\alpha})) \cap f_{2\alpha\lambda}^{-1}(q^{-1}(V_{\lambda})) = f_{2\alpha\lambda}^{-1}(p^{-1}(U_{\alpha})) \cap f_{2\alpha\lambda}^{-1}(q^{-1}(V_{\lambda})) = f_{2\alpha\lambda}^{-1}(W_{\alpha\lambda})\]

by hypothesis, and it thus reduces to noting that the the restrictions of \(f_{1}\) and \(f_{2}\) to \(f_{1\alpha\lambda}^{-1}(W_{\alpha\lambda}) = f_{2\alpha\lambda}^{-1}(W_{\alpha\lambda})\) are identical for each pair of indices. But since these restrictions can be considered as \(S\)-morphisms from \(f_{1\alpha\lambda}^{-1}(W_{\alpha\lambda})\) to \(W_{\alpha\lambda}\), our claim follows from the hypotheses and Definition (3.2.1).

Suppose now that we are given \(S\)-morphisms \(g : T \to X\) and \(h : T \to Y\). Let \(T_{\alpha\lambda} = g^{-1}(U_{\alpha}) \cap h^{-1}(V_{\lambda})\); then the \(T_{\alpha\lambda}\) form an open cover of \(T\). By hypothesis, there exists an \(S\)-morphism \(f_{\alpha\lambda}\) such that \(p \circ f_{\alpha\lambda}\) and \(q \circ f_{\alpha\lambda}\) are the restrictions of \(g\) and \(h\) to \(T_{\alpha\lambda}\) (respectively). Now, we will show that

\[\text{The notation } \text{Hom}_A \text{ denotes here the set of homomorphisms of } A\text{-algebras.}\]
the restrictions of \( f_{\alpha \lambda} \) and \( f_{\beta \mu} \) to \( T_{\alpha \lambda} \cap T_{\beta \mu} \) coincide, which will finish the proof of Lemma (3.2.6.2). The images of \( T_{\alpha \lambda} \cap T_{\beta \mu} \) under \( f_{\alpha \lambda} \) and \( f_{\beta \mu} \) are contained in \( W_{\alpha \lambda} \cap W_{\beta \mu} \) by definition. Since
\[
W_{\alpha \lambda} \cap W_{\beta \mu} = p^{-1}(U_\alpha \cap U_\beta) \cap q^{-1}(V_\lambda \cap V_\mu),
\]
it follows from Lemma (3.2.6.1) that \( W_{\alpha \lambda} \cap W_{\beta \mu} \) and the restrictions to this prescheme of \( p \) and \( q \) form a product of \( U_\alpha \cap U_\beta \) and \( V_\lambda \cap V_\mu \). Since \( p \circ f_{\alpha \lambda} \) and \( p \circ f_{\beta \mu} \) coincide on \( T_{\alpha \lambda} \cap T_{\beta \mu} \) and similarly for \( q \circ f_{\alpha \lambda} \) and \( q \circ f_{\beta \mu} \), we see that \( f_{\alpha \lambda} \) and \( f_{\beta \mu} \) coincide on \( T_{\alpha \lambda} \cap T_{\beta \mu} \).

**Lemma (3.2.6.3).** — Let \((U_\alpha)\) be an open cover of \( X \), \((V_\lambda)\) an open cover of \( Y \), and suppose that, for each pair \((\alpha, \lambda)\), there exists a product of \( U_\alpha \) and \( V_\lambda \); then there exists a product of \( X \) and \( Y \).

Applying Lemma (3.2.6.1) to the open sets \( U_\alpha \cap U_\beta \) and \( V_\lambda \cap V_\mu \), we see that there exists a product of \( S \)-preschemes induced, respectively, by \( X \) and \( Y \) on these open sets; in addition, the uniqueness of the product shows that, if we set \( i = (\alpha, \lambda) \) and \( j = (\beta, \mu) \), there is a canonical isomorphism \( h_{ij} \) (resp. \( h_{ji} \)) from this product to an \( S \)-prescheme \( W_{ij} \) (resp. \( W_{ji} \)) induced by \( U_\alpha \times_S V_\lambda \) (resp. \( U_\beta \times_S V_\mu \)) on open sets; then \( f_{ij} = h_{ij} \circ h_{ji}^{-1} \) is an isomorphism from \( W_{ji} \) to \( W_{ij} \). In addition, for a third pair \( k = (\gamma, \nu) \), we have \( f_{jk} = f_{ij} \circ f_{ik} \) on \( W_{ki} \cap W_{kj} \), by applying Lemma (3.2.6.1) to the open sets \( U_\alpha \cap U_\gamma \cap U_\beta \) and \( V_\lambda \cap V_\nu \cap V_\mu \) in \( U_\alpha \cap U_\beta \) and \( V_\lambda \cap V_\mu \), respectively. It follows that we have a prescheme \( Z \), an open cover \( (Z_\alpha) \) of the underlying space of \( Z \), and, for each \( i \), an isomorphism \( g_i \) from the induced prescheme \( Z_i \) to the prescheme \( U_\alpha \times_S V_\lambda \), so that, for each pair \((i, j)\), we have \( f_{ij} = g_i \circ g_j^{-1} \) (2.3.1); in addition, we have \( g_i(Z_i \cap Z_j) = W_{ij} \). If \( p_\nu \), \( q_\mu \), and \( \theta_i \) are the projections and the structure morphism of the \( S \)-prescheme \( U_\alpha \times_S V_\lambda \) (respectively), we immediately see that \( p_\nu \circ g_i = p_\nu \circ g_j \) on \( Z_\alpha \cap Z_\beta \), and similarly for the two other morphisms. We can thus define the morphisms \( p : Z \to X \) (resp. \( q : Z \to Y \), \( \theta : Z \to S \)) by the condition that \( p \) (resp. \( q \), \( \theta \)) coincide with \( p_\nu \circ g_i \) (resp. \( q_\mu \circ g_i \), \( \theta_i \circ g_i \)) on each of the \( Z_\alpha \); \( Z \), equipped with \( \theta \), is then an \( S \)-prescheme. We now show that \( Z_j' = p^{-1}(U_\alpha) \cap q^{-1}(V_\lambda) \) is equal to \( Z_j \). For each index \( j = (\beta, \mu) \), we have \( Z_j \cap Z_j' = g_j^{-1}(p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda)) \). We have, by Lemma (3.2.6.1),
\[
p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda) = p_j^{-1}(U_\alpha \cap U_\beta) \cap q_j^{-1}(V_\lambda \cap V_\mu);
\]
with the restrictions of \( p_j \) and \( q_j \) to \( p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda) \) defining, on this \( S \)-prescheme, the structure of a product of \( U_\alpha \cap U_\beta \) and \( V_\lambda \cap V_\mu \), but the uniqueness of the product then implies that \( p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda) = W_{ji} \). As a result, we have \( Z_j \cap Z_j' = Z_j \cap Z_j' \) for each \( j \), hence \( Z_j' = Z_j \). We then deduce from Lemma (3.2.6.2) that \((Z, p, q)\) is a product of \( X \) and \( Y \).

**Lemma (3.2.6.4).** — Let \( \phi : X \to S \) and \( \psi : Y \to S \) be the structure morphisms of \( X \) and \( Y \), \((S_i)\) an open cover of \( S \), and let \( X_i = \phi^{-1}(S_i) \), \( Y_i = \psi^{-1}(S_i) \). If each of the products \( X_i \times_S Y_i \) exists, then \( X \times_S Y \) exists.

According to Lemma (3.2.6.3), everything follows from proving that the products \( X_i \times_S Y_i \) exist, and if \( i \) and \( j \). Set \( X_{ij} = X_i \cap X_j = \phi^{-1}(S_i \cap S_j) \), \( Y_{ij} = Y_i \cap Y_j = \psi^{-1}(S_i \cap S_j) \); by Lemma (3.2.6.1), the product \( Z_{ij} = X_{ij} \times_S Y_{ij} \) exists. We now note that, if \( T \) is an \( S \)-prescheme, and if \( g : T \to X_i \) and \( h : T \to Y_i \) are \( S \)-morphisms, then we necessarily have that \( \phi(g(T)) \subset S_i \cap S_j \) by the definition of an \( S \)-morphism, and thus that \( g(T) \subset X_{ij} \) and \( h(T) \subset Y_{ij} \); it is then immediate that \( Z_{ij} \) is the product of \( X_i \) and \( Y_i \).

(3.2.6.5). We can now complete the proof of Theorem (3.2.6). If \( S \) is an affine scheme, then there are covers \((U_\alpha)\) and \((V_\lambda)\) of \( X \) and \( Y \) (respectively) consisting of affine open subsets; since \( U_\alpha \times_S V_\lambda \) exists, by (3.2.2), \( X \times_S Y \) exists similarly, by Lemma (3.2.6.3). If \( S \) is any prescheme, then there is a cover \((S_i)\) of \( S \) consisting of affine open subsets. If \( \phi : X \to S \) and \( \psi : Y \to S \) are the structure morphisms, and if we set \( X_i = \phi^{-1}(S_i) \) and \( Y_i = \psi^{-1}(S_i) \), then the products \( X_i \times_S Y_i \) exist, by the above; but then the products \( X_i \times_S Y_i \) also exist (3.2.5), therefore \( X \times_S Y \) exists similarly, by Lemma (3.2.6.4).

**Corollary (3.2.7).** — Let \( Z = X \times_S Y \) be the product of two \( S \)-preschemes, \( p \) and \( q \) the projections from \( Z \) to \( X \) and to \( Y \) (respectively), and \( \phi \) (resp. \( \psi \)) the structure morphism of \( X \) (resp. \( Y \)). Let \( S' \) be an open subset of \( S \), and \( U \) (resp. \( V \)) an open subset of \( X \) (resp. \( Y \)) contained in \( \phi^{-1}(S') \) (resp. \( \psi^{-1}(S') \)). Then the product
of all, if \( f : T \to X \) and \( g : T \to Y \) are \( S \)-morphisms such that \( f(T) \subset U \) and \( g(T) \subset V \), then the \( S' \)-morphism \((f, g)_S\) can be identified with the restriction of \((f, g)_S\) to \( p^{-1}(U) \cap q^{-1}(V) \).

**Proof.** This follows from Corollary (3.2.5) and Lemma (3.2.6.1). \( \square \)

(3.2.8). Let \((X_a)\) and \((Y_\lambda)\) be families of \( S \)-preschemes, and \( X \) (resp. \( Y \)) the sum of the family \((X_a)\) (resp. \((Y_\lambda)\)) (3.1). Then \( X \times_S Y \) can be identified with the sum of the family \((X_a \times_S Y_\lambda)\); this follows immediately from Lemma (3.2.6.3).

(3.2.9). \(^6\) It follows from (1.8.1) that we can state (3.2.2) in the following manner: \( Z = \text{Spec}(B \otimes_A C) \) is not only a product of \( X = \text{Spec}(B) \) and \( Y = \text{Spec}(C) \) in the category of \( S \)-preschemes, but also in the category of locally ringed spaces over \( S \) (with a definition of \( S \)-morphisms modelled on that of (2.5.2)). The proof of (3.2.6) also proves that, for any two \( S \)-preschemes \( X \) and \( Y \), the prescheme \( X \times_S Y \) is not only the product of \( X \) and \( Y \) in the category of \( S \)-preschemes, but also in the category of locally ringed spaces over the prescheme \( S \).

3.3. Formal properties of the product; change of the base prescheme

(3.3.1). The reader will notice that all the properties stated in this section, except (3.3.13) and (3.3.15), are true without modification in any category, whenever the products involved in the statements exist (since it is clear that the notions of an \( S \)-object and of an \( S \)-morphism can be defined exactly as in (2.5) for any object \( S \) of the category).

(3.3.2). First of all, \( X \times_S Y \) is a covariant bifunctor in \( X \) and \( Y \) on the category of \( S \)-preschemes: it suffices in fact to note that the diagram

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{f \times 1} & X' \times Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & X'
\end{array}
\quad \begin{array}{ccc}
f' \times 1 & \xrightarrow{} & X'' \times Y \\
\downarrow & & \downarrow \\
f' & \xrightarrow{} & X''
\end{array}
\]

is commutative.

**Proposition (3.3.3).** — For each \( S \)-prescheme \( X \), the first (resp. second) projection from \( X \times_S S \) (resp. \( S \times_S X \)) is a functorial isomorphism from \( X \times_S S \) (resp. \( S \times_S X \)) to \( X \), whose inverse isomorphism is \((1_X, \phi)_S\) (resp. \((\phi, 1_X)_S\)), where we denote by \( \phi \) the structure morphism \( X \to S \); we can therefore write, up to a canonical isomorphism,

\[
X \times_S S = S \times_S X = X.
\]

**Proof.** It suffices to prove that the triple \((X, 1_X, \phi)\) is a product of \( X \) and \( S \). If \( T \) is an \( S \)-prescheme, then the only \( S \)-morphism from \( T \) to \( S \) is necessarily the structure morphism \( \psi : T \to S \). If \( f \) is an \( S \)-morphism from \( T \) to \( X \), we necessarily have \( \psi = \phi \circ f \), hence our claim. \( \square \)

**Corollary (3.3.4).** — Let \( X \) and \( Y \) be \( S \)-preschemes, with structure morphisms \( \phi : X \to S \) and \( \psi : Y \to S \). If we canonically identify \( X \) with \( X \times_S S \), and \( Y \) with \( S \times_S Y \), then the projections \( X \times_S Y \to X \) and \( X \times_S Y \to Y \) are identified with \( 1_X \times \psi \) and \( \phi \times 1_Y \) (respectively).

The proof is immediate and is left to the reader.

(3.3.5). We can define, in a manner similar to (3.2), the product of a finite number \( n \) of \( S \)-preschemes, and the existence of these products follows from (3.2.6) by induction on \( n \), and by noting that \((X_1 \times_S X_2 \times_S \cdots \times_S X_n) \times_S X_n \) satisfies the definition of a product. The uniqueness of the product implies, as in any category, its commutativity and associativity properties. If, for example, \( p_1, p_2, \) and \( p_3 \) denote the projections from \( X_1 \times_S X_2 \times_S X_3 \), and if we identify this prescheme with \((X_1 \times_S X_2) \times_S X_3\), then the projection to \( X_1 \times_S X_2 \) is identified with \((p_1, p_2)_S\).

\(^6\) Trans. (3.2.9) is from the errata of EGA II, on page 221, whence the change in page numbering.
(3.3.6). Let $S$ and $S'$ be preschemes, and $\phi : S' \to S$ a morphism, which lets us consider $S'$ as an $S$-prescheme. For each $S$-prescheme $X$, consider the product $X \times_S S'$, and let $p$ and $\pi'$ be the projections to $X$ and to $S'$ (respectively). Equipped with $\pi$, this product is an $S'$-prescheme; when we consider it as such, we denote it by $X_{(S')}$ or $X_{(\phi)}$, and we say that this is the prescheme obtained by base change (or a change of base) from $S$ to $S'$ by means of the morphism $\phi$, or the inverse image of $X$ by $\phi$. We note that, if $\pi$ is the structure morphism of $X$, and $\theta$ the structure morphism of $X \times_S S'$, considered as an $S$-prescheme, then the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{p} & X_{(S')} \\
\downarrow{\pi} & & \downarrow{\pi'} \\
S & \xleftarrow{\phi} & S'
\end{array}
\]

is commutative.

(3.3.7). With the notation of (3.3.6), for each $S$-morphism $f : X \to Y$, we denote by $f_{(S')}$ the $S'$-morphism $f \times_S 1 : X_{(S')} \to Y_{(S')}$, and we say that $f_{(S')}$ is the base change (or inverse image) of $f$ by $\phi$. Therefore $X_{(S')}$ is a covariant functor in $X$, from the category of $S$-preschemes to that of $S'$-preschemes.

(3.3.8). The prescheme $X_{(S')}$ can be considered as a solution to a universal mapping problem: each $S'$-prescheme $T$ is also an $S$-prescheme via $\phi$; each $S$-morphism $g : T \to X$ is then uniquely written as $g = p \circ f$, where $f$ is an $S'$-morphism $T \to X_{(S')}$, as follows from the definition of the product applied to the $S$-morphisms $f$ and $\psi : T \to S'$ (the structure morphism of $T$).

**Proposition (3.3.9).** — ("Transitivity of base change"). Let $S''$ be a prescheme, and $\phi' : S'' \to S$ a morphism. For each $S$-prescheme $X$, there exists a canonical functorial isomorphism from the $S''$-prescheme $(X_{(S')})_{(\phi')}$ to the $S''$-prescheme $X_{(\phi')}$.

**Proof.** Let $T$ be a $S''$-prescheme, $\psi$ its structure morphism, and $g$ an $S$-morphism from $T$ to $X$ ($T$ being considered as an $S$-prescheme with structure morphism $\phi \circ \phi' \circ \psi$). Since $T$ is also an $S'$-prescheme with structure morphism $\phi' \circ \psi$, we can write $g = p \circ g'$, where $g'$ is an $S'$-morphism $T \to X_{(\psi)}$, and then $g'' = p' \circ g''$, where $g''$ is an $S''$-morphism $T \to (X_{(\phi)})_{(\phi')}$:

\[
\begin{array}{ccc}
X & \xrightarrow{p} & X_{(\phi)} \\
\downarrow{\pi} & & \downarrow{\pi'} \\
S & \xleftarrow{\phi} & S'
\end{array}
\]

\[
\begin{array}{ccc}
X_{(\phi)} & \xleftarrow{p'} & (X_{(\phi)})_{(\phi')} \\
\downarrow{\pi'} & & \downarrow{\pi''} \\
S' & \xleftarrow{\phi'} & S''
\end{array}
\]

So the result follows by the uniqueness of the solution to a universal mapping problem. □

This result can be written as the equality (up to a canonical isomorphism) $(X_{(S')})_{(S'')} = X_{(S'')}$ (if there is no chance of confusion), or also as

(3.3.9.1) \[ (X \times_S S') \times_S S'' = X \times_S S''; \]

the functorial nature of the isomorphism defined in (3.3.9) can similarly be expressed by the transitivity formula for base change morphisms

(3.3.9.2) \[ (f_{(S')})_{(S'')} = f_{(S'')} \]

for each $S$-morphism $f : X \to Y$.

**Corollary (3.3.10).** — If $X$ and $Y$ are $S$-preschemes, then there exists a canonical functorial isomorphism from the $S'$-prescheme $X_{(S') \times_S Y_{(S')}}$ to the $S'$-prescheme $(X \times_S Y)_{(S')}$. 

**Proof.** We have, up to canonical isomorphism,

\[ (X \times_S S') \times_S Y \times_S S' = (X \times_S Y) \times_S S' = (X \times_S Y) \times_S S' \]

according to (3.3.9.1) and the associativity of products of $S$-preschemes. □
The functorial nature of the isomorphism defined in Corollary (3.3.10) can be expressed by the formula
\[(3.3.10.1) \quad (u_{(s')}, v_{(s')})_{s'} = ((u, v)_s)_{(s')}
\]
for each pair of \(S\)-morphisms \(u : T \to X, v : T \to Y\).

In other words, the base change functor \(X_{(s')} \text{ commutes with products;}\) it also commutes with sums (3.2.8).

**Corollary (3.3.11).** — *Let \(Y\) be an \(S\)-prescheme, and \(f : X \to Y\) a morphism which makes \(X\) a \(Y\)-prescheme (and, as a result, also an \(S\)-prescheme). The prescheme \(X_{(s')}\) is then identified with the product \(X \times_Y Y_{(s')}\), the projection \(X \times_Y Y_{(s')} \to Y_{(s')}\) being identified with \(f_{(s')}\).*

**Proof.** Let \(\psi : Y \to S\) be the structure morphism of \(Y\); we have the commutative diagram
\[
\begin{array}{ccc}
S' & \xleftarrow{s'} & Y_{(s')} \\
\downarrow & & \downarrow \quad f_{(s')} \\
S & \xleftarrow{\psi} & Y \xrightarrow{f} X. \\
\end{array}
\]
We have that \(Y_{(s')}\) is identified with \(S'_{(\psi)}\), and \(X_{(s')}\) with \(S'_{(\phi \circ f)}\); taking (3.3.9) and (3.3.4) into account, we thus deduce the corollary. \(\square\)

(3.3.12). Let \(f : X \to X'\) and \(g : Y \to Y'\) be \(S\)-morphisms which are *monomorphisms* of preschemes (1, 1.1); then \(f \times_S g\) is a *monomorphism*. Indeed, if \(p\) and \(q\) are the projections of \(X \times_S Y, p'\) and \(q'\) the projections of \(X' \times_S Y',\) and \(u\) and \(v\) both \(S\)-morphisms \(T \to X \times_S Y\), then the equation \((f \times_S g) \circ u = (f \times_S g) \circ v\) implies that \(p' \circ (f \times_S g) \circ u = p' \circ (f \times_S g) \circ v\) or, in other words, that \(f \circ p \circ u = f \circ p \circ v\), and since \(f\) is a monomorphism, \(p \circ u = p \circ v\); using the fact that \(g\) is a monomorphism, we similarly obtain \(q \circ u = q \circ v\), hence \(u = v\).

It follows that, for each base change \(S' \to S\),
\[f_{(s')} : X_{(s')} \to Y_{(s')},\]
is a monomorphism.

(3.3.13). Let \(S\) and \(S'\) be affine schemes of rings \(A\) and \(A'\) (respectively); a morphism \(S' \to S\) then corresponds to a ring homomorphism \(A \to A'\). If \(X\) is an \(S\)-prescheme, we denote by \((X_{(s')})_{s'}\) or \(X \otimes_A A'\) the \(S'\)-prescheme \(X_{(s')}\); when \(X\) is also affine of ring \(B\), \(X_{(s')}\) is affine of ring \(B_{(s')} = B \otimes_A A'\) obtained by extension of scalars from the \(A\)-algebra \(B\) to \(A'\).

(3.3.14). With the notation of (3.3.6), for each \(S\)-morphism \(f : S' \to X\), we have that \(f' = (f, 1_{s'})_S\) is an \(S'\)-morphism \(S' \to X' = X_{(s')}\) such that \(p \circ f' = f, p' \circ f' = 1_{s'}\), or, in other words, an \(S'\)-section of of \(X'\); conversely, if \(f'\) is such an \(S'\)-section, then \(f = p \circ f'\) is an \(S\)-morphism \(S' \to X\). We thus define a canonical *bijective correspondence*
\[\text{Hom}_S(S', X) \simeq \text{Hom}_{S'}(S', X').\]

We say that \(f'\) is the *graph morphism* of \(f\), and we denote it by \(\Gamma_f\).

(3.3.15). Given a prescheme \(X\), which we can always consider as a \(Z\)-prescheme, it follows, in particular, from (3.3.14) that the \(X\)-sections of \(X \otimes_Z Z[T]\) (where \(T\) is an indeterminate) correspond bijectively with morphisms \(Z[T] \to X\). We will show that these \(X\)-sections also correspond bijectively with sections of the structure sheaf \(\mathcal{O}_X\) over \(X\). Indeed, let \((U_a)\) be a cover of \(X\) by affine open subsets; let \(u : X \to X \otimes_Z Z[T]\) be an \(X\)-morphism, and let \(u_{a}\) be its restriction to \(U_a\); if \(A_a\) is the ring of the affine scheme \(U_a\), then \(U_a \otimes_Z Z[T]\) is an affine scheme of ring \(A_a[T]\) (3.2.2), and \(u_{a}\) canonically corresponds to an \(A_a\)-homomorphism \(A_a[T] \to A_a\) (1.7.3). Now, since such a homomorphism is completely determined by the data of the image of \(T\) in \(A_a\), let \(s_a \in A_a = \Gamma(U_a, \mathcal{O}_X)\), and if we suppose that the restrictions of \(u_a\) and \(u_b\) to an affine open subset \(V \subset U_a \cap U_b\) coincide, then we see immediately that \(s_a = s_b\) coincide on \(V\); thus the family \((s_a)\) consists of the restrictions to \(U_a\) of a section \(s\) of \(\mathcal{O}_X\) over \(X\); conversely, it is clear that such a section defines a family \((u_a)\) of morphisms which are the restrictions to \(U_a\) of an \(X\)-morphism \(X \to X \otimes_Z Z[T]\). This result is generalized in (II, 1.7.12).
3.4. Points of a prescheme with values in a prescheme; geometric points

(3.4.1). Let \( X \) be a prescheme; for each prescheme \( T \), we then denote by \( X(T) \) the set \( \text{Hom}(T, X) \) of morphisms \( T \to X \), and the elements of this set are called the points of \( X \) with values in \( T \). If we associate to each morphism \( f : T \to T' \) the map \( u' \mapsto u' \circ f \) from \( X(T') \) to \( X(T) \), we see, for fixed \( X \), that \( X(T) \) is a contravariant functor in \( T \), from the category of preschemes to that of sets. In addition, each morphism of preschemes \( g : X \to Y \) defines a functorial homomorphism \( X(T) \to Y(T) \), which sends \( v \in X(T) \) to \( g \circ v \).

(3.4.2). Given sets \( P, Q \), and \( R \), and maps \( \phi : P \to R \) and \( \psi : Q \to R \), we define the fibre product of \( P \) and \( Q \) over \( R \) (relative to \( \phi \) and \( \psi \)) as the subset of the product set \( P \times Q \) consisting of the pairs \((p, q)\) such that \( \phi(p) = \psi(q) \); we denote it by \( P \times_R Q \). Definition (3.2.1) of the product of \( S \)-preschemes can be interpreted, with the notation of (3.4.1), via the formula

\[
(X \times_S Y)(T) = X(T) \times_{S(T)} Y(T).
\]

the maps \( X(T) \to S(T) \) and \( Y(T) \to S(T) \) corresponding to the structure morphisms \( X \to S \) and \( Y \to S \).

(3.4.3). If we are given a prescheme \( S \) and we consider only the \( S \)-preschemes and \( S \)-morphisms, then we will denote by \( X(S) \) the set \( \text{Hom}_S(T, X) \) of \( S \)-morphisms \( T \to X \), and suppress the subscript \( S \) when there is no chance of confusion; we say that the elements of \( X(T)_S \) are the points (or \( S \)-points, when there is a possibility of confusion) of the \( S \)-prescheme \( X \) with values in the \( S \)-prescheme \( T \). In particular, an \( S \)-section of \( X \) is none other than a point of \( X \) with values in \( S \). The formula (3.4.2.1) can then be written as

\[
(X \times_S Y)_S = X(T)_S \times Y(T)_S.
\]

more generally, if \( Z \) is an \( S \)-prescheme, and \( X \), \( Y \), and \( T \) are \( Z \)-preschemes (thus ipso facto \( S \)-preschemes), then we have

\[
(X \times_Z Y)_S = X(T)_S \times_{Z(T)_S} Y(T)_S.
\]

We note that, to show that a triple \((W, r, s)\) consisting of an \( S \)-prescheme \( W \) and \( S \)-morphisms \( r : W \to X \) and \( s : W \to Y \) is a product of \( X \) and \( Y \) (over \( Z \)), it suffices, by definition, to check that, for each \( S \)-prescheme \( T \), the diagram

\[
\begin{array}{ccc}
W(T)_S & \xrightarrow{r'} & X(T)_S \\
\downarrow s' & & \downarrow \phi' \\
Y(T)_S & \xrightarrow{\psi'} & Z(T)_S
\end{array}
\]

makes \( W(T)_S \) the fibre product of \( X(T)_S \) and \( Y(T)_S \) over \( Z(T)_S \), where \( r' \) and \( s' \) correspond to \( r \) and \( s \), and \( \phi' \) and \( \psi' \) to the structure morphisms \( \phi : X \to Z \) and \( \psi : Y \to Z \).

(3.4.4). When \( T \) (resp. \( S \)) in the above is an affine scheme of ring \( B \) (resp. \( A \)), we replace \( T \) (resp. \( S \)) by \( B \) (resp. \( A \)) in the above notation, and we then call the elements of \( X(B) \) the points of \( X \) with values in the ring \( B \), and the elements of \( X(B)_A \) the points of the \( A \)-prescheme \( X \) with values in the \( A \)-algebra \( B \). We note that \( X(B) \) and \( X(B)_A \) are covariant functors in \( B \). We similarly write \( X(T)_A \) for the set of points of the \( A \)-prescheme \( X \) with values in the \( A \)-prescheme \( T \).

(3.4.5). Consider, in particular, the case where \( T \) is of the form \( \text{Spec}(A) \), where \( A \) is a local ring; the elements of \( X(A) \) then correspond bijectively to local homomorphisms \( \mathcal{O}_x \to A \) for \( x \in X(2.2.4) \); we say that the point \( x \) of the underlying space of \( X \) is the location of the point of \( X \) with values in \( A \) to which it corresponds.

More specifically, we define the geometric points of a prescheme \( S \) to be the points of \( X \) with values in a field \( K \): the data of such a point is equivalent to the data of its location \( x \) in the underlying subspace of \( X \), and of an extension \( K \) of \( k(x) \); \( K \) will be called the field of values of the corresponding geometric point, and we say that this geometric point is located at \( x \). We also define a map \( X(K) \to X \), sending a geometric point with values in \( K \) to its location.

\[\text{[Trans.] We also say that the geometric point lies over this } x.\]
If \( S' = \text{Spec}(K) \) is an \( S \)-prescheme (in other words, if \( K \) is considered as an extension of the residue field \( k(s) \), where \( s \in S \)), and if \( X \) is an \( S \)-prescheme, then an element of \( X(K) \), or, as we say, a geometric point of \( X \) lying over \( s \) with values in \( K \), consists of the data of a \( k(s) \)-monomorphism from the residue field \( k(x) \) to \( K \), where \( x \) is a point of \( X \) lying over \( s \) (therefore \( k(x) \) is an extension of \( k(s) \)).

In particular, if \( S = \text{Spec}(K) = \{ \xi \} \), then the geometric points of \( X \) with values in \( K \) can be identified with the points \( x \in X \) such that \( k(x) = K \); we say that these latter points are the \( K \)-rational points of the \( K \)-prescheme \( X \); if \( K' \) is an extension of \( K \), then the geometric points of \( X \) with values in \( K' \) bijectively correspond to the \( K' \)-rational points of \( X' = X \otimes K' \) (3.3.14).

**Lemma (3.4.6).** — Let \( X_i (1 \leq i \leq n) \) be \( S \)-preschemes, \( s \) a point of \( S \), and \( x_i (1 \leq i \leq n) \) points of \( X_i \) lying over \( s \). Then there exists an extension \( K \) of \( k(s) \) and a geometric point of the product \( Y = X_1 \times_S X_2 \times_S \cdots \times_S X_n \), with values in \( K \), whose projections to the \( X_i \) are localized at the \( x_i \).

**Proof.** There exist \( k(s) \)-monomorphisms \( k(x_i) \rightarrow K \), all in the same extension \( K \) of \( k(s) \) (Bourbaki, Alg., chap. V, §4, prop. 2). The compositions \( k(s) \rightarrow k(x_i) \rightarrow K \) are all identical, and so the morphisms \( \text{Spec}(K) \rightarrow X_i \) corresponding to the \( k(x_i) \rightarrow K \) are all \( S \)-morphisms, and we thus conclude that they define a unique morphism \( \text{Spec}(K) \rightarrow Y \). If \( y \) is the corresponding point of \( Y \), it is clear that its projection in each of the \( X_i \) is \( x_i \). \( \square \)

**Proposition (3.4.7).** — Let \( X_i (1 \leq i \leq n) \) be \( S \)-preschemes, and, for each index \( i \), let \( x_i \) be a point of \( X_i \). For there to exist a point \( y \) of \( Y = X_1 \times_S X_2 \times_S \cdots \times_S X_n \) whose image is \( x_i \) under the \( i \)th projection for each \( 1 \leq i \leq n \), it is necessary and sufficient that the \( x_i \) all lie above the same point \( s \) of \( S \).

**Proof.** The condition is evidently necessary; Lemma (3.4.6) proves that it is sufficient. \( \square \)

In other words, if we denote by \( (X) \) the underlying set of \( X \), we see that we have a canonical surjective function \( (X \times_S Y) \rightarrow (X) \times_S (Y) \); we must point out that this function is not injective in general; in other words, there can exist multiple distinct points \( z \) in \( X \times_S Y \) that have the same projections \( x \in X \) and \( y \in Y \); we have already seen this when \( S, X, \) and \( Y \) are prime spectra of fields \( k, K, \) and \( K' \) (respectively), since the tensor product \( K \otimes_k K' \) has, in general, multiple distinct prime ideals (cf. (3.4.9)).

**Corollary (3.4.8).** — Let \( f : X \rightarrow Y \) be an \( S \)-morphism, and \( f_{(S')} : X_{(S')} \rightarrow Y_{(S')} \) the \( S' \)-morphism induced by \( f \) by an extension \( S' \rightarrow S \) of the base prescheme. Let \( p \) (resp. \( q \)) be the projection \( X_{(S')} \rightarrow X \) (resp. \( Y_{(S')} \rightarrow Y \)); for every subset \( M \) of \( X \), we have

\[
q^{-1}(f(M)) = f_{(S')}(p^{-1}(M)).
\]

**Proof.** Indeed (3.3.11), \( X_{(S')} \) can be identified with the product \( X \times_Y Y_{(S')} \) thanks to the commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{p} & X_{(S')} \\
\downarrow{f} & & \downarrow{f_{(S')}} \\
Y & \xleftarrow{q} & Y_{(S')}
\end{array}
\]

By (3.4.7), the equation \( q(y') = f(x) \) for \( x \in M \) and \( y' \in Y_{(S')} \) is equivalent to the existence of some \( x' \in X_{(S')} \) such that \( p(x') = x \) and \( f_{(S')}(x') = y' \), whence the corollary. \( \square \)

**Lemma (3.4.6)** can be made clearer in the following manner:

**Proposition (3.4.9).** — Let \( X \) and \( Y \) be \( S \)-preschemes, \( x \) a point of \( X \), and \( y \) a point of \( Y \), with both \( x \) and \( y \) lying above the same point \( s \in S \). The set of points of \( X \times_S Y \) with projections \( x \) and \( y \) is in bijective correspondence with the set of types of extensions (7) composed of \( k(x) \) and \( k(y) \) considered as extensions of \( k(s) \) (Bourbaki, Alg., chap. VIII, §8, prop. 2).

**Proof.** Let \( p \) (resp. \( q \)) be the projection from \( X \times_S Y \) to \( X \) (resp. \( Y \)), and \( E \) the subspace \( p^{-1}(x) \cap q^{-1}(y) \) of the underlying space of \( X \times_S Y \). First, note that the morphisms \( \text{Spec}(k(x)) \rightarrow S \)
and Spec($k(y))$ → $S$ factor as Spec($k(x)) → Spec(k(s)) → S$ and Spec($k(y)) → Spec(k(s)) → S$; since Spec($k(s)) → S$ is a monomorphism (2.4.7), it follows from (3.2.4) that we have

$$P = \text{Spec}(k(x)) \times_S \text{Spec}(k(y)) = \text{Spec}(k(x)) \times_{\text{Spec}(k(s))} \text{Spec}(k(y)) = \text{Spec}(k(x) \otimes_{k(s)} k(y)).$$

We will define two maps, $\alpha : P_0 \to E$ and $\beta : E \to P_0$ inverse to one another (where $P_0$ denotes the underlying set of the prescheme $P$). If $i : \text{Spec}(k(x)) → X$ and $j : \text{Spec}(k(y)) → Y$ are the canonical morphisms (2.4.5), we take $\alpha$ to be the map of underlying spaces corresponding to the morphism $i \times j$. On the other hand, every $z \in E$ defines, by hypothesis, two $k(s)$-monomorphisms, $k(x) → k(z)$ and $k(y) → k(z)$, and thus a $k(s)$-monomorphism $k(x) \otimes_{k(s)} k(y) → k(z)$, and thus a morphism $\text{Spec}(k(z)) → P$; $\beta(z)$ will be the image of $z$ in $P_0$ under this morphism. The verification of the fact that $\alpha \circ \beta$ and $\beta \circ \alpha$ are the identity maps follows from (2.4.5) and the definition of the product (3.2.1). Finally, we know that $P_0$ is in bijective correspondence with the set of types of extensions (?) composed of $k(x)$ and $k(y)$ (Bourbaki, Alg., chap. VIII, §8, prop. 1).

3.5. Surjections and injections

(3.5.1). In a general sense, consider a property $P$ of morphisms of preschemes, and the following two propositions:

(i) If $f : X → X'$ and $g : Y → Y'$ are $S$-morphisms that have property $P$, then $f \times_S g$ also has property $P$.

(ii) If $f : X → Y$ is an $S$-morphism that has property $P$, then every $S'$-morphism $f_{S'} : X_{S'} → Y_{S'}$, induced by $f$ by an extension of the base prescheme, also has property $P$.

Since $f_{S'} = f \times 1_{S'}$, we see that, if, for every prescheme $X$, the identity $1_X$ has property $P$, then (i) implies (ii); on the other hand, since $f \times_S g$ is the composite morphism

$$X \times_S Y \xrightarrow{f \times 1_Y} X' \times_S Y \xrightarrow{1_{X'} \times g} X' \times_S Y',$$

we see that, if $f$ has property $P$, then so does the product $f \times_S g$, and so (ii) implies (i).

A first application of this remark is

**Proposition (3.5.2).** —

(i) If $f : X → X'$ and $g : Y → Y'$ are surjective $S$-morphisms, then $f \times_S g$ is surjective.

(ii) If $f : X → Y$ is a surjective $S$-morphism, then $f_{S'}$ is surjective for every extension $S'$ of the base prescheme.

**Proof.** The composition of any two surjections being a surjection, it suffices to prove (ii); but this proposition follows immediately from (3.4.8) applied to $M = X$. □

**Proposition (3.5.3).** — For a morphism $f : X → Y$ to be surjective, it is necessary and sufficient that, for every field $K$ and every morphism $\text{Spec}(K) → Y$, there exist an extension $K'$ of $K$ and a morphism $\text{Spec}(K') → X$ that make the following diagram commute:

$$\begin{array}{ccc}
X & \xleftarrow{\text{Spec}(K')} & \\
\downarrow{f} & & \downarrow{f} \\
Y & \xleftarrow{\text{Spec}(K)} & \\
\end{array}$$

**Proof.** The condition is sufficient because, for all $y \in Y$, it suffices to apply it to a morphism $\text{Spec}(K) → Y$ corresponding to a monomorphism $k(y) → K$, with $K$ being an extension of $k(y)$ (2.4.6). Conversely, suppose that $f$ is surjective, and let $y \in Y$ be the image of the unique point of $\text{Spec}(K)$; there exists some $x \in X$ such that $f(x) = y$; we will consider the corresponding monomorphism $k(y) → k(x)$ (2.2.1); it then suffices to take $K'$ to be the extension of $k(y)$ such that there exist $k(y)$-monomorphisms from $k(x)$ and $K$ to $K'$ (Bourbaki, Alg., chap. V, §4, prop. 2); the morphism $\text{Spec}(K') → X$ corresponding to $k(x) → K'$ is exactly that for which we are searching. □

With the language introduced in (3.4.5), we can say that every geometric point of $Y$ with values in $K$ comes from a geometric point of $X$ with values in an extension of $K$. 

Definition (3.5.4). — We say that a morphism \( f : X \to Y \) of preschemes is universally injective, or a radicial morphism, if, for every field \( K \), the corresponding map \( X(K) \to Y(K) \) is injective.

It follows also from the definitions that every monomorphism of preschemes (T, 1.1) is radicial.

(3.5.5). For a morphism \( f : X \to Y \) to be radicial, it suffices that the condition of Definition (3.5.4) hold for every algebraically closed field. In fact, if \( K \) is an arbitrary field, and \( K' \) an algebraically-closed extension of \( K \), then the diagram

\[
\begin{array}{ccc}
X(K) & \xrightarrow{\alpha} & Y(K) \\
\Phi \downarrow & & \downarrow \phi' \\
X(K') & \xrightarrow{\alpha'} & Y(K')
\end{array}
\]

commutes, where \( \phi \) and \( \phi' \) come from the morphism \( \text{Spec}(K') \to \text{Spec}(K) \), and \( \alpha \) and \( \alpha' \) corresponding to \( f \). However, \( \phi \) is injective, and so too is \( \alpha' \), by hypothesis; hence \( \alpha \) is necessarily injective.

Proposition (3.5.6). — Let \( f : X \to Y \) and \( g : Y \to Z \) be two morphisms of preschemes.

(i) If \( f \) and \( g \) are radicial, then so is \( g \circ f \).

(ii) Conversely, if \( g \circ f \) is radicial, then so is \( f \).

Proof. Taking into account Definition (3.5.4), the proposition reduces to the corresponding claims for the maps \( X(K) \to Y(K) \to Z(K) \), and these claims are evident.

Proposition (3.5.7). —

(i) If the \( S \)-morphisms \( f : X \to X' \) and \( g : X \to X' \) are radicial, then so is \( f \times_S g \).

(ii) If the \( S \)-morphism \( f : X \to Y \) is radicial, then so is \( f(s) : X(s) \to Y(s) \) for every extension \( S' \to S \) of the base prescheme.

Proof. Given (3.5.1), it suffices to prove (i). We have seen (3.4.2.1) that

\[
(X \times_S Y)(K) = X(K) \times_{S(K)} Y(K),
\]

\[
(X' \times_S Y')(K) = X'(K) \times_{S(K)} Y'(K),
\]

with the map \( (X \times_S Y)(K) \to (X' \times_S Y')(K) \) corresponding to \( f \times_S g \) thus being identified with \( (u, v) \to (f \circ u, g \circ v) \), and the proposition then follows.

Proposition (3.5.8). — For a morphism \( f = (\psi, \theta) : X \to Y \) to be radicial, it is necessary and sufficient for \( \psi \) to be injective and for the monomorphism \( \theta^x : k(\psi(x)) \to k(x) \) to make \( k(x) \) a radicial extension of \( k(\psi(x)) \) for every \( x \in X \).

Proof. We suppose that \( f \) is radicial and first show that the equation \( \psi(x_1) = \psi(x_2) = y \) necessarily implies that \( x_1 = x_2 \). Indeed, there exists a field \( K \), and an extension of \( k(y) \), along with \( k(y) \)-monomorphisms \( k(x_1) \to K \) and \( k(x_2) \to K \) (Bourbaki, Alg., chap. V, §4, prop. 2); the corresponding morphisms \( u_1 : \text{Spec}(K) \to X \) and \( u_2 : \text{Spec}(K) \to X \) are then such that \( f \circ u_1 = f \circ u_2 \), and so \( u_1 = u_2 \) by hypothesis, and this implies, in particular, that \( x_1 = x_2 \). We now consider \( k(x) \) as the extension of \( k(\psi(x)) \) by means of \( \theta^x \); if \( k(x) \) is not a radicial algebraically-closed extension, then there exist two distinct \( k(\psi(x)) \)-monomorphisms from \( k(x) \) to an algebraically-closed extension \( K' \) of \( k(\psi(x)) \), and the two corresponding morphisms \( \text{Spec}(K) \to X \) would contradict the hypothesis. Conversely, taking (2.4.6) into account, it is immediate that the conditions stated are sufficient for \( f \) to be radicial.

Corollary (3.5.9). — If \( A \) is a ring, and \( S \) is a multiplicative set of \( A \), then the canonical morphism \( \text{Spec}(S^{-1}A) \to \text{Spec}(A) \) is radicial.

Proof. Indeed, this morphism is a monomorphism (1.6.2).

Corollary (3.5.10). — Let \( f : X \to Y \) be a radicial morphism, \( g : Y' \to Y \) a morphism, and \( X' = X_{g^{-1}} = X \times_Y Y' \). Then the radicial morphism \( f_{g(Y')} \) (3.5.7, ii) is a bijection from the underlying space of \( X \) to \( g^{-1}(f(X)) \); further, for every field \( K \), the set \( X'(K) \) can be identified with the subset of \( Y'(K) \) given by the inverse image of the map \( Y'(K) \to Y(K) \) (corresponding to \( g \)) from the subset \( X(K) \) of \( Y(K) \).
Proof. The first claim follows from (3.5.8) and (3.4.8); the second, from the commutativity of the following diagram:

\[
\begin{array}{ccc}
X'(K) & \longrightarrow & Y'(K) \\
\downarrow & & \downarrow \\
X(K) & \longrightarrow & Y(K)
\end{array}
\]

Remark (3.5.11). — We say that a morphism \( f = (\psi, \theta) \) of preschemes is injective if the map \( \psi \) is injective. For a morphism \( f = (\psi, \theta) : X \to Y \) to be radical, it is necessary and sufficient that, for every morphism \( Y' \to Y \), the morphism \( f_{(Y')} : X_{(Y')} \to Y' \) be injective (which justifies the terminology of a universally injective morphism). In fact, the condition is necessary by (3.5.7, ii) and (3.5.8). Conversely, the condition implies that \( \psi \) is injective; if, for some \( x \in X \), the monomorphism \( \theta^* : k(\psi(x)) \to k(x) \) were not radical, then there would be an extension \( K \) of \( k(\psi(x)) \), and two distinct morphisms \( \text{Spec}(K) \to X \) corresponding to the same morphism \( \text{Spec}(K) \to Y \) (3.3.14). But then, setting \( Y' = \text{Spec}(K) \), there would be two distinct \( Y' \)-sections of \( X_{(Y')} \) which contradicts the hypothesis that \( f_{(Y')} \) is injective.

3.6. Fibres

Proposition (3.6.1). — Let \( f : X \to Y \) be a morphism, \( y \) a point of \( Y \), and \( \mathfrak{a}_y \) an ideal of definition for \( \mathcal{O}_y \) for the \( m_y \)-adic topology. Then the projection \( p : X \times Y \text{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \to X \) is a homeomorphism from the underlying space of the prescheme \( X \times Y \text{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \) to the fibre \( f^{-1}(y) \) equipped with the topology induced from that of the underlying space of \( X \).

Proof. Since \( \text{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \to Y \) is radical ((3.5.4) and (2.4.7)), since \( \text{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \) is a single point, and since the ideal \( \mathfrak{m}_y/\mathfrak{a}_y \) is nilpotent by hypothesis (1.1.12), we already know (3.5.10) and (3.5.11) that \( p \) identifies, as sets, the underlying space of \( X \times Y \text{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \) with \( f^{-1}(y) \); everything reduces to proving that \( p \) is a homeomorphism. By (3.2.7), the question is local on \( X \) and \( Y \), and so we can suppose that \( X = \text{Spec}(B) \) and \( Y = \text{Spec}(A) \), with \( B \) being an \( A \)-algebra. The morphism \( p \) then corresponds to the homomorphism \( 1 \otimes \phi : B \to B \otimes_A A' \), where \( A' = A_y/\mathfrak{a}_y \) and \( \phi \) is the canonical map from \( A \) to \( A' \). Then every element of \( B \otimes_A A' \) can be written as

\[
\sum_i b_i \otimes \phi(a_i)/\phi(s) = \left( \sum_i (a_i b_i \otimes 1) \right) (1 \otimes \phi(s))^{-1},
\]

where \( s \not\in \mathfrak{j}_y \), and Proposition (1.2.4) applies.

(3.6.2). Throughout the rest of this treatise, whenever we consider a fibre \( f^{-1}(y) \) of a morphism as having the structure of a \( k(y) \)-prescheme, it will always be the prescheme obtained by transporting the structure of \( X \times_Y \text{Spec}(k(y)) \) by the projection to \( X \). We will also write this (latter) product as \( X \times_Y k(y) \), or \( X \otimes_{\mathcal{O}_y} k(y) \); more generally, if \( B \) is an \( \mathcal{O}_y \)-algebra, we will denote by \( X \times_Y B \) or \( X \otimes_{\mathcal{O}_y} B \) the product \( X \times_Y \text{Spec}(B) \).

With the preceding convention, it follows from (3.5.10) that the points of \( X \) with values in an extension \( K \) of \( k(y) \) are identified with the points of \( f^{-1}(y) \) with values in \( K \).

(3.6.3). Let \( f : X \to Y \) and \( g : Y \to Z \) be two morphisms, and \( h = g \circ f \) their composition; for all \( z \in Z \), the fibre \( h^{-1}(z) \) is a prescheme isomorphic to

\[
X \times_Z \text{Spec}(k(z)) = (X \times_Y Y) \times_Z \text{Spec}(k(z)) = X \times_Y g^{-1}(z).
\]

In particular, if \( U \) is an open subset of \( X \), then the prescheme induced on \( U \cap f^{-1}(y) \) by the prescheme \( f^{-1}(y) \) is isomorphic to \( f^{-1}(y) \) (\( f_U \) being the restriction of \( f \) to \( U \)).

Proposition (3.6.4). — (Transitivity of fibres) Let \( f : X \to Y \) and \( g : Y' \to Y \) be morphisms; let \( X' = X \times_Y Y' = X_{(Y')} \) and \( f' = f_{(Y')} : X' \to Y' \). For every \( y' \in Y' \), if we let \( y = g(y') \), then the prescheme \( f'^{-1}(y') \) is isomorphic to \( f^{-1}(y) \otimes_{k(y)} k(y') \).
Proof. Indeed, it suffices to remark that the two preschemes \((X \otimes Y, k(y)) \otimes_{k(y)} k(y')\) and 
\((X \times Y', \otimes_{Y'} k(y')\) are both canonically isomorphic to \(X \times Y \text{Spec}(k(y'))\) by (3.3.9.1).

In particular, if \(V\) is an open neighborhood of \(y\) in \(Y\), and we denote by \(f_V\) the restriction of \(f\) to the induced prescheme on \(f^{-1}(V)\), then the preschemes \(f^{-1}(y)\) and \(f_V^{-1}(y)\) are canonically identified.

**Proposition (3.6.5).** — Let \(f : X \to Y\) be a morphism, \(y\) a point of \(Y\), \(Z\) the local prescheme \(\text{Spec}(O_y)\), and \(p = (\psi, \theta)\) the projection \(X \times Y \to X\); then \(p\) is a homeomorphism from the underlying space of \(X \times Y\) to the subspace \(f^{-1}(Z)\) of \(X\) (when the underlying space of \(Z\) is identified with a subspace of \(Y\), cf. (2.4.2)), and, for all \(t \in X \times Y\), letting \(z = \psi(t)\), we have that \(\theta_t^z\) is an isomorphism from \(O_z\) to \(O_t\).

**Proof.** Since \(Z\) (identified as a subspace of \(Y\)) is contained inside every affine open containing \(y\) (2.4.2), we can, as in (3.6.1), reduce to the case where \(X = \text{Spec}(A)\) and \(Y = \text{Spec}(B)\) are affine schemes, with \(A\) being a \(B\)-algebra. Then \(X \times Y\) is the prime spectrum of \(A \otimes_B B_y\), and this ring is canonically identified with \(S^{-1}A\), where \(S\) is the image of \(B - y\) in \(A\) (0, 1.5.2); since \(p\) then corresponds to the canonical homomorphism \(A \to S^{-1}A\), the proposition follows from (1.6.2). \(\square\)

**3.7. Application: reduction of a prescheme mod. \(\mathfrak{J}\)** This section, which makes use of notions and results from Chapter I and Chapter II, will not be used in what follows in this treatise, and is only intended for readers familiar with classical algebraic geometry.

**Lemma (3.7.1).** Let \(A\) be a ring, \(X\) an \(A\)-prescheme, and \(\mathfrak{J}\) an ideal of \(A\); then \(X_0 = X \otimes_A (A/\mathfrak{J})\) is an \((A/\mathfrak{J})\)-prescheme, which we sometimes say is induced from \(X\) by reduction mod. \(\mathfrak{J}\).

This terminology is used foremost when \(A\) is a local ring and \(\mathfrak{J}\) its maximal ideal, in such a way that \(X_0\) is a prescheme over the residue field \(k = A/\mathfrak{J}\) of \(A\).

When \(A\) is also integral, with field of fractions \(K\), we can consider the \(K\)-prescheme \(X' = X \otimes_A K\). By an abuse of language which we will not use, it has been said, up until now, that \(X_0\) is induced by \(X\) by reduction mod. \(\mathfrak{J}\). In the case where this language was used, \(A\) was a local ring of dimension 1 (most often a discrete valuation ring) and it was implied (be it more or less explicitly) that the given \(K\)-prescheme \(X'\) was a closed subscheme of a \(K\)-prescheme \(P'\) (in fact, a projective space of the form (7) \(P'_K\), cf. (II, 4.1.1)), itself of the form \(P' = P \otimes_A K\), where \(P\) is a given \(A\)-prescheme (in fact, the \(A\)-scheme \(P'_{A'}\), with the notation of (II, 4.1.1)). In our language, the definition of \(X_0\) in terms of \(X'\) is formulated as follows:

We consider the affine scheme \(Y = \text{Spec}(A)\), formed of two points, the unique closed point \(y = \mathfrak{J}\) and the generic point \((0)\), the singleton set \(U\) of the generic point being thus an open \(U = \text{Spec}(K)\) in \(Y\). If \(X\) is an \(A\)-prescheme (or, in other words, a \(Y\)-prescheme), then \(X \otimes_A K = X'\) is exactly the prescheme induced by \(X\) on \(\psi^{-1}(U)\), denoting by \(\psi\) the structure morphism \(X \to Y\). In particular, if \(\phi\) is the structure morphism \(P \to Y\), then a closed subscheme \(X'\) of \(P' = \phi^{-1}(U)\) is a (locally closed) subscheme of \(P\). If \(P\) is Noetherian (for example, if \(A\) is Noetherian and \(P\) is of finite type over \(A\)), then there exists a smaller closed subscheme \(X = X'\) of \(P\) that through which \(X'\) factors (9.5.10), and \(X'\) is the prescheme induced by \(X\) on the open \(\phi^{-1}(U) \cap X\), and so is isomorphic to \(X \otimes_A K\) (9.5.10). The immersion of \(X'\) into \(P' = P \otimes_A K\) thus lets us canonically consider \(X'\) as being of the form \(X' = X \otimes_A K\), where \(X\) is an \(A\)-prescheme. We can then consider the reduced mod. \(\mathfrak{J}\) prescheme \(X_0 = X \otimes_A k\), which is exactly the fibre \(\psi^{-1}(y)\) of the closed point \(y\). Up until now, lacking the adequate terminology, we had avoided explicitly introducing the \(A\)-prescheme \(X\). One ought to, however, note that all the claims normally made about the “reduced mod. \(\mathfrak{J}\)” prescheme \(X_0\) should be seen as consequences of more complicated claims about \(X\) itself, and cannot be satisfactorily formulated or understood except by interpreting them as such. It seems also that any hypotheses made on \(X_0\) always reduce to hypotheses on \(X\) itself (independent of the prior data of an immersion of \(X'\) in \(P'_{K'}\)), which lets us give more intrinsic statements.

**Remark (3.7.3).** Lastly, we draw attention to a very particular fact, which has undoubtedly contributed to slowing the conceptual clarification of the situation envisaged here: if \(A\) is a discrete valuation ring, and if \(X\) is proper over \(A\) (which is indeed the case if \(X\) is a closed subscheme of some \(P'_{A'}\), cf. (II, 5.5.4)), then the points of \(X\) with values in \(A\) and the points of \(X'\) with values in \(k\) are in bijective correspondence (II, 7.3.8). This is why we often believe that facts about \(X'\) have been proved, when
in reality we have proved facts about $X$, and these remain valuable (in this form) whenever we no longer suppose that the base local ring is of dimension 1.

§4. SUBPRESCHEMES AND IMMERSION MORPHISMS

4.1. Subpreschemes

(4.1.1). As the notion of a quasi-coherent sheaf (0, 5.1.3) is local, a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ over a prescheme $X$ can be defined by the following condition: for each affine open $V$ of $X$, $\mathcal{F}|V$ is isomorphic to the sheaf associated to a $\Gamma(V, \mathcal{O}_X)$-module (4.1.4). It is clear that, over a prescheme $X$, the structure sheaf $\mathcal{O}_X$ is quasi-coherent, and that the kernels, cokernels, and images of homomorphisms of quasi-coherent $\mathcal{O}_X$-modules, as well as inductive limits and direct sums of quasi-coherent $\mathcal{O}_X$-modules, are also quasi-coherent (Theorem (1.3.7) and Corollary (1.3.9)).

Proposition (4.1.2). — Let $X$ be a prescheme, and $\mathcal{I}$ a quasi-coherent sheaf of ideals of $\mathcal{O}_X$. Then the support $Y$ of the sheaf $\mathcal{O}_X/\mathcal{I}$ is closed, and if we denote by $\mathcal{O}_Y$ the restriction of $\mathcal{O}_X/\mathcal{I}$ to $Y$, then $(Y, \mathcal{O}_Y)$ is a prescheme.

**Proof.** It evidently suffices (2.1.3) to consider the case where $X$ is an affine scheme, and to show that, in this case, $Y$ is closed in $X$, and is an affine scheme. Indeed, if $X = \text{Spec}(A)$, then we have $\mathcal{O}_X = \mathcal{A}$ and $\mathcal{I} = \mathcal{I}$, where $\mathcal{I}$ is an ideal of $A$ (1.4.1); $Y$ is then equal to the closed subset $V(\mathcal{I})$ of $X$ and can be identified with the prime spectrum of the ring $B = A/\mathcal{I}$ (1.11.11); in addition, if $\phi$ is the canonical homomorphism $A \to B = A/\mathcal{I}$, then the direct image $\phi_* (\mathcal{B})$ is canonically identified with the sheaf $\mathcal{A}/\mathcal{I} = \mathcal{O}_X/\mathcal{I}$ (Proposition (1.6.3) and Corollary (1.3.9)), which finishes the proof. □

We say that $(Y, \mathcal{O}_Y)$ is the subprescheme of $(X, \mathcal{O}_X)$ defined by the sheaf of ideals $\mathcal{I}$; this is a particular case of the more general notion of a subprescheme:

**Definition (4.1.3).** — We say that a ringed space $(Y, \mathcal{O}_Y)$ is a subprescheme of a prescheme $(X, \mathcal{O}_X)$ if:

1st. $Y$ is a locally closed subspace of $X$;
2nd. if $U$ denotes the largest open subset of $X$ containing $Y$ such that $Y$ is closed in $U$ (equivalently, the complement in $X$ of the boundary of $Y$ with respect to $Y$), then $(Y, \mathcal{O}_Y)$ is a subprescheme of $(U, \mathcal{O}_X|U)$ defined by a quasi-coherent sheaf of ideals of $\mathcal{O}_X|U$.

We say that the subprescheme $(Y, \mathcal{O}_Y)$ of $(X, \mathcal{O}_X)$ is closed if $Y$ is closed in $X$ (in which case $U = X$).

It follows immediately from this definition and Proposition (4.1.2) that the closed subpreschemes of $X$ are in canonical bijective correspondence with the quasi-coherent sheaves of ideals $\mathcal{I}$ of $\mathcal{O}_X$, since if two such sheaves $\mathcal{I}$ and $\mathcal{I}'$ have the same (closed) support $Y$, and if the restrictions of $\mathcal{O}_X/\mathcal{I}$ and $\mathcal{O}_X/\mathcal{I}'$ to $Y$ are identical, then we have $\mathcal{I}' = \mathcal{I}$.

(4.1.4). Let $(Y, \mathcal{O}_Y)$ be a subprescheme of $X$, $U$ the largest open subset of $X$ such that $Y$ is closed (and thus contained) in $U$, and $V$ an open subset of $X$ contained in $U$; then $V \cap Y$ is closed in $V$. In addition, if $Y$ is defined by the quasi-coherent sheaf of ideals $\mathcal{I}$ of $\mathcal{O}_X|U$, then $\mathcal{I}|V$ is a quasi-coherent sheaf of ideals of $\mathcal{O}_X|V$, and it is immediate that the prescheme induced by $Y$ on $Y \cap V$ is the closed subprescheme of $V$ defined by the sheaf of ideals $\mathcal{I}|V$. Conversely:

**Proposition (4.1.5).** — Let $(Y, \mathcal{O}_Y)$ be a ringed space such that $Y$ is a subspace of $X$, and there exists a cover $(V_a)$ of $Y$ by open subsets of $X$ such that, for each $a$, $Y \cap V_a$ is closed in $V_a$, and the ringed space $(Y \cap V_a, \mathcal{O}_Y|_a (Y \cap V_a))$ is a closed subprescheme of the prescheme induced on $V_a$ by $X$. Then $(Y, \mathcal{O}_Y)$ is a subprescheme of $X$.

**Proof.** The hypotheses imply that $Y$ is locally closed in $X$ and that the largest open $U$ in which $Y$ is closed (and thus contained) contains all the $V_a$; we can thus reduce to the case where $U = X$ and $Y$ is closed in $X$. We then define a quasi-coherent sheaf of ideals $\mathcal{I}$ of $\mathcal{O}_X$ by taking $\mathcal{I}|V_a$ to be the sheaf of ideals of $\mathcal{O}_X|V_a$ which define the closed subprescheme $(Y \cap V_a, \mathcal{O}_Y|_a (Y \cap V_a))$, and, for each open subset $W$ of $X$ not intersecting $Y$, $\mathcal{I}|W = \mathcal{O}_X|W$. We see immediately, by Definition (4.1.3) and (4.1.4), that there exists a unique sheaf of ideals $\mathcal{I}$ satisfying these conditions, and that it defines the closed subprescheme $(Y, \mathcal{O}_Y)$.
In particular, the induced (by $X$) prescheme on an open subset of $X$ is a subprescheme of $X$.

**Proposition (4.1.6).** — A subprescheme (resp. a closed subprescheme) of a prescheme (resp. closed subprescheme) of $X$ is canonically identified with a subprescheme (resp. closed subprescheme) of $X$.

**Proof.** Since a locally closed subset of a locally closed subspace of $X$ is a locally closed subspace of $X$, it is clear (4.1.5) that the question is local and that we can thus suppose that $X$ is affine; the proposition then follows from the canonical identification of $A/\mathfrak{J}$ with $(A/\mathfrak{J})/(\mathfrak{J}/\mathfrak{J})$, where $A$ is some ring, and $\mathfrak{J}$ and $\mathfrak{J}'$ are ideals of $A$ such that $\mathfrak{J} \subset \mathfrak{J}'$.

In what follows, we will always make use of the above identification.

**Proposition (4.1.9).** — For an injection morphism $Z \to X$, where $\psi$ is the canonical injection, we sometimes (but as little as we can) use ‘majoré’.

**Corollary (4.1.10).** — For an injection morphism $Z \to X$ of preschemes (2.2.1), the hypotheses imply that, for each $z \in Z$, we have $(\psi^* (\mathcal{O}_X))_z \subset \mathcal{O}_Z$, and, as a result, that $\psi^* (\mathcal{O}_X) \subset \mathcal{O}_Z$. Thus $\psi^*$ factors as

$$\psi^* (\mathcal{O}_X) \to \mathcal{O}_Z,$$

the first arrow being the canonical homomorphism. Let $\psi'$ be the continuous map $Z \to Y$ coinciding with $\psi$; it is clear that we have $\psi'^* (\mathcal{O}_Y) = \psi^* (\mathcal{O}_X / \mathcal{J})$; on the other hand, $\omega$ is evidently a local homemorphism, so $g = (\psi', \omega)$ is a morphism of preschemes $Z \to Y (2.2.1)$, which, according to the above, is such that $f = j \circ g$, hence the proposition.

**Corollary (4.1.10).** — For an injection morphism $Z \to X$ to be factor through the injection morphism $Y \to X$, it is necessary and sufficient for $Z$ to be a subprescheme of $Y$.

We then write $Z \subseteq Y$, and this condition is evidently an ordering on the set of subpreschemes of $X$.  

[Trans.] There doesn’t seem to be an English equivalent of this, except for ‘bounded above’, which doesn’t make much sense in this context. We would normally just say that $f$ factors through $j$, but to avoid having to entirely restructure the often-lengthy sentences in the original, we sometimes (but as little as we can) use ‘majoré’.  

---

\[\mathcal{O}_X|_Z \to \mathcal{O}_Z\]
4.2. Immersion morphisms

Definition (4.2.1). — We say that a morphism \( f : Y \to X \) is an immersion (resp. a closed immersion, an open immersion) if it factors as \( Y \xrightarrow{g} Z \xrightarrow{j} X \), where \( g \) is an isomorphism, \( Z \) is a subprescheme of \( X \) (resp. a closed subprescheme, a subprescheme induced by an open set), and \( j \) is the injection morphism.

The subprescheme \( Z \) and the isomorphism \( g \) are then determined in a unique way, since if \( Z' \) is a second subprescheme of \( X \), \( j' \) the injection \( Z' \to X \), and \( g' \) an isomorphism \( Y \to Z' \) such that \( j \circ g = j' \circ g' \), then we have \( j' = j \circ g \circ g'^{-1} \), hence \( Z' \subseteq Z \) (4.1.10), and we can similarly show that \( Z \subseteq Z' \), hence \( Z' = Z \), and, since \( j \) is a monomorphism of preschemes, \( g' = g \).

We say that \( f = j \circ g \) is the canonical factorization of the immersion \( f \), and the subprescheme \( Z \) and the isomorphism \( g \) are those associated to \( f \).

It is clear that an immersion is a monomorphism of preschemes (4.1.7) and a fortiori a radical morphism (3.5.4).

Proposition (4.2.2). —

(a) For a morphism \( f = (\psi, \theta) : Y \to X \) to be an open immersion, it is necessary and sufficient for \( \psi \) to be a homeomorphism from \( Y \) to some open subset of \( X \), and, for all \( y \in Y \), that the homomorphism \( \theta_y^\sharp : \mathcal{O}_{\psi(y)} \to \mathcal{O}_y \) be bijective.

(b) For a morphism \( f = (\psi, \theta) : Y \to X \) to be an immersion (resp. a closed immersion), it is necessary and sufficient for \( \psi \) to be a homeomorphism from \( Y \) to some locally closed (resp. closed) subset of \( X \), and, for all \( y \in Y \), that the homomorphism \( \theta_y^\sharp : \mathcal{O}_{\psi(y)} \to \mathcal{O}_y \) be surjective.

Proof.

(a) The conditions are evidently necessary. Conversely, if they are satisfied, then it is clear that \( \theta^\sharp \) is an isomorphism from \( \mathcal{O}_Y \) to \( \psi^*(\mathcal{O}_X) \), and \( \psi^*(\mathcal{O}_X) \) is the sheaf induced by “transport of structure” via \( \psi^{-1} \) from \( \mathcal{O}_X|\psi(Y) \); hence the conclusion.

(b) The conditions are evidently sufficient—we prove that they are sufficient. Consider first the particular case where we suppose that \( X \) is an affine scheme, and that \( Z = \psi(Y) \) is closed in \( X \). We then know (0, 3.4.6) that \( \psi_* (\mathcal{O}_Y) \) has support equal to \( Z \), and that, denoting its restriction to \( Z \) by \( \mathcal{O}_Z \), the ringed space \( (Z, \mathcal{O}_Z) \) is induced from \( (Y, \mathcal{O}_Y) \) by transport of structure via the homeomorphism \( \psi \) considered as a map from \( Y \) to \( Z \). Let us now show that \( f_* (\mathcal{O}_Y) = \psi_* (\mathcal{O}_Y) \) is a quasi-coherent \( \mathcal{O}_X \)-module. Indeed, for all \( x \not\in Z \), \( \psi_* (\mathcal{O}_Y) \) restricted to a suitable neighborhood of \( x \) is zero. On the contrary, if \( z \in Z \), then we have \( x = \psi(y) \) for some well-defined \( y \in Y \); let \( V \) be an affine open neighborhood of \( y \) in \( Y \); \( \psi(V) \) is then open in \( Z \), and so equal to the intersection of \( Z \) with an open subset \( U \) of \( X \), and the restriction of \( U \) to \( \psi_* (\mathcal{O}_Y) \) is identical to the restriction of \( U \) to the direct image \( (\psi_V)_* (\mathcal{O}_Y|V) \), where \( \psi_V \) is the restriction of \( \psi \) to \( V \). The restriction of the morphism \( (\psi, \theta) \) to \( (V, \mathcal{O}_Y|V) \) is a morphism from this aforementioned prescheme to \( (X, \mathcal{O}_X) \), and, as a result, is of the form \((\theta^\phi, \phi)\), where \( \phi \) is the homomorphism from the ring \( A = \Gamma(X, \mathcal{O}_X) \) to the ring \( \Gamma(V, \mathcal{O}_Y) \) (1.7.3); we conclude that \( (\psi_V)_* (\mathcal{O}_Y|V) \) is a quasi-coherent \( \mathcal{O}_X \)-module (1.6.3), which proves our assertion, due to the local nature of quasi-coherent sheaves. In addition, the hypothesis that \( \psi \) is a homeomorphism implies (0, 3.4.5) that, for all \( y \in Y \), \( \psi_y \) is an isomorphism \( (\mathcal{O}_Y)_y \to \mathcal{O}_y \); since the diagram

\[
\begin{array}{ccc}
\mathcal{O}_y & \xrightarrow{\theta^\sharp_y} & \mathcal{O}_y \\
\psi_y \circ \phi_{\psi(y)} \downarrow & & \downarrow \psi_y \\
(\mathcal{O}_X)_y & \xrightarrow{\vartheta^\sharp_y} & \mathcal{O}_y
\end{array}
\]

is commutative, and the vertical arrows are the isomorphisms (0, 3.7.2), the hypothesis that \( \theta^\sharp_y \) is surjective implies that \( \theta_{\psi(y)} \) is surjective as well. Since the support of \( \psi_* (\mathcal{O}_Y) \)
is $Z = \psi(Y)$, $\theta$ is a surjective homomorphism from $\mathcal{O}_X = \bar{A}$ to the quasi-coherent $\mathcal{O}_X$-module $f_*(\mathcal{O}_\mathcal{Y})$. As a result, there exists a unique isomorphism $\omega$ from a sheaf quotient $\bar{A}/\bar{\mathfrak{a}}$ (with $\bar{\mathfrak{a}}$ being an ideal of $A$) to $f_*(\mathcal{O}_\mathcal{Y})$ which, when composed with the canonical homomorphism $A \to \bar{A}/\bar{\mathfrak{a}}$, gives $\theta$ (1.3.8); if $\mathcal{O}_Z$ denotes the restriction of $\bar{A}/\bar{\mathfrak{a}}$ to $Z$, then $(Z, \mathcal{O}_Z)$ is a subscheme of $(X, \mathcal{O}_X)$, and $f$ factors through the canonical injection of this subscheme into $X$ and the isomorphism $(\psi_0, \omega_0)$, where $\psi_0$ is $\psi$ considered as a map from $Y$ to $Z$, and $\omega_0$ the restriction of $\omega$ to $\mathcal{O}_Z$.

We now pass to the general case. Let $U$ be an affine open subset of $X$ such that $U \cap \psi(Y)$ is closed in $U$ and nonempty. By restricting $f$ to the prescheme induced by $Y$ on the open subset $\psi^{-1}(U)$, and by considering it as a morphism from this prescheme to the prescheme induced by $X$ on $U$, we reduce to the first case; the restriction of $f$ to $\psi^{-1}(U)$ is thus a closed immersion $\psi^{-1}(U) \to U$, canonically factoring as $j_U \circ g_U$, where $g_U$ is an isomorphism from the prescheme $\psi^{-1}(U)$ to a subscheme $Z_U$ of $U$, and $j_U$ is the canonical injection $Z_U \to U$. Let $V$ be a second affine open subset of $X$ such that $V \subset U$; since the restriction $Z_V$ of $Z_U$ to $V$ is a subscheme of the prescheme $V$, the restriction of $f$ to $\psi^{-1}(V)$ factors as $j'_V \circ g'_V$, where $j'_V$ is the canonical injection $Z'_V \to V$ and $g'_V$ is an isomorphism from $\psi^{-1}(V)$ to $Z'_V$. By the uniqueness of the canonical factorization of an immersion (4.2.1), we necessarily have that $Z'_V = Z_V$ and $g'_V = g_V$. We conclude (4.1.5) that there is a subscheme $Z$ of $X$ whose underlying space is $\psi(Y)$, and whose restriction to each $U \cap \psi(Y)$ is $Z_U$; the $g_U$ are then the restrictions to $\psi^{-1}(U)$ of an isomorphism $g : Y \to Z$ such that $f = j \circ g$, where $j$ is the canonical injection $Z \to X$.

\[ \square \]

**Corollary (4.2.3).** — Let $X$ be an affine scheme. For a morphism $f = (\psi, \theta) : Y \to X$ to be a closed immersion, it is necessary and sufficient for $Y$ to be an affine scheme, and the homomorphism $\Gamma(\psi) : \Gamma(\mathcal{O}_X) \to \Gamma(\mathcal{O}_Y)$ to be surjective.

**Corollary (4.2.4).**

(a) Let $f$ be a morphism $Y \to X$, and $(V_\lambda)$ a cover of $f(Y)$ by open subsets of $X$. For $f$ to be an immersion (resp. an open immersion), it is necessary and sufficient for its restriction to each of the induced preschemes $f^{-1}(V_\lambda)$ to be an immersion (resp. an open immersion) into $V_\lambda$.

(b) Let $f$ be a morphism $Y \to X$, and $(V_\lambda)$ an open cover of $X$. For $f$ to be a closed immersion, it is necessary and sufficient for its restriction to each of the induced preschemes $f^{-1}(V_\lambda)$ to be a closed immersion into $V_\lambda$.

**Proof.** Let $f = (\psi, \theta)$; in the case (a), $\theta^0_\psi$ is surjective (resp. bijective) for all $y \in Y$, and in the case (b), $\theta^0_\psi$ is surjective for all $y \in Y$; it thus suffices to check, in case (a), that $\psi$ is a homeomorphism from $Y$ to a locally closed (resp. open) subset of $X$, and, in case (b), that $\psi$ is a homeomorphism from $Y$ to a closed subset of $X$. Now $\psi$ is evidently injective, and sends each neighborhood of $y$ in $Y$ to a neighborhood of $\psi(y)$ in $\psi(Y)$ for all $y \in Y$, by virtue of the hypothesis; in case (a), $\psi(Y) \cap V_\lambda$ is locally closed (resp. open) in $V_\lambda$, so $\psi(Y)$ is locally closed (resp. open) in the union of the $V_\lambda$, and a fortiori in $X$; in case (b), $\psi(Y) \cap V_\lambda$ is closed in $V_\lambda$, so $\psi(Y)$ is closed in $X$ since $X = \bigcup_\lambda V_\lambda$.

\[ \square \]

**Proposition (4.2.5).** — The composition of any two immersions (resp. of two open immersions, of two closed immersions) is an immersion (resp. an open immersion, a closed immersion).

**Proof.** This follows easily from (4.1.6).
4.3. Products of immersions

**Proposition (4.3.1).** — Let \( \alpha : X' \to X \), \( \beta : Y' \to Y \) be two \( S \)-morphisms; if \( \alpha \) and \( \beta \) are immersions (resp. open immersions, closed immersions), then \( \alpha \times_S \beta \) is an immersion (resp. an open immersion, a closed immersion). In addition, if \( \alpha \) (resp. \( \beta \)) identifies \( X' \) (resp. \( Y' \)) with a subscheme \( X'' \) (resp. \( Y'' \)) of \( X \) (resp. \( Y \)), then \( \alpha \times_S \beta \) identifies the underlying space of \( X' \times_S Y' \) with the subspace \( p^{-1}(X'') \cap q^{-1}(Y'') \) of the underlying space of \( X \times_S Y \), where \( p \) and \( q \) denote the projections from \( X \times_S Y \) to \( X \) and \( Y \) respectively.

**Proof.** According to Definition (4.2.1), we can restrict to the case where \( X' \) and \( Y' \) are subschemes, and \( \alpha \) and \( \beta \) the injection morphisms. The proposition has already been proven for the subschemes induced by open sets (3.2.7); since every subscheme is a closed subscheme of a prescheme induced by an open set (4.1.3), we can reduce to the case where \( X' \) and \( Y' \) are closed subschemes.

Let us first show that we can assume \( S \) to be affine. Let \((S,\lambda)\) be a cover of \( S \) by affine open sets; if \( \phi \) and \( \psi \) are the structure morphisms of \( X \) and \( Y \), then \( X_\lambda = \phi^{-1}(S_\lambda) \) and \( Y_\lambda = \psi^{-1}(S_\lambda) \). The restriction \( X'_\lambda \) (resp. \( Y'_\lambda \)) of \( X' \) (resp. \( Y' \)) to \( X_\lambda \cap X' \) (resp. \( Y_\lambda \cap Y' \)) is a closed subscheme of \( X_\lambda \) (resp. \( Y_\lambda \)), the preschemes \( X_\lambda, Y_\lambda, X'_\lambda, \) and \( Y'_\lambda \) can be considered as \( S_\lambda \)-preschemes, and the products \( X_\lambda \times S_\lambda Y_\lambda \) and \( X'_\lambda \times S_\lambda Y'_\lambda \) are identical (3.2.5). If the proposition is true when \( S \) is affine, then the restriction of \( \alpha \times_S \beta \) to each of the \( X'_\lambda \times S_\lambda Y'_\lambda \) is thus an immersion (3.2.7). Since the product \( X'_\lambda \times S_\lambda Y'_\lambda \) (resp. \( X_\lambda \times S_\lambda Y_\lambda \)) can be identified with \( (X_\lambda \times S_\lambda Y_\lambda) \times_S (Y_\lambda \times S_\lambda Y_\lambda) \) (resp. \( (X'_\lambda \times S_\lambda Y'_\lambda) \times_S (Y'_\lambda \times S_\lambda Y'_\lambda) \), the restriction of \( \alpha \times S \beta \) to each of the \( X'_\lambda \times S_\lambda Y'_\lambda \) is also an immersion; the same is true for \( \alpha \times_S \beta \) by (4.2.4).

Next, we show that we can assume \( X \) and \( Y \) to be affine. Indeed, let \((U_i, V_i)\) (resp. \((V_j, V_j)\)) be a cover of \( X \) (resp. \( Y \)) by affine open sets, and let \( X'_i \) (resp. \( Y'_j \)) be the restriction of \( X' \) (resp. \( Y' \)) to \( X' \cap U_i \) (resp. \( Y' \cap V_j \)), which is a closed subscheme of \( U_i \) (resp. \( V_j \)); \((U_i \times S V_j)\) can be identified with the restriction of \( X \times_S Y \) to \( p^{-1}(U_i) \cap q^{-1}(V_j) \) (3.2.7); similarly, if \( p' \) and \( q' \) are the projections from \( X' \times_S Y' \), then \( X'_i \times_S Y'_j \) can be identified with the restriction of \( X' \times_S Y' \) to \( p'^{-1}(X'_i) \cap q'^{-1}(Y'_j) \).

Set \( \gamma = \alpha \times_S \beta \); we have, by definition, \( p \circ \gamma = \alpha \circ p' \) and \( q \circ \gamma = \beta \circ q' \); since \( X'_i = \alpha^{-1}(U_i) \) and \( Y'_j = \beta^{-1}(V_j) \), we also have \( p'^{-1}(X'_i) = \gamma^{-1}(p^{-1}(U_i)) \) and \( q'^{-1}(Y'_j) = \gamma^{-1}(q^{-1}(V_j)) \), hence

\[
p'^{-1}(X'_i) \cap q'^{-1}(Y'_j) = \gamma^{-1}(p^{-1}(U_i)) \cap q^{-1}(V_j) = \gamma^{-1}(U_i \times S V_j),
\]

and we then conclude as in the previous part of the proof.

So suppose \( X, Y, \) and \( S \) are affine, and let \( B, C, \) and \( A \) be their respective rings. Then \( B \) and \( C \) are \( A \)-algebras, and \( X' \) and \( Y' \) are affine schemes whose rings are quotient algebras \( B' \) and \( C' \) of \( B \) and \( C \) respectively. In addition, we have \( \alpha = (\alpha \times \beta) \) and \( \beta = (\beta \times \alpha) \), where \( \alpha \) and \( \beta \) are (respectively) the canonical homomorphisms \( B \to B' \) and \( C \to C' \) (1.7.3). With that in mind, we know that \( X \times_S Y \) (resp. \( X' \times_S Y' \)) is an affine scheme with ring \( B \otimes_A C \) (resp. \( B' \otimes_A C' \)), and \( \alpha \times_S \beta \) is \( (\alpha \otimes \beta) \), where \( \alpha \) is the homomorphism \( \rho \otimes \sigma \) from \( B \otimes_A C \) to \( B' \otimes_A C' \) (Proposition (3.2.2) and Corollary (3.2.3)); since this homomorphism is surjective, \( \alpha \times_S \beta \) is an immersion. In addition, if \( \beta \) (resp. \( \epsilon \)) is the kernel of \( \rho \) (resp. \( \sigma \)), then the kernel of \( \tau \) is \( u(\beta) + v(\epsilon) \), where \( u \) (resp. \( v \)) is the homomorphism \( B \to B \otimes 1 \) (resp. \( c \to 1 \otimes c \)). Since \( p = (p, \overline{u}) \) and \( q = (q, \overline{v}) \), this kernel corresponds, in the prime spectrum of \( B \otimes_A C \), to the closed set \( p^{-1}(X') \cap q^{-1}(Y') \) (1.2.2.1) and Proposition (1.1.2, iii)), which finishes the proof. \( \square \)

**Corollary (4.3.2).** — If \( f : X \to Y \) is an immersion (resp. an open immersion, a closed immersion) and an \( S \)-morphism, then \( f_!(S') \) is an immersion (resp. an open immersion, a closed immersion) for every extension \( S' \to S \) of the base prescheme.
4.4. Inverse images of a subprescheme

Proposition (4.4.1). — Let \( f : X \to Y \) be a morphism, \( Y' \) a subprescheme (resp. a closed subprescheme, a prescheme induced by an open set) of \( Y \), and \( j' : Y' \to Y \) the injection morphism. Then the projection \( p : X \times_Y Y' \to X \) is an immersion (resp. a closed immersion, an open immersion); the underlying space of the subprescheme of \( X \) associated to \( p \) is \( f^{-1}(Y') \); in addition, if \( j' \) is the injection morphism of this subprescheme into \( X \), then for a morphism \( h : Z \to X \) to be such that \( f \circ h : Z \to Y \) factors through \( j \), it is necessary and sufficient for \( h \) to factor through \( j' \).

Proof. Since \( p = 1_X \times_Y j \) (3.3.4), the first claim follows from Proposition (4.3.1); the second is a particular case of Corollary (3.5.10) (after swapping the roles of \( X \) and \( Y' \)). Finally, if we have \( f \circ h = j \circ h' \), where \( h' \) is a morphism \( Z \to Y' \), then it follows from the definition of the product that we have \( h = p \circ u \), where \( u \) is a morphism \( Z \to X \times_Y Y' \), whence the last claim.

We say that the subprescheme of \( X \) thus defined is the inverse image of the subprescheme \( Y' \) of \( Y \) under the morphism \( f \), terminology which is consistent with that introduced more generally in (3.3.6). When we speak of \( f^{-1}(Y') \) as a subprescheme of \( X \), this will always be the subprescheme we mean.

When the preschemes \( f^{-1}(Y') \) and \( X \) are identical, \( j' \) is the identity and each morphism \( h : Z \to X \) thus factors through \( j' \), so the morphism \( f : X \to Y \) factors as \( X \xrightarrow{\delta} Y' \xrightarrow{\beta} Y \).

When \( y \) is a closed point of \( Y \) and \( Y' = \text{Spec}(k(y)) \) is the smallest closed subprescheme of \( Y \) having \( \{y\} \) as its underlying space (4.1.9), the closed subprescheme \( f^{-1}(Y') \) is canonically isomorphic to the fibre \( f^{-1}(y) \) defined in (3.6.2), and we will use this identification in all that follows.

Corollary (4.4.2). — Let \( f : X \to Y \) and \( g : Y \to Z \) be morphisms, and \( h = g \circ f \) their composition. For each subprescheme \( Z' \) of \( Z \), the subpreschemes \( f^{-1}(g^{-1}(Z')) \) and \( h^{-1}(Z') \) of \( X \) are identical.

Proof. This follows from the existence of the canonical isomorphism \( X \times_Y (Y \times_Z Z') \cong X \times_Z Z' \) (3.3.9.1).

Corollary (4.4.3). — Let \( X' \) and \( X'' \) be subpreschemes of \( X \), and \( j' : X' \to X \), and \( j'' : X'' \to X \) their injection morphisms; then \( j'^{-1}(X'') \) and \( j''^{-1}(X') \) are both equal to the greatest lower bound \( \inf(X', X'') \) of \( X' \) and \( X'' \) for the ordering \( \leq \) on subpreschemes, and this is canonically isomorphic to \( X' \times_X X'' \).

Proof. This follows immediately from Proposition (4.4.1) and Corollary (4.1.10).

Corollary (4.4.4). — Let \( f : X \to Y \) be a morphism, and \( Y' \) and \( Y'' \) subpreschemes of \( Y \); then we have \( \inf(f^{-1}(Y'), f^{-1}(Y'')) = \inf(f^{-1}(Y'), f^{-1}(Y'')) \).

Proof. This follows from the existence of the canonical isomorphism between \( (X \times_Y Y') \times_X (X \times_Y Y'') \) and \( X \times_Y (Y' \times_Y Y'') \) (3.3.9.1).

Proposition (4.4.5). — Let \( f : X \to Y \) be a morphism, and \( Y' \) a closed subprescheme of \( Y \) defined by a quasi-coherent sheaf of ideals \( \mathcal{H} \) of \( \mathcal{O}_Y \) (4.1.3); the closed subprescheme \( f^{-1}(Y') \) of \( X \) is then defined by the quasi-coherent sheaf of ideals \( f^*(\mathcal{H}) \mathcal{O}_X \) of \( \mathcal{O}_X \).

Proof. The statement is evidently local on \( X \) and \( Y \); it thus suffices to note that if \( A \) is a \( B \)-algebra and \( \mathfrak{a} \) an ideal of \( B \), then we have \( A \otimes_B (B/\mathfrak{a}) = A/\mathfrak{a}A \), and to then apply (1.6.9).

Corollary (4.4.6). — Let \( X' \) be a closed subprescheme of \( X \) defined by a quasi-coherent sheaf of ideals \( \mathcal{J} \) of \( \mathcal{O}_X \), and \( i \) the injection \( X' \to X \); for the restriction \( f \circ i \) of \( f \) to \( X' \) to factor through the injection \( j' : Y' \to Y \) (in other words, for it to factor as \( j \circ g \), with \( g \) a morphism \( X' \to Y' \)), it is necessary and sufficient that \( f^*(\mathcal{H}) \subseteq \mathcal{J} \).

Proof. It suffices to apply Proposition (4.4.1) to \( i \), taking Proposition (4.4.5) into account.
4.5. Local immersions and local isomorphisms

**Definition (4.5.1).** — Let $f : X \to Y$ be a morphism of preschemes. We say that $f$ is a local immersion at a point $x \in X$ if there exists an open neighborhood $U$ of $x$ in $X$ and an open neighborhood $V$ of $f(x)$ in $Y$ such that the restriction of $f$ to the induced prescheme $U$ is a closed immersion of $U$ into the induced prescheme $V$. We say that $f$ is a local immersion if $f$ is a local immersion at each point of $X$.

**Definition (4.5.2).** — We say that a morphism $f : X \to Y$ is a local isomorphism at a point $x \in X$ if there exists an open neighborhood $U$ of $x$ in $X$ such that the restriction of $f$ to the induced prescheme $U$ is an open immersion of $U$ into $Y$. We say that $f$ is a local isomorphism if $f$ is a local isomorphism at each point of $X$.

**Remark (4.5.3).** An immersion (resp. a closed immersion) $f : X \to Y$ can be characterized as a local immersion such that $f$ is a homeomorphism from the underlying space of $X$ to a subset (resp. a closed subset) of $Y$. An open immersion $f$ can be characterized as an injective local isomorphism.

**Proposition (4.5.4).** — Let $X$ be an irreducible prescheme, and $f : X \to Y$ a dominant injective morphism. If $f$ is a local immersion, then $f$ is an immersion, and $f(X)$ is open in $Y$.

**Proof.** Let $x \in X$, and let $U$ be an open neighborhood of $x$, and $V$ an open neighborhood of $f(x)$ in $Y$ such that the restriction of $f$ to $U$ is a closed immersion into $V$; since $U$ is dense in $X$, $f(U)$ is dense in $Y$ by hypothesis, so $f(U) = V$, and $f$ is a homeomorphism from $U$ to $V$; the hypothesis that $f$ is injective implies that $f^{-1}(V) = U$, hence the proposition. □

**Proposition (4.5.5).**

(i) The composition of any two local immersions (resp. two local isomorphisms) is a local immersion (resp. a local isomorphism).

(ii) Let $f : X \to X'$ and $g : Y \to Y'$ be two $S$-morphisms. If $f$ and $g$ are local immersions (resp. local isomorphisms), then so too is $f \times_S g$.

(iii) If an $S$-morphism $f$ is a local immersion (resp. a local isomorphism), then so too is $f|_{(S')}$ for every extension $S' \to S$ of the base prescheme.

**Proof.** According to (3.5.1), it suffices to prove (i) and (ii).

(i) follows immediately from the transitivity of closed (resp. open) immersions (4.2.5) and from the fact that if $f$ is a homeomorphism from $X$ to a closed subset of $Y$, then for every open $U \subset X$, $f(U)$ is open in $f(X)$, so there exists an open subset $V$ of $Y$ such that $f(U) = V \cap f(X)$, and, as a result, $f(U)$ is closed in $V$.

To prove (ii), let $p$ and $q$ be the projections from $X \times_Y Y$, and $p'$ and $q'$ the projections from $X' \times_S Y'$. There exists, by hypothesis, open neighborhoods $U, U', V, V'$ of $x = p(z), x' = p'(z'), y = q(z)$, and $y' = q'(z')$ (respectively), such that the restrictions of $f$ and $g$ to $U$ and $V$ (respectively) are closed (resp. open) immersions into $U'$ and $V'$ (respectively). Since the underlying spaces of $U \times_S V$ and $U' \times_S V'$ can be identified with the open neighborhoods $p^{-1}(U) \cap q^{-1}(V)$ and $p'^{-1}(U') \cap q'^{-1}(V')$ of $z$ and $z'$ (respectively) (3.2.7), the proposition follows from Proposition (4.3.1). □

§5. Reduced preschemes; the separation condition

5.1. Reduced preschemes

**Proposition (5.1.1).** — Let $(X, O_X)$ be a prescheme, and $\mathcal{B}$ a quasi-coherent $O_X$-algebra. Then there exists a unique quasi-coherent $O_X$-module $\mathcal{N}$ whose stalk $\mathcal{N}_x$ at any $x \in X$ is the nilradical of the ring $\mathcal{B}_x$. When $X$ is affine, and, consequently, $\mathcal{B} = \mathcal{B}$, where $\mathcal{B}$ is an algebra over $A(X)$, then we have $\mathcal{N} = \mathfrak{N}$, where $\mathfrak{N}$ is the nilradical of $B$.

**Proof.** The statement is local, so it suffices to show the latter claim. We know that $\mathfrak{N}$ is a quasi-coherent $O_X$-module (1.4.1), and that its stalk at a point $x \in X$ is the ideal $\mathfrak{N}_x$ of the ring of fractions $B_x$; it remains to prove that the nilradical of $B_x$ is contained in $\mathfrak{N}_x$, the converse inclusion being evident. Let $z/s$ be an element of the nilradical of $B_x$, with $z \in B$, and $s \notin j_x$; by hypothesis, there exists an integer $k$ such that $(z/s)^k = 0$, which implies that there exists some $t \notin j_x$ such that $tz^k = 0$. We conclude that $(tz)^k = 0$, and, as a result, that $z/s = (tz)/(ts) \in \mathfrak{N}_x$. □
We say that the quasi-coherent $\mathcal{O}_X$-module $\mathcal{N}$ thus defined is the nilradical of the $\mathcal{O}_X$-algebra $\mathcal{B}$; in particular, we denote by $\mathcal{N}_X$ the nilradical of $\mathcal{O}_X$.

**Corollary (5.1.2).** — Let $X$ be a prescheme; the closed subprescheme of $X$ defined by the sheaf of ideals $\mathcal{N}_X$ is the only reduced subprescheme (0.4.14) of $X$ that has $X$ as its underlying space; it is also the smallest subprescheme of $X$ that has $X$ as its underlying space.

**Proof.** Since the structure sheaf of the closed subprescheme of $Y$ defined by $\mathcal{N}_X$ is $\mathcal{O}_X/\mathcal{N}_X$, it is immediate that $Y$ is reduced and has $X$ as its underlying space, because $\mathcal{N}_X \neq \mathcal{O}_X$ for any $x \in X$. To show the other claims, note that a subprescheme $Z$ of $X$ that has $X$ as its underlying space is defined by a sheaf of ideals $\mathcal{I}$ (4.1.3) such that $\mathcal{I}_x \neq \mathcal{O}_x$ for any $x \in X$. We can restrict to the case where $X$ is affine, say $X = \text{Spec}(A)$ and $\mathcal{I} = \mathfrak{I}$, where $\mathfrak{I}$ is an ideal of $A$; then, for every $x \in X$, we have $\mathfrak{I}_x \subset \mathfrak{m}_x$, and so $\mathfrak{I}$ is contained in every prime ideal of $A$, and so also in their intersection $\mathfrak{n}_l$, the nilradical of $A$. This proves that $Y$ is the small subprescheme of $X$ that has $X$ as its underlying space (4.1.9); furthermore, if $Z$ is distinct from $Y$, we necessarily have $\mathcal{I}_X \neq \mathcal{N}_X$ for at least one $x \in X$, and so (5.1.1) $Z$ is not reduced. \qed

**Definition (5.1.3).** — We define the reduced prescheme associated to a prescheme $X$, denoted by $X_{\text{red}}$, to be the unique reduced subprescheme of $X$ that has $X$ as its underlying space.

Saying that a prescheme $X$ is reduced thus implies that $X = X_{\text{red}}$.

**Proposition (5.1.4).** — For the prime spectrum of a ring $A$ to be a reduced (resp. integral) prescheme (2.1.7), it is necessary and sufficient for $A$ to be a reduced (resp. integral) ring.

**Proof.** Indeed, it follows immediately from (5.1.1) that the condition $\mathcal{N} = (0)$ is necessary and sufficient for $X = \text{Spec}(A)$ to be reduced; the claim corresponding to integral rings is then a consequence of (1.1.13). \qed

Since every ring of fractions $\neq \{0\}$ of an integral ring is integral, it follows from (5.1.4) that, for every locally integral prescheme $X$, $\mathcal{O}_X$ is an integral ring for every $x \in X$. The converse is true whenever the underlying space of $X$ is locally Noetherian: indeed, $X$ is then reduced, and if $U$ is an affine open subset of $X$, which is a Noetherian space, then $U$ has only a finite number of irreducible components, and so its ring $A$ has only a finite number of minimal prime ideals (1.1.14). If two of the irreducible components of $U$ had a common point $x$, then $\mathcal{O}_X$ would have at least two distinct minimal prime ideals, and would thus not be integral; the components of $U$ are thus open subsets that are pairwise disjoint, and each of them is thus integral.

**Lemma (5.1.5).** Let $f = (\psi, \theta) : X \to Y$ be a morphism of preschemes; the homomorphism $\theta^0 : \mathcal{O}_{\psi(x)} \to \mathcal{O}_x$, sends each nilpotent element of $\mathcal{O}_{\psi(x)}$ to a nilpotent element of $\mathcal{O}_x$; by passing to the quotients, $\theta^0$ induces a homomorphism 

$$\omega : \psi^0(\mathcal{O}_Y/\mathcal{N}_Y) \longrightarrow \mathcal{O}_X/\mathcal{N}_X.$$ 

It is clear that, for every $x \in X$, $\omega_x : \mathcal{O}_{\psi(x)}/\mathcal{N}_{\psi(x)} \to \mathcal{O}_x/\mathcal{N}_x$ is a local homomorphism, and so $(\psi, \omega^0)$ is a morphism of preschemes $X_{\text{red}} \to Y_{\text{red}}$, which we denote by $f_{\text{red}}$, and call the reduced morphism associated to $f$. It is immediate that, for morphisms $f : X \to Y$ and $g : Y \to Z$, we have $(g \circ f)_{\text{red}} = g_{\text{red}} \circ f_{\text{red}}$, and so we have defined $X_{\text{red}}$ as a functor, covariant in $X$.

The preceding definition shows that the diagram

$$\begin{array}{ccc}
X_{\text{red}} & \xrightarrow{f_{\text{red}}} & Y_{\text{red}} \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}$$

is commutative, where the vertical arrows are the injection morphisms; in other words, $X_{\text{red}} \to X$ is a functorial morphism. We note in particular that, if $X$ is reduced, then every morphism $f : X \to Y$ factors as $X \xrightarrow{f_{\text{red}}} Y_{\text{red}} \to Y$; in other words, $f$ factors through the injection morphism $Y_{\text{red}} \to Y$. 

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**Proposition (5.1.6).** — Let \( f : X \to Y \) be a morphism; if \( f \) is surjective (resp. radicial, an immersion, a closed immersion, an open immersion, a local immersion, a local isomorphism), then so too is \( f_{\text{red}} \). Conversely, if \( f_{\text{red}} \) is surjective (resp. radicial), then so too is \( f \).

**Proof.** The proposition is trivial if \( f \) is surjective; if \( f \) is radicial, then the proposition follows from the fact that, for every \( x \in X \), the field \( k(x) \) is the same for the preschemes \( X \) and \( X_{\text{red}} \) (3.5.8). Finally, if \( f = (\psi, \theta) \) is an immersion, a closed immersion, or a local immersion (resp. an open immersion, or a local isomorphism), then the proposition follows from the fact that, if \( \Theta_{\psi(x)} \) is surjective (resp. bijective), then so too is the homomorphism obtained by passing to the quotients by the nilradicals \( \Theta_{\psi(x)} \) and \( \Theta_{x} \) ((5.1.2) and (4.2.2)) (cf. (5.5.12)). \( \square \)

**Proposition (5.1.7).** — If \( X \) and \( Y \) are \( S \)-preschemes, then the preschemes \( X_{\text{red}} \times_{\text{red}} Y_{\text{red}} \) and \( X_{\text{red}} \times_{S} Y_{\text{red}} \) are identical, and canonically identified with a subscheme of \( X \times_{S} Y \) that has the same underlying subspace as the two aforementioned products.

**Proof.** The canonical identification of \( X_{\text{red}} \times_{S} Y_{\text{red}} \) with a subscheme of \( X \times_{S} Y \) that has the same underlying space follows from (4.3.1). Furthermore, if \( \phi \) and \( \psi \) are the structure morphisms \( X_{\text{red}} \to S \) and \( Y_{\text{red}} \to S \) (respectively), then they factor through \( S_{\text{red}} \) (5.1.5), and since \( S_{\text{red}} \to S \) is a monomorphism, the first claim of the proposition follows from (3.2.4). \( \square \)

**Corollary (5.1.8).** — The preschemes \( (X \times_{S} Y)_{\text{red}} \) and \( (X_{\text{red}} \times_{S_{\text{red}}} Y_{\text{red}})_{\text{red}} \) are canonically identified with one another.

**Proof.** This follows from (5.1.2) and (5.1.7). \( \square \)

We note that, even if \( X \) and \( Y \) are reduced preschemes, \( X \times_{S} Y \) might not be reduced, because the tensor product of two reduced algebras can have nilpotent elements.

**Proposition (5.1.9).** — Let \( X \) be a prescheme, and \( \mathcal{I} \) a quasi-coherent sheaf of ideals of \( \mathcal{O}_{X} \) such that \( \mathcal{I}^{n} = 0 \) for some integer \( n > 0 \). Let \( X_{0} \) be the closed subscheme \( (X, \mathcal{O}_{X}/\mathcal{I}) \) of \( X \); for \( X \) to be an affine scheme, it is necessary and sufficient for \( X_{0} \) to be an affine scheme.

The condition is clearly necessary, so we will show that it is sufficient. If we set \( X_{k} = (X, \mathcal{O}_{X}/\mathcal{I}^{k+1}) \), it is enough to prove by induction on \( k \) that \( X_{k} \) is affine, and so we are led to consider the base case, where \( \mathcal{I}^{2} = 0 \). We set

\[
\begin{align*}
A &= \Gamma(X, \mathcal{O}_{X}) \\
A_{0} &= \Gamma(X_{0}, \mathcal{O}_{X_{0}}) = \Gamma(X, \mathcal{O}_{X}/\mathcal{I}).
\end{align*}
\]

The canonical homomorphism \( \mathcal{O}_{X} \to \mathcal{O}_{X}/\mathcal{I} \) induces a homomorphism of rings \( \phi : A \to A_{0} \). We will see below that \( \phi \) is surjective, which implies that

(5.1.9.1) \[
0 \longrightarrow \Gamma(X, \mathcal{I}) \longrightarrow \Gamma(X, \mathcal{O}_{X}) \longrightarrow \Gamma(X, \mathcal{O}_{X}/\mathcal{I}) \longrightarrow 0
\]

is an exact sequence. We now prove, assuming that this is true, the proposition. Note that \( \mathcal{R} = \Gamma(X, \mathcal{I}) \) is an ideal whose square is zero in \( A \), and thus a module over \( A_{0} = A/\mathcal{R} \). By hypothesis, we have \( X_{0} = \text{Spec}(A) \), and, since the underlying spaces of \( X_{0} \) and \( X \) are identical, \( \mathcal{R} = \Gamma(X_{0}, \mathcal{I}) \); additionally, since \( \mathcal{I}^{2} = 0 \), \( \mathcal{I} \) is a quasi-coherent \( (\mathcal{O}_{X}/\mathcal{I}) \)-module, so we have \( \mathcal{I} \cong \mathcal{R} \) and \( \mathcal{R}_{x} \) for all \( x \in X_{0} \) (1.4.1). With this in mind, let \( X' = \text{Spec}(A) \), and consider the morphism \( f = (\psi, \theta) : X \to X' \) of preschemes that corresponds to the identity map \( A \to \Gamma(X, \mathcal{O}_{X}) \) (2.2.4). For every affine open subset \( V \) of \( X \), the diagram

\[
\begin{array}{ccc}
A & \longrightarrow & \Gamma(V, \mathcal{O}_{X}|_{V}) \\
\downarrow & & \downarrow \\
A_{0} = A/\mathcal{R} & \longrightarrow & \Gamma(V, \mathcal{O}_{X_{0}}|_{V})
\end{array}
\]
commutes, whence the diagram
\[
\begin{array}{ccc}
X' & \leftarrow & X \\
\uparrow & & \uparrow \\
Y' & \leftarrow & Y \\
\downarrow & & \downarrow \\
X_0' & \leftarrow & X_0
\end{array}
\]
also commutes ($X_0'$ being the closed subscheme induced on an open subset of $X'$ defined by the quasi-coherent sheaf of ideals $\mathcal{R}$, and $j$ and $j'$ being the canonical injection morphisms). But since $X_0$ is affine, $f_0$ is an isomorphism, and since the underlying continuous maps of $j$ and $j'$ are identity maps, we see straight away that $\psi : X \to X'$ is a homeomorphism. Furthermore, the equation $\mathfrak{r}_x = \mathcal{F}$, shows that the restriction of $\theta^\sharp : \psi^* (\mathcal{O}_{X'}) \to \mathcal{O}_X$ is an isomorphism from $\psi^* (\mathcal{R})$ to $\mathcal{F}$; additionally, by passing to the quotients, $\theta^\sharp$ gives an isomorphism $\psi^* (\mathcal{O}_X/\mathcal{R}) \to \mathcal{O}_X/\mathcal{F}$, because $f_0$ is an isomorphism; we thus immediately conclude, by the 5 lemma (M, I, 1.1), that $\theta^\sharp$ is itself an isomorphism, and thus that $f$ is an isomorphism, and thus that $X$ is affine. So everything reduces to proving the exactitude of (5.1.9.1), which will follow from showing that $H^1(X, \mathcal{F}) = 0$. But $H^1(X, \mathcal{F}) = H^1(X_0, \mathcal{F})$, and we have seen that $\mathcal{F}$ is a quasi coherent $\mathcal{O}_{X_0}$-module. Our proof will thus follow from

Lemma (5.1.9.2). — If $Y$ is an affine scheme, and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_Y$-module, then $H^1(Y, \mathcal{F}) = 0$.

Proof. This lemma will be proven in Chapter III, §1, as a consequence of the more general theorem that $H^i(Y, \mathcal{F}) = 0$ for all $i > 0$. To give an independent proof, note that $H^1(Y, \mathcal{F})$ can be identified with the module $\text{Ext}^1_{\mathcal{O}_Y}(Y, \mathcal{F})$ of extensions classes of the $\mathcal{O}_Y$-module $\mathcal{F}$ by the $\mathcal{O}_Y$-module $\mathcal{F}$ (T, 4.2.3); so everything reduces to proving that such an extension $\mathcal{F}$ is trivial. But, for all $y \in Y$, there is a neighbourhood $V$ of $y$ in $Y$ such that $\mathcal{F}|V$ is isomorphic to $\mathcal{F}|Y \oplus \mathcal{O}_V|V$ (0, 5.4.9); from this we conclude that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module. If $A$ is the ring of $Y$, then we have $\mathcal{F} = M$ and $\mathcal{F} = N$, where $M$ and $N$ are $A$-modules, and, by hypothesis, $N$ is an extension of the $A$-module $A$ by the $A$-module $M$ (1.3.11). Since this extension is necessarily trivial, the lemma is proven, and thus so too is (5.1.9).

Corollary (5.1.10). — Let $X$ be a prescheme such that $\mathcal{N}_X$ is nilpotent. For $X$ to be an affine scheme, it is necessary and sufficient for $X_{\text{red}}$ to be an affine scheme.

5.2. Existence of a subscheme with a given underlying space

Proposition (5.2.1). — For every locally closed subspace $Y$ of the underlying space of a prescheme $X$, there exists exactly one reduced subscheme of $X$ that has $Y$ as its underlying space.

Proof. The uniqueness follows from (5.1.2), so it remains only to show the existence of the prescheme in question.

If $X$ is affine, given by some ring $A$, and $Y$ closed in $X$, then the proposition is immediate: $j(Y)$ is the largest ideal $a \subset A$ such that $V(a) = Y$, and it is equal to its radical (1.1.4, i), so $A/j(Y)$ is a reduced ring.

In the general case, for every affine open $U \subset X$ such that $U \cap Y$ is closed in $U$, consider the closed subscheme $Y_U$ of $U$ defined by the sheaf of ideals associated to the ideal $j(U \cap Y)$ of $A(U)$, which is reduced. We can show that, if $V$ is an affine open subset of $X$ contained in $U$, then $Y_U$ is induced by $Y_U$ on $V \cap Y$; but this induced prescheme is a closed subscheme of $V$ which is reduced and has $V \cap Y$ as its underlying space; the uniqueness of $Y_U$ thus implies our claim.

Proposition (5.2.2). — Let $X$ be a reduced subscheme of a prescheme $Y$; if $Z$ is the closed reduced subscheme of $Y$ that has $X$ as its underlying space, then $X$ is a subscheme induced on an open subset of $Z$.

Proof. There is indeed an open subset $U$ of $Y$ such that $X = U \cap \overline{X}$; since, by (5.2.2), $X$ is a reduced subscheme of $Z$, the subscheme $X$ is induced by $Z$ on the open subspace $X$ by uniqueness (5.2.1).

Corollary (5.2.4). — Let $f : X \to Y$ be a morphism, and $X'$ (resp. $Y'$) a closed subscheme of $X$ (resp. $Y$) defined by a quasi-coherent sheaf of ideals $\mathcal{J}$ (resp. $\mathcal{K}$) of $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$). Suppose that $X'$ is reduced, and that $f(X') \subset Y'$. Then $f^*(\mathcal{K})\mathcal{O}_X \subset \mathcal{J}$. 

Proof. Since, by (5.2.2), the restriction of \( f \) to \( X' \) factors as \( X' \to Y' \to Y \), it suffices to apply (4.4.6).

5.3. Diagonal; graph of a morphism

(5.3.1). Let \( X \) be an \( S \)-prescheme; we define the diagonal morphism of \( X \) in \( X \times_S X \), denoted by \( \Delta_X \), or \( \Delta_X |_{S'} \), or even \( \Delta \) if no confusion may arise, to be the \( S \)-morphism \( (1_X, 1_X)_S \), or, in other words, the unique \( S \)-morphism \( \Delta_X \) such that

\[
\begin{align*}
p_1 \circ \Delta_X &= p_2 \circ \Delta_X = 1_X, \\
\end{align*}
\]

where \( p_1 \) and \( p_2 \) are the projections of \( X \times_S X \) (Definition (3.2.1)). If \( f : T \to X \) and \( g : T \to Y \) are \( S \)-morphisms, we immediately have that

\[
(f, g)_S = (f \times_S g) \circ \Delta_T |_{S'}.
\]

The reader will note that the preceding definition and the results stated in (5.3.1) to (5.3.8) are true in any category, provided that the products used within exist in the category.

Proposition (5.3.2). — Let \( X \) and \( Y \) be \( S \)-preschemes, and \( \phi : S \to T \) a morphism of preschemes, which lets us consider every \( S \)-prescheme as a \( T \)-prescheme. Let \( f : X \to S \) and \( g : Y \to S \) be the structure morphisms, \( p \) and \( q \) the projections of \( X \times_S Y \), and \( \pi = f \circ p = g \circ q \) the structure morphism \( X \times_S Y \to S \). Then the diagram

\[
\begin{array}{ccc}
X \times_S Y & \xrightarrow{(p, q)_T} & X \times_T Y \\
\downarrow_{\pi} & & \downarrow_{f \times_T g} \\
S & \xrightarrow{\Delta_{S|T}} & S \times_T S
\end{array}
\]

commutes, and identifies \( X \times_S Y \) with the product of the \((S \times_S T)\)-preschemes \( S \) and \( X \times_T Y \), and the projections with \( \pi \) and \((p, q)_T\).

Proof. By (3.4.3), we are led to proving the corresponding proposition in the category of sets, replacing \( X, Y, \) and \( S \) by \( X(Z)_T, Y(Z)_T, \) and \( S(Z)_T \) (respectively), with \( Z \) being an arbitrary \( T \)-prescheme. But, in the category of sets, the proof is immediate and left to the reader.

Corollary (5.3.6). — The morphism \((p, q)_T\) can be identified (letting \( P = S \times_T S \)) with \( 1_{X \times_Y} \times_P \Delta_S \).

Proof. This follows from (5.3.5) and (3.3.4).

Corollary (5.3.7). — If \( f : X \to Y \) is an \( S \)-morphism, then the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{(1_X, f)} & X \times_S Y \\
\downarrow_{f} & & \downarrow_{f \times_{S} 1_Y} \\
Y & \xrightarrow{\Delta_Y} & Y \times_S Y
\end{array}
\]

commutes.
commutes, and identifies $X$ with the product of the $(Y \times_S Y)$-preschemes $Y$ and $X \times_S Y$.

**Proof.** It suffices to apply (5.3.5), replacing $S$ by $Y$, and $T$ by $S$, and noting that $X \times_Y Y = X$ (3.3.3).

**Proposition (5.3.8).** — For $f : X \to Y$ to be a monomorphism of preschemes, it is necessary and sufficient for $\Delta_{X|Y}$ to be an isomorphism from $X$ to $X \times_Y X$.

**Proof.** Indeed, to say that $f$ is a monomorphism implies that, for every $Y$-prescheme $Z$, the corresponding map $f' : X(Z)_Y \to Y(Z)_Y$ is an injection, and, since $Y(Z)_Y$ consists of a single element, this implies that $X(Z)_Y$ consists of a single element as well. But this can also be expressed by saying that $X(Z)_Y \times X(Z)_Y$ is canonically isomorphic to $X(Z)_Y$; the former is exactly the set $(X \times_Y X)(Z)_Y$ (3.4.3.1), which implies that $\Delta_{X|Y}$ is an isomorphism.

**Proposition (5.3.9).** — The diagonal morphism $\Delta_X$ is an immersion from $X$ to $X \times_S X$.

**Proof.** Indeed, since the continuous maps $p_1$ and $\Delta_X$ from the underlying spaces are such that $p_1 \circ \Delta_X$ is the identity, $\Delta_X$ is a homeomorphism from $X$ to $\Delta_X(X)$. Similarly, the composite homomorphism $\mathcal{O}_S \to \mathcal{O}_{\Delta_X(X)} \to \mathcal{O}_S$ (composed of the homomorphisms corresponding to $p_1$ and $\Delta_X$) is the identity, which means that the homomorphism corresponding to $\Delta_X$ is surjective; the proposition thus follows from (4.2.2).

We say that the subscheme of $X \times_S X$ associated to the immersion $\Delta_X$ (4.2.1) is the diagonal of $X \times_S X$.

**Corollary (5.3.10).** — Under the hypotheses of (5.3.5), $(p,q)_T$ is an immersion.

**Proof.** This follows from (5.3.6) and (4.3.1).

We say (under the hypotheses of (5.3.5)) that $(p,q)_T$ is the canonical immersion of $X \times_S Y$ into $X \times_T Y$.

**Corollary (5.3.11).** — Let $X$ and $Y$ be $S$-preschemes, and $f : X \to Y$ an $S$-morphism; then the graph morphism $\Gamma_f = (1_X,f)_S$ of $f$ (3.3.14) is an immersion of $X$ into $X \times_S Y$.

**Proof.** This is a particular case of Corollary (5.3.10), where we replace $S$ by $Y$, and $T$ by $S$ (cf. (3.3.7)).

The subscheme of $X \times_S Y$ associated to the immersion $\Gamma_f$ (4.2.1) is called the graph of the morphism $f$; the subschemes of $X \times_S Y$ that are graphs of morphisms $X \to Y$ are characterised by the property that the restriction to such a subscheme $G$ of the projection $p_1 : X \times_S Y \to X$ is an isomorphism $g$ from $G$ to $X$: $G$ is the graph of the morphism $p_2 \circ g^{-1}$, where $p_2$ is the projection $X \times_S Y \to Y$.

When we take, in particular, $X = S$, then the $S$-morphisms $S \to Y$ (which are exactly the $S$-sections of $Y$ (2.5.5)) are equal to their graph morphisms; the subschemes of $Y$ that are the graphs of $S$-sections (in other words, those that are isomorphic to $S$ by the restriction of the structure morphism $Y \to S$) are then also called the images of these sections, or, by an abuse of language, the $S$-sections of $Y$.

**Corollary (5.3.12).** — With the hypotheses and notation of (5.3.11), for every morphism $g : S' \to S$, let $f'$ be the inverse image of $f$ under $g$ (3.3.7); then $\Gamma_{f'}$ is the inverse image of $\Gamma_f$ under $g$.

**Proof.** This is a particular case of (3.3.10.1).

**Corollary (5.3.13).** — Let $f : X \to Y$ and $g : Y \to Z$ be morphisms; if $g \circ f$ is an immersion (resp. a local immersion), then so too is $f$.

**Proof.** Indeed, $f$ factors as $X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$. Furthermore, $p_2$ can be identified with $(g \circ f) \times_Y 1_Y$ (3.3.4); if $g \circ f$ is an immersion (resp. a local immersion), then so too is $p_2$ ((4.3.1) and (4.5.5)), and since $\Gamma_f$ is an immersion (5.3.11), we are done, by (4.2.4) (resp. (4.5.5)).

**Corollary (5.3.14).** — Let $j : X \to Y$ and $g : Z \to Z$ be $S$-morphisms. If $j$ is an immersion (resp. a local immersion), then so too is $(j,g)_S$. 


Proof. Indeed, if $p : Y \times_S Z \to Y$ is the projection onto the first component, then we have $j = p \circ (j, g)_S$, and it suffices to apply (5.3.13).

**Proposition (5.3.15).** — If $f : X \to Y$ is an $S$-morphism, then the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta_X} & X \times_S X \\
f & \downarrow & \downarrow f \times_S f \\
Y & \xrightarrow{\Delta_Y} & Y \times_S Y
\end{array}
\]

commutes (in other words, $\Delta_X$ is a functorial morphism in the category of preschemes).

Proof. The proof is immediate and left to the reader.

**Corollary (5.3.16).** — If $X$ is a subprescheme of $Y$, then the diagonal $\Delta_X(X)$ can be identified with a subprescheme of $\Delta_Y(Y)$, and the underlying space can be identified with

$\Delta_Y(Y) \cap p_1^{-1}(X) = \Delta_Y \cap p_2^{-1}(X)$

($p_1$ and $p_2$ being the projections of $Y \times_S Y$).

Proof. Applying (5.3.15) to the injection morphism $f : X \to Y$, we see that $f \times_S f$ is an immersion that identifies the underlying space of $X \times_S X$ with the subspace $p_1^{-1}(X) \cap p_2^{-1}(X)$ of $Y \times_S Y$ (4.3.1); further, if $z \in \Delta_Y(Y) \cap p_1^{-1}(X)$, then we have $z = \Delta_Y(y)$ and $y = p_1(z) \in X$, so

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$y = f(y)$, and $z = \Delta_Y(f(y))$ belongs to $\Delta_X(X)$ by the commutativity of (5.3.15.1).

**Corollary (5.3.17).** — Let $f_1 : Y \to X$ and $f_2 : Y \to X$ be $S$-morphisms, and $y$ a point of $Y$ such that $f_1(y) = f_2(y) = x$, and such that the homomorphisms $k(x) \to k(y)$ corresponding to $f_1$ and $f_2$ are identical. Then, if $f = (f_1, f_2)_S$, the point $f(y)$ belongs to the diagonal $\Delta_X(X)$.

Proof. The two homomorphisms $k(x) \to k(y)$ corresponding to $f_i$ ($i = 1, 2$) define two $S$-morphisms $g_i : \text{Spec}(k(y)) \to \text{Spec}(k(x))$ such that the diagrams

\[
\begin{array}{ccc}
\text{Spec}(k(y)) & \xrightarrow{g_i} & \text{Spec}(k(x)) \\
\downarrow & & \downarrow \\
Y & \xrightarrow{f_i} & X
\end{array}
\]

commute. The diagram

\[
\begin{array}{ccc}
\text{Spec}(k(y)) & \xrightarrow{(g_1, g_2)_S} & \text{Spec}(k(x)) \times_S \text{Spec}(k(x)) \\
\downarrow & & \downarrow \\
Y & \xrightarrow{(f_1, f_2)_S} & X \times_S X
\end{array}
\]

thus also commutes. But it follows from the equality $g_1 = g_2$ that the image under $(g_1, g_2)_S$ of the unique point of $\text{Spec}(k(y))$ belongs to the diagonal of $\text{Spec}(k(x)) \times_S \text{Spec}(k(x))$; the conclusion then follows from (5.3.15).
5. Reduced Preschemes; The Separation Condition

5.4. Separated morphisms and separated preschemes

Definition (5.4.1). — We say that a morphism of preschemes \( f : X \to Y \) is separated if the diagonal morphism \( X \to X \times_Y Y \) is a closed immersion; we then also say that \( X \) is a separated prescheme over \( Y \), or a \( Y \)-scheme. We say that a prescheme \( X \) is separated if it is separated over \( \text{Spec}(\mathbb{Z}) \); we then also say that \( X \) is a scheme\(^9\) (cf. (5.5.7)).

By (5.3.9), for \( X \) to be separated over \( Y \), it is necessary and sufficient for \( \Delta_X(X) \) to be a closed sublocale of the underlying space of \( X \times_Y X \).

Proposition (5.4.2). — Let \( S \to T \) be a separated morphism. If \( X \) and \( Y \) are \( S \)-preschemes, then the canonical morphism \( X \times_S Y \to X \times_T Y \) (5.3.10) is closed.

Proof. Indeed, if we refer to the diagram in (5.3.5.1), we see that \( (p, q)_T \) can be considered as being obtained from \( \Delta_{S/T} \) by the extension \( f \times_T g : X \times_T Y \to S \times_T S \) of the base prescheme \( S \times_T S \); the proposition then follows from (4.3.2). \( \square \)

Corollary (5.4.3). — Let \( Y \) be an \( S \)-scheme, and \( f : X \to Y \) an \( S \)-morphism. Then the graph morphism \( \Gamma_f : X \to X \times_S Y \) (5.3.11) is a closed immersion.

Proof. This is a particular case of (5.4.2), where we replace \( S \) by \( Y \), and \( T \) by \( S \). \( \square \)

Corollary (5.4.4). — Let \( f : X \to Y \) and \( g : Y \to Z \) be morphisms, with \( g \) separated. If \( g \circ f \) is a closed immersion, then so too is \( f \).

Proof. The proof using (5.4.3) is the same as that of (5.3.13) using (5.3.11). \( \square \)

Corollary (5.4.5). — Let \( Z \) be an \( S \)-scheme, and \( j : X \to Y \) and \( g : X \to Z \) \( S \)-morphisms. If \( j \) is a closed immersion, then so too is \( (j, g)_S : X \to Y \times_S Z \).

Proof. The proof using (5.4.4) is the same as that of (5.3.14) using (5.3.13). \( \square \)

Corollary (5.4.6). — If \( X \) is an \( S \)-scheme, then every \( S \)-section of \( X \) (2.5.5) is a closed immersion.

Proof. If \( \phi : X \to S \) is the structure morphism, and \( \psi : S \to X \) an \( S \)-section of \( X \), it suffices to apply (5.4.5) to \( \phi \circ \psi = 1_S \). \( \square \)

Corollary (5.4.7). — Let \( X \) be an integral prescheme with generic point \( s \), and \( Y \) an \( S \)-scheme. If two \( S \)-sections \( f \) and \( g \) are such that \( f(s) = g(s) \), then \( f = g \).

Proof. Indeed, if \( x = f(s) = g(s) \), then the homomorphisms \( k(x) \to k(s) \) corresponding to \( f \) and \( g \) are necessarily identical. If \( h = (f, g)_S \), we thus deduce (5.3.17) that \( h(s) \) belongs to the diagonal \( Z = \Delta_X(X) \); but since \( S = \{s\} \), and since \( Z \) is closed by hypothesis, we have \( h(S) \subset Z \). It then follows from (5.2.2) that \( h \) factors as \( S \to Z \to X \times_S X \), and we thus conclude that \( f = g \), by definition of the diagonal. \( \square \)

Remark (5.4.8). — If we suppose, conversely, that the conclusion of (5.4.3) is true when \( f = 1_Y \), then we can conclude that \( Y \) is separated over \( S \); similarly, if we suppose that the conclusion of (5.4.5) applies to the two morphisms \( Y \xrightarrow{\Delta_Y} Y \times_Z Y \xrightarrow{\text{pr}_1} Y \), then we can conclude that \( \Delta_Y \) is a closed immersion, and thus that \( Y \) is separated over \( Z \); finally, if we assume that the conclusion of (5.4.6) is true for the \( Y \) section \( \Delta_Y \) of the \( Y \)-prescheme \( X \times_S Y \to Y \), then this implies that \( Y \) is separated over \( S \).

\(^9\)[Trans.] We repeat here the warning given at the very start of this translation: the early versions of the EGA use prescheme to mean is now usually called a scheme, and scheme for what is now usually called a separated scheme. Grothendieck himself later said that the more modern terminology was preferable, but we have decided to keep this translation ‘historically accurate’ by using the older nomenclature.
5.5. Separation criteria

**Proposition (5.5.1).** —

(i) Every monomorphism of preschemes (and, in particular, every immersion) is a separated morphism.

(ii) The composition of any two separated morphisms is separated.

(iii) If \( f : X \to X' \) and \( g : Y \to Y' \) are separated \( S \)-morphisms, then \( f \times_S g \) is separated.

(iv) If \( f : X \to Y \) is a separated \( S \)-morphism, then the \( S' \)-morphism \( f_{(S')} \) is separated for every extension \( S' \to S \) of the base prescheme.

(v) If the composition \( g \circ f \) is separated, then \( f \) is separated.

(vi) For a morphism \( f \) to be separated, it is necessary and sufficient for \( f_{\text{red}} \) (5.1.5) to be separated.

**PROOF.** Note that (i) is an immediate consequence of (5.3.8). If \( f : X \to Y \) and \( g : Y \to Z \) are morphisms, then the diagram

\[
\begin{array}{ccc}
X & \to & X \times_Z X \\
\downarrow \Delta_{X/Z} & & \downarrow \Delta_{X/Y} \\
X \times_Y X & \to & X \times_Y X
\end{array}
\]

commutes (where \( j \) denotes the canonical immersion (5.3.10)), as can be immediately verified. If \( f \) and \( g \) are separated, then \( \Delta_{X/Y} \) is a closed immersion by definition, and \( j \) is a closed immersion by (5.4.2), whence \( \Delta_{X/Z} \) is a closed immersion by (4.2.4), which proves (ii). Given (i) and (ii), (iii) and (iv) are equivalent (3.5.1), so it suffices to prove (iv). But \( X_{(S')} \times_{Y_{(S')}} X_{(S')} \) is canonically identified with \( (X \times_Y X) \times_Y Y_{(S')} \) by (3.3.11) and (3.3.9.1), and we immediately see that the diagonal morphism \( \Delta_{X_{(S')}} \) can then be identified with \( \Delta_{X \times_Y Y_{(S')}} \); the proposition then follows from (4.3.1).

To prove (v), consider, as in (5.3.13), the factorisation \( X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y \) of \( f \), noting that \( p_2 = (g \circ f) \times_Z 1_Y \); the hypothesis (that \( g \circ f \) is separated) implies that \( g_2 \) is separated, by (iii) and (i), and, since \( \Gamma_f \) is an immersion, \( \Gamma_f \) is separated, by (i), whence \( f \) is separated, by (ii). Finally, to prove (vi), recall that the preschemes \( X_{\text{red}} \times_{Y_{\text{red}}} X_{\text{red}} \) and \( X_{\text{red}} \times_Y X_{\text{red}} \) are canonically identified with one another (5.1.7); if we denote by \( j \) the injection \( X_{\text{red}} \to X \), then the diagram

\[
\begin{array}{ccc}
X_{\text{red}} & \to & X_{\text{red}} \times_Y X_{\text{red}} \\
\downarrow j & & \downarrow j \times_Y j \\
X & \to & X \times_Y X
\end{array}
\]

commutes (5.3.15), and the proposition follows from the fact that the vertical arrows are homeomorphisms of the underlying spaces (4.3.1). \( \square \)

**Corollary (5.5.2).** — If \( f : X \to Y \) is separated, then the restriction of \( f \) to any subprescheme of \( X \) is separated.

**PROOF.** This follows from (5.5.1, i and ii). \( \square \)

**Corollary (5.5.3).** — If \( X \) and \( Y \) are \( S \)-preschemes such that \( Y \) is separated over \( S \), then \( X \times_S Y \) is separated over \( X \).

**PROOF.** This is a particular case of (5.5.1, iv). \( \square \)

**Proposition (5.5.4).** — Let \( X \) be a prescheme, and assume that its underlying space is a finite union of closed subsets \( X_k \) (\( 1 \leq k \leq n \)); for each \( k \), consider the reduced subprescheme of \( X \) that has \( X_k \) as its underlying space (5.2.1), and denote this again by \( X_k \). Let \( f : X \to Y \) be a morphism, and, for each \( k \), let \( Y_k \) be a closed subset of \( Y \) such that \( f(X_k) \subset Y_k \); we again denote by \( Y_k \) the reduced subprescheme of \( Y \) that has \( Y_k \) as its underlying space, so that the restriction \( X_k \to Y \) of \( f \) to \( X_k \) factors as \( X_k \xrightarrow{f_k} Y_k \to Y \) (5.2.2). For \( f \) to be separated, it is necessary and sufficient for all the \( f_k \) to be separated.
The necessity follows from (5.5.1, i, ii, and v). Conversely, if the condition of the statement is satisfied, then each of the restrictions $X_k \to Y$ of $f$ is separated (5.5.1, (i) and (ii)); if $p_1$ and $p_2$ are the projections of $X \times_X Y$, then the subspace $\Delta_X(X_k)$ can be identified with the subspace $\Delta_X(X) \cap p_1^{-1}(X_k)$ of the underlying space of $X \times_X Y$ (5.3.16); these subspaces are closed in $X \times_X Y$, and thus so too is their union $\Delta_X(X)$. □

Suppose, in particular, that the $X_k$ are the irreducible components of $X$; then we can suppose that the $Y_k$ are the irreducible components of $Y$ (0.2.1.5); Proposition (5.5.4) then, in this case, leads to the idea of separation in the case of integral preschemes (2.1.7).

**Proposition (5.5.5).** — Let $(\lambda_k)$ be an open cover of a prescheme $Y$; for a morphism $f : X \to Y$ to be separated, it is necessary and sufficient for each of its restrictions $f^{-1}(Y_k) \to Y_k$ to be separated.

**Proof.** If we set $X_k = f^{-1}(Y_k)$, everything reduces, by taking (4.2.4, b) and the identification of the products $X_\lambda \times_Y X_\lambda$ and $X_\lambda \times_Y X_\lambda$ into account, to proving that the $X_\lambda \times_Y X_\lambda$ form a cover of $X \times_Y Y$. But if we set $Y_\mu = Y_\lambda \cap Y_\mu$ and $X_\lambda \cap X_\mu = f^{-1}(Y_\lambda \cap Y_\mu)$, then $X_\lambda \times_Y X_\mu$ can be identified with the product $X_\lambda \times_Y X_\mu \times_Y X_\mu$ (3.2.6.4), and so also with $X_\lambda \times_Y X_\mu$ (3.2.5), and finally with an open subset of $X_\lambda \times_Y X_\lambda$, which proves our claim (3.2.7). □

Proposition (5.5.4) allows us, by taking an affine open cover of $Y$, to restrict our study of separated morphisms to just those that take values in affine schemes.

**Proposition (5.5.6).** — Let $Y$ be an affine scheme, $X$ a prescheme, and $(U_\alpha)$ a cover of $X$ by affine open subsets. For a morphism $f : X \to Y$ to be separated, it is necessary and sufficient for $U_\alpha \cap U_\beta$ to be, for every pair of indices $(\alpha, \beta)$, an affine open subset, and for the ring $\Gamma(U_{\alpha \cap U_\beta}, \mathcal{O}_X)$ to be generated by the union of the canonical images of the rings $\Gamma(U_\alpha, \mathcal{O}_X)$ and $\Gamma(U_\beta, \mathcal{O}_X)$.

**Proof.** The $U_\alpha \times_Y U_\beta$ form an open cover of $X \times_Y X$ (3.2.7); denoting the projections of $X \times_Y X$ by $p$ and $q$, we have

$$\Delta^{-1}(U_\alpha \times_Y U_\beta) = \Delta^{-1}(p^{-1}(U_\alpha) \cap q^{-1}(U_\beta))$$

$$= \Delta^{-1}(p^{-1}(U_\alpha)) \cap \Delta^{-1}(q^{-1}(U_\beta)) = U_\alpha \cap U_\beta;$$

so everything reduces to proving that the restriction of $\Delta_X$ to $U_\alpha \cap U_\beta$ is a closed immersion into $U_\alpha \times_Y U_\beta$. But this restriction is exactly $(j_\alpha, j_\beta)_Y$, where $j_\alpha \,(\text{resp.} \, j_\beta)$ denotes the injection morphism from $U_\alpha \cap U_\beta$ to $U_\alpha$ (resp. $U_\beta$), as follows from the definitions. Since $U_\alpha \times_Y U_\beta$ is an affine scheme whose ring is canonically isomorphic to $\Gamma(U_\alpha, \mathcal{O}_X) \otimes_{\Gamma(Y, \mathcal{O}_Y)} \Gamma(U_\beta, \mathcal{O}_X)$ (3.2.2), we see that $U_\alpha \cap U_\beta$ must be an affine scheme, and that the map $h_\alpha \otimes h_\beta \mapsto h_\alpha h_\beta$ from the ring $A(U_\alpha \times_Y U_\beta)$ to $\Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X)$ must be surjective (4.2.3), which finishes the proof. □

**Corollary (5.5.7).** — An affine scheme is separated (and is thus a scheme, which justifies the terminology of (5.4.1)).

**Corollary (5.5.8).** — Let $Y$ be an affine scheme; for $f : X \to Y$ to be a separated morphism, it is necessary and sufficient for $X$ to be separated (in other words, for $X$ to be a scheme).

**Proof.** Indeed, we see that the criteria of (5.5.6) do not depend on $f$. □

**Corollary (5.5.9).** — For a morphism $f : X \to Y$ to be separated, it is necessary and sufficient for the induced prescheme $f^{-1}(U)$ to be separated, for every open subset of $U$ on which $Y$ induces a separated prescheme, and it is sufficient for it to be the case for every affine open subset $U \subset Y$.

**Proof.** The necessity of the condition follows from (5.5.4) and (5.5.1, ii); the sufficiency follows from (5.5.4) and (5.5.8), taking into account the existence of affine open covers of $Y$. □

In particular, if $X$ and $Y$ are affine schemes, then every morphism $X \to Y$ is separated.

**Proposition (5.5.10).** — Let $Y$ be a scheme, and $f : X \to Y$ a morphism. For every affine open subset $U$ of $X$, and every affine open subset $V$ of $Y$, $U \cap f^{-1}(V)$ is affine.
PROOF. Let $p_1$ and $p_2$ be the projections of $X \times_Z Y$; the subspace $U \cap f^{-1}(V)$ is the image of $\Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$ under $p_1$. But $p_1^{-1}(U) \cap p_2^{-1}(V)$ can be identified with the underlying space of the prescheme $U \times_Z V$ (3.2.7), and thus is an affine scheme (3.2.2); since $\Gamma_f(X)$ is closed in $X \times_Z Y$ (5.4.3), $\Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$ is closed in $U \times_Z V$, and so the prescheme induced by the subscheme of $X \times_Z Y$ associated to $\Gamma_f$ (4.2.1) on the open subset $\Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$ of its underlying space is a closed subscheme of an affine scheme, and thus an affine scheme (4.2.3). The proposition then follows from the fact that $\Gamma_f$ is an immersion.

\[\square\]

Examples (5.5.11). — The prescheme from Example (2.3.2) (“the projective line over a field $K$”) is separated, because, for the cover $(X_1, X_2)$ of $X$ by affine open subsets, $X_1 \cap X_2 = U_{12}$ is affine, and $\Gamma(U_{12}, \mathcal{O}_X)$, the ring of rational fractions of the form $f(s)/s^m$ with $f \in K[s]$, is generated by $K[s]$ and $1/s$, so the conditions of (5.5.6) are satisfied.

With the same choice of $X_1, X_2, U_{12}$, and $U_{21}$ as in Example (2.3.2), now take $u_{12}$ to be the isomorphism which sends $f(s)$ to $f(t)$; we now obtain, by gluing, a non-separated integral prescheme $X$, because the first condition of (5.5.6) is satisfied, but not the second. It is immediate here that $\Gamma(X, \mathcal{O}_X) = K[s]$ is an isomorphism; the inverse isomorphism defines a morphism $f : X \to \text{Spec}(K[s])$ that is surjective, and for every $y \in \text{Spec}(K[s])$ such that $iy \neq (0), f^{-1}(y)$ consists of a single point, but for $i_y = (0), f^{-1}(y)$ consists of two distinct points (we say that $X$ is the “affine line over $K$ with the point 0 doubled”).

We can also give examples where neither of the two conditions of (5.5.6) are satisfied. First note that, in the prime spectrum $Y$ of the ring $A = K[s, t]$ of polynomials in two indeterminates over a field $K$, the open subset $U$ given by the union of $D(s)$ and $D(t)$ is not an affine open subset. Indeed, if $z$ is a section of $\mathcal{O}_Y$ over $U$, there exist two integers $m, n \geq 0$ such that $s^m z$ and $t^n z$ are the restrictions of polynomials in $s$ and $t$ to $U$ (1.4.1), which is clearly possible only if the section $z$ extends to a section over the whole of $Y$, identified with a polynomial in $s$ and $t$. If $U$ were an affine open subset, then the injection morphism $U \to Y$ would be an isomorphism (1.7.3), which is a contradiction, since $U \neq Y$.

With the above in mind, take two affine schemes $Y_1$ and $Y_2$, prime spectra of the rings $A_1 = K[s_1, t_2]$ and $A_2 = K[s_2, t_2]$ (respectively); take $U_{12} = D(s_1) \cup D(t_1)$ and $U_{21} = D(s_2) \cup D(t_2)$, and take $u_{12}$ to be the restriction of an isomorphism $Y_2 \to Y_1$ to $U_{21}$ corresponding to the isomorphism of rings that sends $f(s_1, t_1)$ to $f(s_2, t_2)$; we then have an example where the conditions of (5.5.6) are not satisfied (the integral prescheme thus obtained is called “the affine plane over $K$ with the point 0 doubled”).

Remark (5.5.12). — Given some property $P$ of morphisms of preschemes, consider the following propositions.

(i) Every closed immersion has property $P$.
(ii) The composition of any two morphisms that both have property $P$ also has property $P$.
(iii) If $f : X \to X'$ and $g : Y \to Y'$ are $S$-morphisms that have property $P$, then $f \times_S g$ has property $P$.
(iv) If $f : X \to Y$ is an $S$-morphism that has property $P$, then every $S'$-morphism $f_{(S')}$ obtained by an extension $S' \to S$ of the base prescheme also has property $P$.
(v) If the composition $g \circ f$ of two morphisms $f : X \to Y$ and $g : Y \to Z$ has property $P$, and $g$ is separated, then $f$ has property $P$.
(vi) If a morphism $f : X \to Y$ has property $P$, then so too does $f_{\text{red}}$ (5.1.5).

If we suppose that (i) and (ii) are both true, then (iii) and (iv) are equivalent, and (v) and (vi) are consequences of (i), (ii), and (iii).

The first claim has already been shown (3.5.1). Consider the factorisation $5.3.13$ $\Gamma_f : X \times_Z Y \xrightarrow{p_2} Y$ of $f$; the equation $p_2 = (g \circ f) \times_Z 1_X$ shows that, if $g \circ f$ has property $P$, then so too does $p_2$, by (iii); if $g$ is separated, then $\Gamma_f$ is a closed immersion (4.4.3), and so also has property $P$, by (i); finally, by (ii), $f$ has property $P$. 
Finally, consider the commutative diagram

\[
\begin{array}{ccc}
X_{\text{red}} & \xrightarrow{f_{\text{red}}} & Y_{\text{red}} \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y,
\end{array}
\]

where the vertical arrows are the closed immersions (5.1.5), and thus have property \( P \), by (i). The hypothesis that \( f \) has property \( P \) implies, by (ii), that \( X_{\text{red}} \xrightarrow{f_{\text{red}}} Y_{\text{red}} \rightarrow Y \) has property \( P \); finally, since a closed immersion is separated (5.5.1, i), \( f_{\text{red}} \) has property \( P \), by (v).

Note that, if we consider the propositions

(i') Every immersion has property \( P \);  
(v') If \( g \circ f \) has property \( P \), then so too does \( f \);  

then the above arguments show that (v') is a consequence of (i'), (ii), and (iii).

(5.5.13). Note that (v) and (vi) are again consequences of (i), (iii), and

(ii') If \( j : X \rightarrow Y \) is a closed immersion, and \( g : Y \rightarrow Z \) is a morphism that has property \( P \), then \( g \circ j \) has property \( P \).

Similarly, (v') is a consequence of (i'), (iii), and

(ii'') If \( j : X \rightarrow Y \) is an immersion, and \( g : Y \rightarrow Z \) is a morphism that has property \( P \), then \( g \circ j \) has property \( P \).

This follows immediately from the arguments of (5.5.12).

§6. FINITENESS CONDITIONS

6.1. Noetherian and locally Noetherian preschemes

Definition (6.1.1). — We say that a prescheme \( X \) is Noetherian (resp. locally Noetherian) if it is a finite union (resp. union) of affine open \( V_\alpha \) in such a way that the ring of the of the induced scheme on each of the \( V_\alpha \) is Noetherian.

It follows immediately from (1.5.2) that, if \( X \) is locally Noetherian, then the structure sheaf \( \mathcal{O}_X \) is a coherent sheaf of rings, since the questions is a local one. Every quasi-coherent \( \mathcal{O}_X \)-module (resp. quasi-coherent quotient \( \mathcal{O}_X \)-module) of a coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \) is coherent, as the question is once again a local one, and it suffices to apply (1.5.1), (1.4.1), and (1.3.10), combined with the fact that a submodule (resp. quotient module) of a module of finite type over a Noetherian ring is of finite type. In particular, every quasi-coherent sheaf of ideals of \( \mathcal{O}_X \) is coherent.

If a prescheme \( X \) is a finite union (resp. union) of open subsets \( W_\lambda \) in such a way that the preschemes induced on the \( W_\lambda \) are Noetherian (resp. locally Noetherian), it is clear that \( X \) is Noetherian (resp. locally Noetherian).

Proposition (6.1.2). — For a prescheme \( X \) to be Noetherian, it is necessary and sufficient for it to be locally Noetherian and have a quasi-compact underlying space. The underlying space itself is then also Noetherian.

Proof. The first claim follows immediately from the definitions and (1.1.10, ii). The second follows from (1.1.6) and the fact that every space that is a finite union of Noetherian subspaces is itself Noetherian (0, 2.2.3).

Proposition (6.1.3). — Let \( X \) be an affine scheme given by a ring \( A \). The following conditions are equivalent: (a) \( X \) is Noetherian; (b) \( X \) is locally Noetherian; (c) \( A \) is Noetherian.

Proof. The equivalence between (a) and (b) follows from (6.1.2) and the fact that the underlying space of every affine scheme is quasi-compact (1.1.10); it is furthermore clear that (c) implies (a). To see that (a) implies (c), we remark that there is a finite cover \( (V_i) \) of \( X \) by affine open subsets such that the ring \( A_i \) of the prescheme induced on \( V_i \) is Noetherian. So let \( (a_n) \) be an increasing sequence of ideals of \( A_i \); by a canonical bijective correspondence, there is a corresponding sequence \( (\tilde{a}_n) \) of sheaves of ideals in \( \tilde{A} = \mathcal{O}_X \); to see that the sequence \( (\tilde{a}_n) \) is stable (?), it suffices to prove that the sequence \( (\tilde{a}_n) \) is. But the restriction \( \tilde{a}_n|_{V_i} \) is a quasi-coherent sheaf of ideals in \( \mathcal{O}_X|_{V_i} \), being
6. FINITENESS CONDITIONS

the inverse image of \( \tilde{a}_n \) under the canonical injection \( V_i \to X \) \((0, 5.1.4)\); \(\tilde{a}_n|_V \) is thus of the form \(\tilde{a}_{ni}\), where \(a_{ni}\) is an ideal of \(A_i\) \((1.3.7)\). Since \(A_i\) is Noetherian, the sequence \((a_{ni})\) is stable for all \(i\), whence the proposition.

We note that the above argument proves also that if \(X\) is a Noetherian prescheme, then every increasing sequence of coherent sheaves of ideals of \(\mathcal{O}_X\) is stable \((?\).

**Proposition (6.1.4).** — Every subprescheme of a Noetherian (resp. locally Noetherian) prescheme is Noetherian (resp. locally Noetherian).

**Proof.** If suffices to give a proof for a Noetherian prescheme \(X\); further, by definition (6.1.1), we can also restrict to the case where \(X\) is an affine scheme. Since every subprescheme of \(X\) is a closed subscheme of a prescheme induced on an open subset \((4.1.3)\), we can restrict to the case of a subscheme \(Y\), either closed or induced on an open subset of \(X\). The proof in the case where \(Y\) is closed is immediate, since if \(A\) is the ring of \(X\), we know that \(Y\) is an affine scheme given by the ring \(A/\mathfrak{J}\), where \(\mathfrak{J}\) is an ideal of \(A\) \((4.2.3)\); since \(A\) is Noetherian \((6.1.3)\), so too is \(A/\mathfrak{J}\). 

Now suppose that \(Y\) is open in \(X\); the underlying space of \(Y\) is Noetherian \((6.1.2)\), hence quasi-locally Noetherian, and thus a finite union of open subsets \(D(f_i) (f_i \in A)\); everything reduces to showing the proposition in the case where \(Y = D(f)\) with \(f \in A\). But then \(Y\) is an affine scheme whose ring is isomorphic to \(A_f\) \((1.3.6)\); since \(A\) is Noetherian \((6.1.3)\), so too is \(A_f\). \(\square\)

(6.1.5). We note that the product of two Noetherian \(S\)-preschemes is not necessarily Noetherian, even if the preschemes are affine, since the tensor product of two Noetherian algebras in not necessarily a Noetherian ring (cf. \((6.3.8)\)).

**Proposition (6.1.6).** — If \(X\) is a Noetherian prescheme, the nilradical \(\mathcal{N}_X\) of \(\mathcal{O}_X\) is nilpotent.

**Proof.** We can in fact cover \(X\) with a finite number of affine open subsets \(U_i\), and it suffices to prove that there exists whole numbers \(n_i\) such that \((\mathcal{N}_X|_U)^{n_i} = 0\); if \(n\) is the largest of the \(n_i\), then we will have \(\mathcal{N}_X^n = 0\). We can thus restrict to the case where \(X = \text{Spec}(A)\) is affine, with \(A\) a Noetherian ring; by \((5.1.1)\) and \((1.3.13)\), it suffices to observe that the nilradical of \(A\) is nilpotent \((\text{Sam53}, \text{p. 127}, \text{cor. 4})\). \(\square\)

**Corollary (6.1.7).** — Let \(X\) be a Noetherian prescheme; for \(X\) to be an affine scheme, it is necessary and sufficient that \(X_{\text{red}}\) be affine.

**Proof.** This follows from \((6.1.6)\) and \((5.1.10)\). \(\square\)

**Lemma (6.1.8).** — Let \(X\) be a topological space, \(x\) a point of \(X\), and \(U\) an open neighbourhood of \(x\) having only a finite number of irreducible components. Then there exists a neighbourhood \(V\) of \(x\) such that every open neighbourhood of \(x\) contained in \(V\) is connected.

**Proof.** Let \(U_i\) \((1 \leq i \leq m)\) be the irreducible components of \(U\) not containing \(x\); the complement (in \(U\)) of the union of the \(U_i\) is an open neighbourhood \(V\) of \(X\) inside \(U\), and thus so too in \(X\); it is also, incidentally, the complement (in \(X\)) of the union of the irreducible components of \(X\) that do not contain \(x\) \((0, 2.1.6)\). So let \(W\) be an open neighbourhood of \(X\) contained in \(V\). The irreducible components of \(W\) are the intersections of \(W\) with the irreducible components of \(U\) \((0, 2.1.6)\), so these components contain \(x\); since they are connected, so too is \(W\). \(\square\)

**Corollary (6.1.9).** — A locally Noetherian topological space is locally connected (which implies, amongst other things, that its connected components are open).

**Proposition (6.1.10).** — Let \(X\) be a locally Noetherian topological space. The following conditions are equivalent.

(a) The irreducible components of \(X\) are open.
(b) The irreducible components of \(X\) are exactly its connected components.
(c) The connected components of \(X\) are irreducible.
(d) Two distinct irreducible components of \(X\) have an empty intersection.

Finally, if \(X\) is a prescheme, then these conditions are also equivalent to

(e) For every \(x \in X\), \(\text{Spec}(\mathcal{O}_x)\) is irreducible (or, in other words, the nilradical of \(\mathcal{O}_x\) is prime).
PROOF. It is immediate that (a) implies (b), because an irreducible space is connected, and (a) implies that the irreducible components of $X$ are the sets that are both open and closed. It is trivial that (b) implies (c); conversely, a closed set $F$ containing a connected component $C$ of $X$, with $C$ distinct from $F$, cannot be irreducible, because not being connected means that $F$ is the union of two disjoint nonempty sets that are both open and closed in $F$, and thus closed in $X$; as a result, (c) implies (b). We immediately conclude from this that (c) implies (d), since two distinct connected components have no points in common.

We have not yet used the fact that $X$ is locally Noetherian. Suppose now that this is indeed the case, and we will show that (d) implies (a): by $(0, 2.1.6)$, we can restrict ourselves to the case where the space $X$ is Noetherian, and so has only a finite number of irreducible components. Since they are closed and pairwise disjoint, they are open.

Finally, the equivalence between (d) and (e) holds true even without the assumption that the underlying space of the prescheme $X$ is locally Noetherian. We can in fact restrict ourselves to the case where $X = \text{Spec}(A)$ is affine, by $(0, 2.1.6)$; to say that $x$ is contained in only one single irreducible component of $X$ is to say that $I_x$ contains only one single minimal ideal of $A$ $(1.1.14)$, which is equivalent to saying that $I_x \mathcal{O}_x$ contains only one single minimal ideal of $\mathcal{O}_x$, whence the conclusion.

Corollary (6.1.11). — Let $X$ be a locally Noetherian space. For $X$ to be irreducible, it is necessary and sufficient that $X$ be connected and nonempty, and that any two distinct irreducible components of $X$ have an empty intersection. If $X$ is a prescheme, this latter condition is equivalent to asking that $\text{Spec}(\mathcal{O}_X)$ be irreducible for all $x \in X$.

PROOF. The second claim has already been shown in $(6.1.10)$; the only thing thus remaining to show is that the conditions in the first claim are sufficient. But by $(6.1.10)$, these conditions imply that the irreducible components of $X$ are exactly its connected components, and since $X$ is connected and nonempty, it is irreducible.

Corollary (6.1.12). — Let $X$ be a locally Noetherian prescheme. For $X$ to be integral, it is necessary and sufficient that $X$ be connected and that $\mathcal{O}_X$ be integral for all $x \in X$.

Proposition (6.1.13). — Let $X$ be a locally Noetherian prescheme, and let $x \in X$ be a point such that the nilradical $\mathcal{N}_x$ of $\mathcal{O}_x$ is prime (resp. such that $\mathcal{O}_x$ is reduced, resp. integral); then there exists an open neighbourhood $U$ of $x$ that is irreducible (resp. reduced, resp. integral).

PROOF. It suffices to consider two cases: where $\mathcal{N}_x$ is prime, and where $\mathcal{N}_x = 0$; the third hypotheses is a combination of the first two. If $\mathcal{N}_x$ is prime, then $x$ belongs to only one single irreducible component $Y$ of $X$ $(6.1.10)$; the union of the irreducible components of $X$ that do not contain $x$ is closed (the set of these components being locally finite), and the complement $U$ of this union is thus open and contained in $Y$, and thus irreducible $(0, 2.1.6)$ If $\mathcal{N}_x = 0$, we also have $\mathcal{N}_y = 0$ for any $y$ in a neighbourhood of $x$, because $\mathcal{N}$ is quasi-coherent $(5.1.1)$, and thus coherent, since $X$ is locally Noetherian, and the conclusion then follows from $(0, 5.2.2)$.

6.2. Artinian preschemes

Definition (6.2.1). — We say that a prescheme is Artinian if it is affine, and given by an Artinian ring.

Proposition (6.2.2). — Given a prescheme $X$, the following conditions are equivalent:

(a) $X$ is an Artinian scheme;
(b) $X$ is Noetherian and its underlying space is discrete;
(c) $X$ is Noetherian and the points of its underlying space are closed (the $T_1$ condition).

When any of the above hold, the underlying space of $X$ is finite, and the ring $A$ of $X$ is the direct sum of local (Artinian) rings of points of $X$.

PROOF. We know that (a) implies the last claim ([SZ60, p. 205, th. 3]), so every prime ideal of $A$ is thus maximal and is the inverse image of a maximal ideal of one of the local components of $A$, and so the space $X$ is finite and discrete; (a) thus implies (b), and (b) clearly implies (c). To see that (c) implies (a), we first show that $X$ is then finite; we can indeed restrict to the case where $X$ is affine,
and we know that a Noetherian ring whose prime ideals are all maximal is Artinian ([SZ60, p. 203]), whence our claim. The underlying space $X$ is then discrete, the topological sum of a finite number of points $x_i$, and the local rings $\mathcal{O}_{x_i} = A_i$ are Artinian; it is clear that $X$ is isomorphic to the prime spectrum affine scheme of the ring $A$ (the direct sum of the $A_i$) (1.7.3).

6.3. Morphisms of finite type

Definition (6.3.1). — We say that a morphism $f : X \to Y$ is of finite type if $Y$ is the union of a family $(V_a)$ of affine open subsets having the following property:

$$(P) \ f^{-1}(V_a) \text{ is a finite union of affine open subsets } U_{ai} \text{ that are such that each ring } A(U_{ai}) \text{ is an algebra of finite type over } A(V_a).$$

We then say that $X$ is a prescheme of finite type over $Y$, or a $Y$-prescheme of finite type.

Proposition (6.3.2). — If $f : X \to Y$ is a morphism of finite type, then every affine open subset $W$ of $Y$ satisfies property (P) of (6.3.1).

We first show

Lemma (6.3.2.1). — If $T \subset Y$ is an affine open subset, satisfying property (P), then, for every $g \in A(T)$, $D(g)$ also satisfies property (P).

Proof. By hypothesis, $f^{-1}(T)$ is a finite union of affine open subsets $Z_j$, that are such that $A(Z_j)$ is an algebra of finite type over $A(T)$; let $\phi_j : A(T) \to A(Z_j)$ be the homomorphism of rings corresponding to the restriction of $f$ to $Z_j$ (2.2.4), and set $g_j = \phi_j(g)$; we then have $f^{-1}(D(g)) \cap Z_j = D(g_j)$ (1.2.2.2). But $A(D(g_j)) = A(Z_j)_{g_j} = A(Z_j)[1/g_j]$ is of finite type over $A(Z_j)$, and a fortiori over $A(T)$ by the hypothesis, and so also over $A(D(g)) = A(T)[1/g]$, which proves the lemma. \(\square\)

Proof. With the above lemma, since $W$ is quasi-compact (1.1.10), there exists a finite covering of $W$ by sets of the form $D(g_i) \subset W$, where each $g_i$ belongs to a ring $A(V_{a(i)})$. Each $D(g_i)$, being quasi-compact, is a finite union of sets $D(h_{ik})$, where $h_{ik} \in A(W)$; if $\phi_i : A(W) \to A(D(g_i))$ is the canonical map, then we have $D(h_{ik}) = D(\phi_i(h_{ik}))$ by (1.2.2.2). By (6.3.2.1), each of the $f^{-1}(D(h_{ik}))$ admits a finite covering by affine open subsets $U_{ijk}$, that are such that the $A(U_{ijk})$ are algebras of finite type over $A(D(h_{ik})) = A(W)[1/h_{ik}]$, whence the proposition. \(\square\)

We can thus say that the notion of a prescheme of finite type over $Y$ is local on $Y$.

Proposition (6.3.3). — Let $X$ and $Y$ be affine schemes; for $X$ to be of finite type over $Y$, it is necessary and sufficient that $A(X)$ be an algebra of finite type over $A(Y)$.

Proof. Since the condition clearly suffices, we show that it is necessary. Set $A = A(Y)$ and $B = A(X)$; by (6.3.2), there exists a finite affine open cover $(V_i)$ of $X$ such that each of the rings $A(V_i)$ is an $A$-algebra of finite type. Further, since the $V_i$ are quasi-compact, we can cover each of them with a finite number of open subsets of the form $D(g_{ij}) \subset V_i$, where $g_{ij} \in B$; if $\phi_i$ is a homomorphism $B \to A(V_i)$ that corresponds to the canonical injection $V_i \to X$, then we have $B_{g_{ij}} = (A(V_i))_{\phi_i(g_{ij})} = A(V_i)[1/\phi_i(g_{ij})]$, so $B_{g_{ij}}$ is an $A$-algebra of finite type. We can thus restrict to the case where $V_i = D(g_i)$ with $g_i \in B$. By hypothesis, there exists a finite subset $F_i$ of $B$ and an integer $n_i \geq 0$ such that $B_{g_i}$ is the algebra generated over $A$ by the elements $b_i/g_i^{n_i}$, where the $b_i$ run over all of $F_i$. Since there are only finitely many of the $g_i$, we can assume that all the $n_i$ are equal to the same integer $n$. Further, since the $D(g_i)$ form a cover of $X$, the ideal generated in $B$ by the $g_i$ is equal to $B$, or, in other words, there exist $h_i \in B$ such that $\sum h_i g_i = 1$. So let $F$ be the finite subset of $B$ given by the union of the $F_i$, the set of the $g_i$, and the set of the $h_i$; we will show that the subring $B' = A[F]$ of $B$ is equal to $B$. By hypothesis, for every $b \in B$ and every $i$, the canonical image of $b$ in $B_{g_i}$ is of the form $b_i/g_i^{m_i}$, where $b_i \in B'$; by multiplying the $b_i$ by suitable powers of the $g_i$, we can again assume that all the $m_i$ are equal to the same integer $m$. By the definition of the ring of fractions, there is thus an integer $N$ (dependant on $b$) such that $N \geq m$ and $g_i^N b \in B'$ for all $i$; but, in the ring $B'$, the $g_i^N$ generate the ideal $B'$, because the $g_i$ do (and the $h_i$ belong to $B'$); there are thus $c_i \in B'$ such that $\sum c_i g_i^N = 1$, whence $b = \sum c_i g_i^N b \in B'$, Q.E.D. \(\square\)
(i) Every closed immersion is of finite type.
(ii) The composition of any two morphisms of finite type is of finite type.
(iii) If \( f : X \to X' \) and \( g : Y \to Y' \) are \( S \)-morphisms of finite type, then \( f \times_S g \) is of finite type.
(iv) If \( f : X \to Y \) is an \( S \)-morphism of finite type, then \( f_{(S')} \) is of finite type for any extension \( g : S' \to S \) of the base prescheme.
(v) If the composition \( g \circ f \) of two morphisms is of finite type, with \( g \) separated, then \( f \) is of finite type.
(vi) If a morphism \( f \) is of finite type, then \( \Gamma \) is of finite type.

**Proof.** By (5.5.12), it suffices to prove (i), (ii), and (iv).

To show (i), we can restrict to the case of a canonical injection \( X \to Y \), with \( X \) being a closed subscheme of \( Y \); further (6.3.2), we can assume that \( Y \) is affine, in which case \( X \) is also affine (4.2.3) and its ring is isomorphic to a quotient ring \( A/\mathfrak{J} \), where \( A \) is the ring of \( Y \) and \( \mathfrak{J} \) is an ideal of \( A \); since \( A/\mathfrak{J} \) is of finite type over \( A \), the conclusion follows.

Now we show (ii). Let \( f : X \to Y \) and \( g : Y \to Z \) be two morphisms of finite type, and let \( U \) be an affine open subset of \( Z \); \( g^{-1} \) admits a finite covering by affine open subsets \( V_i \) that are such that each \( A(V_i) \) is an algebra of finite type over \( A(U) \); similarly, each of \( f^{-1} \) admits a finite cover by affine open subsets \( W_{ij} \) that are such that each \( A(W_{ij}) \) is an algebra of finite type over \( A(V_i) \), and so also an algebra of finite type over \( A(U) \), whence the conclusion.

Finally, to show (iv), we can restrict to the case where \( S = Y \); then \( f_{(S')} \) is also equal to \( f_{(S')} \), where we consider \( f \) as a \( Y \)-morphism, and the base extension is \( Y_{(S')} \to Y \) (3.3.9). So let \( p \) and \( q \) be the projections \( X_{(S')} \to X \) and \( X_{(S')} \to S' \). Let \( V \) be an affine open subset of \( S \); \( f^{-1}(V) \) is a finite union of affine open subsets \( W_i \), each of which is such that \( A(W_i) \) is an algebra of finite type over \( A(V) \). Let \( V' \) be an affine open subset of \( S' \) contained in \( g^{-1}(V) \); since \( f \circ p = g \circ q \), \( q^{-1}(V') \) is contained in the union of the \( p^{-1}(W_i) \); on the other hand, the intersection \( p^{-1}(W_i) \cap q^{-1}(V') \) can be identified with the product \( W_i \times V' \) (3.2.7), which is an affine scheme whose ring is isomorphic to \( A(W_i \otimes_{A(V)} A(V')) \) (3.2.2); this ring is, by hypothesis, an algebra of finite type over \( A(V') \), which proves the proposition.

**Corollary (6.3.5).** — Let \( f : X \to Y \) be an immersion morphism. If the underlying space of \( Y \) (resp. \( X \)) is locally Noetherian (resp. Noetherian), then \( f \) is of finite type.

**Proof.** We can always assume that \( Y \) is affine (6.3.2); if the underlying space of \( Y \) is locally Noetherian, then we can further assume that it is Noetherian, and then the underlying space of \( X \), which is a subspace, is also Noetherian. In other words, we can assume that \( Y \) is affine and that the underlying space of \( X \) is Noetherian; there then exists a covering of \( X \) by a finite number of affine open subsets \( D(g_i) \subset Y \), where \( g_i \in A(Y) \), that are such that \( X \cap D(g_i) \) are closed in \( D(g_i) \) (and thus affine schemes (4.2.3)), because \( X \) is locally closed in \( Y \) (4.1.3). Then \( A(X \cap D(g_i)) \) is an algebra of finite type over \( A(D(g_i)) \), by (6.3.4, i) and (6.3.3), and \( A(D(g_i)) = A(Y)_{g_i} = A(Y)[1/g_i] \) is of finite type over \( A(Y) \), which finishes the proof.

**Corollary (6.3.6).** — Let \( f : X \to Y \) and \( g : Y \to Z \) be morphisms. If \( g \circ f \) of finite type, with either \( X \) Noetherian or \( X \times_Z Y \) locally Noetherian, then \( f \) is of finite type.

**Proof.** This follows immediately from the proof of (5.5.12) and from (6.3.5) applied to the immersion morphism \( \Gamma_f \).

**Proposition (6.3.7).** — Let \( f : X \to Y \) be a morphism of finite type; if \( Y \) is Noetherian (resp. locally Noetherian), then \( X \) is Noetherian (resp. locally Noetherian).

**Proof.** We can restrict to proving the proposition for when \( Y \) is Noetherian. Then \( Y \) is a finite union of affine open subsets \( V_i \) that are such that \( A(V_i) \) are Noetherian rings. By (6.3.2), each of the \( f^{-1}(V_i) \) is a finite union of affine open subsets \( W_{ij} \) that are such that \( A(W_{ij}) \) are algebras of finite type over \( A(V_i) \), and thus Noetherian rings; this proves that \( X \) is Noetherian.

**Corollary (6.3.8).** — Let \( X \) be a presheaf of finite type over \( S \). For every base extension \( S' \to S \) with \( S' \) Noetherian (resp. locally Noetherian), \( X_{(S')} \) is Noetherian (resp. locally Noetherian).

**Proof.** This follows from (6.3.7), since \( X_{(S')} \) is of finite type over \( S' \) by (6.3.4, iv).
We can also say that, for a product $X \times_S Y$ of $S$-preschemes, if one of the factors $X$ or $Y$ is of finite type over $S$ and the other is Noetherian (resp. locally Noetherian), then $X \times_S Y$ is Noetherian (resp. locally Noetherian).

**Corollary (6.3.9).** — Let $X$ be a prescheme of finite type over a locally Noetherian prescheme $S$. Then every $S$-morphism $f : X \to Y$ is of finite type.

**Proof.** In fact, we can assume that $S$ is Noetherian; if $\phi : X \to S$ and $\psi : Y \to S$ are the structure morphisms, then we have $\phi = \psi \circ f$, and $X$ is Noetherian by (6.3.7); $f$ is thus of finite type by (6.3.6).

**Proposition (6.3.10).** — Let $f : X \to Y$ be a morphism of finite type. For $f$ to be surjective, it is necessary and sufficient that, for every algebraically closed field $\Omega$, the map $X(\Omega) \to Y(\Omega)$ that corresponds to $f$ (3.4.1) be surjective.

**Proof.** The condition suffices, as we can see by considering, for all $y \in Y$, an algebraically closed extension $\Omega$ of $k(y)$, and the commutative diagram

![Diagram](image)

(cf. (3.5.3)). Conversely, suppose that $f$ is surjective, and let $g : \{\xi\} = \text{Spec}(\Omega) \to Y$ be a morphism, where $\Omega$ is an algebraically closed field. If we consider the diagram

![Diagram](image)

then it suffices to show that there exists a rational point over $\Omega$ in $X(\Omega)$ ((3.3.14), (3.4.3), and (3.4.4)). Since $f$ is surjective, $X(\Omega)$ is nonempty (3.5.10), and since $f$ is of finite type, so too is $f(\Omega)$ (6.3.4, iv); thus $X(\Omega)$ contains a nonempty affine open subset $Z$ such that $A(Z)$ is an non-null algebra of finite type over $\Omega$. By Hilbert’s Nullstellensatz [Zar47], there exists an $\Omega$-homomorphism $A(Z) \to \Omega$, and thus a section of $X(\Omega)$ over $\text{Spec}(\Omega)$, which proves the proposition.

### 6.4. Algebraic preschemes

**Definition (6.4.1).** — Given a field $K$, we define an algebraic $K$-prescheme to be a prescheme $X$ of finite type over $K$; $K$ is called the base field of $X$. If in addition $X$ is a scheme (or if $X$ is a $K$-scheme, which is equivalent (5.5.8)), we say that $X$ is an algebraic $K$-scheme.

Every algebraic $K$-prescheme is Noetherian (6.3.7).

**Proposition (6.4.2).** — Let $X$ be an algebraic $K$-prescheme. For a point $x \in X$ to be closed, it is necessary and sufficient that $k(x)$ be an algebraic extension of $K$ of finite degree.

**Proof.** We can assume that $X$ is affine, with the ring $A$ of $X$ being a $K$-algebra of finite type. Indeed, the affine open subsets $U$ of $X$ such that $A(U)$ is a $K$-algebra of finite type form a finite cover of $X$ (6.3.1). The closed points of $X$ are thus the points such that $I_x$ is a maximal ideal of $A$, or in other words, such that $A/I_x$ is a field (necessarily equal to $k(x)$). Since $A/I_x$ is a $K$-algebra of finite type, we see that if $x$ is closed, then $k(x)$ is a field that is an algebra of finite type over $K$, and so necessarily a $K$-algebra of finite rank [Zar47]. Conversely, if $k(x)$ is of finite rank over $K$, then so is $A/I_x \subset k(x)$, and since every integral ring that is also a $K$-algebra of finite rank is a field, we have that $A/I_x = k(x)$, and hence $x$ is closed.
Corollary (6.4.3). — Let $K$ be an algebraically-closed field, and $X$ an algebraic $K$-prescheme; the closed points of $X$ are then the rational points over $K$ (3.4.4) and can be canonically identified with the points of $X$ with values in $K$.

Proposition (6.4.4). — Let $X$ be an algebraic prescheme over a field $K$. The following properties are equivalent.

(a) $X$ is Artinian.
(b) The underlying space of $X$ is discrete.
(c) The underlying space of $X$ has only a finite number of closed points.
(c') The underlying space of $X$ is finite.
(d) The points of $X$ are closed.
(e) $X$ is isomorphic to $\text{Spec}(A)$, where $A$ is a $K$-algebra of finite type.

Proof. Since $X$ is Noetherian, it follows from (6.2.2) that the conditions (a), (b), and (d) are equivalent, and imply (c) and (c'); it is also clear that (e) implies (a). It remains to see that (c) implies (d) and (e); we can restrict to the case where $X$ is affine. Then $A(X)$ is a $K$-algebra of finite type (6.3.3), and thus a Jacobson ring ([CC, p. 3-11 and 3-12]), in which there are, by hypothesis, only a finite number of maximal ideals. Since a finite intersection of prime ideals can only be a prime ideal if it is equal to one of the prime ideals being intersected, every prime ideal of $A(X)$ is thus maximal, whence (d). Further, we then know (6.2.2) that $A(X)$ is an Artinian $K$-algebra of finite type, and so necessarily of finite rank [Zar47].

(6.4.5). When the conditions of (6.4.4) are satisfied, we say that $X$ is a scheme finite over $K$ (cf. (II, 6.1.1)), or a finite $K$-scheme, of rank $[A : K]$, which we also denote by $\text{rg}_K(X)$; if $X$ and $Y$ are finite $K$-schemes, we have

\[(6.4.5.1) \quad \text{rg}_K(X \sqcup Y) = \text{rg}_K(X) + \text{rg}_K(Y),\]

\[(6.4.5.2) \quad \text{rg}_K(X \times_K Y) = \text{rg}_K(X) \text{rg}_K(Y),\]

as a result of (3.2.2).

Corollary (6.4.6). — Let $X$ be a finite $K$-scheme. For every extension $K'$ of $K$, $X \otimes_K K'$ as a finite $K'$-scheme, and its rank over $K'$ is equal to the rank of $X$ over $K$.

Proof. If $A = A(X)$, then we have $[A \otimes_K K' : K'] = [A : K]$.

Corollary (6.4.7). — Let $X$ be a scheme finite over a field $K$; we let $n = \sum_{x \in X} [k(x) : K]_\mathfrak{s}$ (we recall that if $K'$ is an extension of $K$, then $[K' : K]_\mathfrak{s}$ is the separable rank of $K'$ over $k$, the rank of the largest algebraic separable extension of $K$ contained in $K'$); then for every algebraically closed extension $\Omega$ of $K$, the underlying space of $X \otimes_K \Omega$ has exactly $n$ points, which can be identified with the points of $X$ with values in $\Omega$.

Proof. We can clearly restrict to the case where the ring $A = A(X)$ is local (6.2.2); let $m$ be its maximal ideal, and $L = A/m$ its residue field, an algebraic extension of $K$. The points of $X$ with values in $\Omega$ then correspond, bijectively, to the $\Omega$-sections of $X \otimes_K \Omega$ ((3.4.1) and (3.3.14)), and also to the $K$-homomorphisms from $L$ to $\Omega$ (1.7.3), whence the proposition (Bourbaki, Alg., chap. V, §7, n° 5, prop. 8), taking (6.4.3) into account.

(6.4.8). The number $n$ defined in (6.4.7) is called the separable rank of $A$ (or of $X$) over $K$, or also the geometric number of points of $X$; it is equal to the number of elements of $X(\Omega)_K$. It follows immediately from this definition that, for every extension $K'$ of $K$, $X \otimes_K K'$ has the same geometric number of points as $X$. If we denote this number by $n(X)$, it is clear that, if $X$ and $Y$ are two schemes, finite over $K$, then

\[(6.4.8.1) \quad n(X \sqcup Y) = n(X) + n(Y),\]

Under the same hypotheses, we also have

\[(6.4.8.2) \quad n(X \times_K Y) = n(X)n(Y)\]

because of the interpretation of $n(X)$ as the number of elements of $X(\Omega)_K$ and Equation (3.4.3.1).
Proposition (6.4.9). — Let $K$ be a field, $X$ and $Y$ algebraic $K$-schemes, $f : X \to Y$ a $K$-morphism, and $\Omega$ an algebraically closed extension of $K$ of infinite transcendence degree over $K$. For $f$ to be surjective, it is necessary and sufficient that the map $X(\Omega)_K \to Y(\Omega)_K$ that corresponds to $f$ (3.4.1) be surjective.

Proof. The necessity follows from (6.3.10), noting that $f$ is necessarily of finite type (6.3.9). To see that the condition is sufficient, we argue as in (6.3.10), noting that, for every $y \in Y$, $k(y)$ is an extension of $K$ of finite type, and so is $K$-isomorphic to a subfield of $\Omega$.

Remark (6.4.10). — We will see in chapter IV that the conclusion of (6.4.9) still holds without the hypothesis on the transcendence degree of $\Omega$ over $K$.

Proposition (6.4.11). — If $f : X \to Y$ is a morphism of finite type, then, for every $y \in Y$, the fibre $f^{-1}(y)$ is an algebraic prescheme over the residue field $k(y)$, and for every $x \in f^{-1}(y)$, $k(x)$ is an extension of $k(y)$ of finite type.

Proof. Since $f^{-1}(y) = X \otimes_Y k(y)$ (6.3.6), the proposition follows from (6.3.4, iv) and (6.3.3).

Proposition (6.4.12). — Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms; set $X' = X \times_Y Y'$, and let $f' = f(y') : X' \to Y'$. Let $y' \in Y'$ and set $y = g(y')$; if the fibre $f^{-1}(y)$ is a finite algebraic scheme over $k(y)$, then the fibre $f'^{-1}(y)$ is a finite algebraic scheme over $k(y')$, and has the same rank and geometric number of points as $f^{-1}(y)$ does.

Proof. Taking into account the transitivity of fibres (3.6.5), this follows immediately from (6.4.6) and (6.4.8).

(6.4.13). Proposition (6.4.11) shows that the morphisms of finite type that correspond, intuitively, to the “algebraic families of algebraic varieties”, with the points of $Y$ playing the role of “parameters”, which gives these morphisms a “geometric” meaning. The morphisms which are not of finite type will show up in the following mostly in questions of “changing the base prescheme”, by localisation or completion, for example.

6.5. Local determination of a morphism

Proposition (6.5.1). — Let $X$ and $Y$ be $S$-schemes, with $Y$ of finite type over $S$; let $x \in X$ and $y \in Y$ lie over the same point $s \in S$.

(i) If two $S$-morphisms $f = (\psi, \theta)$ and $f' = (\psi', \theta')$ from $X$ to $Y$ are such that $\psi(x) = \psi'(x) = y$, and the (local) $\mathcal{O}_s$-homomorphisms $\theta_x^f$ and $\theta_x^{f'}$ from $\mathcal{O}_y$ to $\mathcal{O}_x$ are identical, then $f$ and $f'$ agree on an open neighbourhood of $x$.

(ii) Suppose further that $S$ is locally Noetherian. For every local $\mathcal{O}_s$-homomorphism $\phi : \mathcal{O}_y \to \mathcal{O}_x$, there exists an open neighbourhood $U$ of $x$ in $X$, and an $S$-morphism $f = (\psi, \theta)$ from $U$ to $Y$ such that $\psi(x) = y$ and $\theta_x^f = \phi$.

Proof.

(i) Since the question is local on $S$, $X$, and $Y$, we can assume that $S$, $X$, and $Y$ are affine, given by rings $A$, $B$, and $C$ (respectively), and with $f$ and $f'$ of the form $(\phi, \theta)$ and $(\phi', \theta')$ (respectively), where $\phi$ and $\phi'$ are $A$-homomorphisms from $C$ to $B$ such that $\phi^{-1}(1_x) = \phi'^{-1}(1_x) = 1_Y$, and the homomorphisms $\phi_x$ and $\phi'_x$ from $C_y$ to $B_x$, induced by $\phi$ and $\phi'$, are identical; we can further suppose that $C$ is an $A$-algebra of finite type. Let $c_i$ ($1 \leq i \leq n$) be the generators of the $A$-algebra $C$, and set $b_i = \phi(c_i)$ and $b'_i = \phi'(c_i)$; by hypothesis, we have $b_i/b'_i = 1$ in the ring of fractions $B_x$ ($1 \leq i \leq n$). This implies that there exist elements $s_i \in B - 1_x$ such that $s_i(b_i - b'_i) = 0$ for $1 \leq i \leq n$, and we can clearly assume that all the $s_i$ are equal to a single element $g \in B - 1_x$. From this, we conclude that we have $b_i/b'_i = 1$ in the ring of fractions $B_g$; if $i_g$ is the canonical homomorphism $B \to B_g$, we then have $i_g \circ \phi = i_g \circ \phi'$; so the restrictions of $f$ and $f'$ to $D(g)$ are identical.

(ii) We can restrict to the situation as in (i), and further assume that the ring $A$ is Noetherian. Let $c_i$ ($1 \leq i \leq n$) be the generators of the $A$-algebra $C$, and let $a : A[X_1, \ldots, X_n] \to C$ be
the homomorphism of polynomial algebras that sends $X_i$ to $c_i$ for $1 \leq i \leq n$. Also let $i_y$ be the canonical homomorphism $C \to C_y$, and consider the composite homomorphism 

$$
\beta : A[X_1, \ldots, X_n] \xrightarrow{a} C \xrightarrow{i_y} C_y \xrightarrow{\phi} B_x.
$$

We denote by $a$ the kernel of $\beta$; since $A$ is Noetherian, so too is $A[X_1, \ldots, X_n]$, and so $a$ admits a finite system of generators $Q_j(X_1, \ldots, X_n)$ $(1 \leq j \leq m)$. Furthermore, each of the elements $\phi(i_y(c_i))$ can be written in the form $b_i/s_i$, where $b_i \in B$ and $s_i \notin 1_X$; we can further assume that all of the $s_i$ are equal to a single element $g \in B - 1_X$. Indeed, we can assume $Q_j(b_i/g, \ldots, b_n/g) = 0$ in $B_X$; set 

$$
Q_j(X_1/T, \ldots, X_n/T) = P_j(X_1, \ldots, X_n, T)/T^k
$$

where $P_j$ is homogeneous of degree $k_j$. Then let $d_j = P_j(b_1, \ldots, b_n, g) \in B$. By hypothesis, we have $tjd_j = 0$ for some $t_j \in B - 1_X$ $(1 \leq j \leq m)$, and we can clearly assume that all the $t_j$ are equal to a single element $h \in B - 1_X$; from this we conclude that $P_j(hb_1, \ldots, hb_n, hg) = 0$ for $1 \leq j \leq m$. With this, consider the homomorphism $\rho$ from $A[X_1, \ldots, X_n]$ to the ring of fractions $B_{hg}$ which sends $X_i$ to $hb_i/hg$ $(1 \leq i \leq n)$; the image of a under this homomorphism is 0, and is a fortiori the same as the image of the kernel $\alpha^{-1}(0)$ under $\rho$. So $\rho$ factors as $A[X_1, \ldots, X_n] \xrightarrow{a} C \xrightarrow{\gamma} B_{hg}$, with $\gamma(c_i) = hb_i/hg$, and it is clear that, if $i_y$ is the canonical homomorphism $B_{hg} \to B_x$, then the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\gamma} & B_{hg} \\
\downarrow{i_y} & & \downarrow{i_y} \\
C_y & \xrightarrow{\phi} & B_x
\end{array}
\]

is commutative; we thus have $\phi = \gamma_{X_x}$, and since $\phi$ is a local homomorphism, $a\gamma(x) = y$; $f = (a\gamma, \tilde{\gamma})$ is thus an $S$-morphism from the neighbourhood $D(hg)$ of $x$ to $Y$ as claimed in the proposition.

\[\square\]

**Corollary (6.5.2).** — Under the hypotheses of (6.5.1, ii), if, further, $X$ is of finite type over $S$, then we can assume that the morphism $f$ is of finite type.

**Proof.** This follows from Corollary (6.3.6).

\[\square\]

**Corollary (6.5.3).** — Suppose that the hypotheses of Proposition (6.5.1, ii), and suppose further that $Y$ is integral, and that $\phi$ is an injective homomorphism. Then we can assume that $f = (\gamma, \tilde{\gamma})$, where $\gamma$ is injective.

**Proof.** Indeed, we can assume $C$ to be integral (5.1.4), hence $i_y$ injective; it then follows from the diagram (6.5.1.1) that $\gamma$ is injective.

\[\square\]

**Proposition (6.5.4).** — Let $f = (\psi, \theta) : X \to Y$ be a morphism of finite type, $x$ a point of $X$, and $y = \psi(x)$.

(i) For $f$ to be a local immersion at the point $x$ (4.5.1), it is necessary and sufficient that $\theta_y : \mathcal{O}_y \to \mathcal{O}_x$ be surjective.

(ii) Assume further that $Y$ is locally Noetherian. For $f$ to be a local isomorphism at the point $x$ (4.5.2), it is necessary and sufficient that $\theta_y$ be an isomorphism.

**Proof.**

(i) By (6.5.1), there exists an open neighbourhood $V$ of $Y$ and a morphism $g : V \to X$ such that $g \circ f$ (resp. $f \circ g$) is defined and agrees with the identity on a neighbourhood of $x$ (resp. $y$), whence we can easily see that $f$ is a local isomorphism.

(ii) Since the question is local on $X$ and $Y$, we can assume that $X$ and $Y$ are affine, given by rings $A$ and $B$ (respectively); we have $f = (\phi, \tilde{\phi})$, where $\phi$ is a homomorphism of rings $B \to A$ that makes $A$ a $B$-algebra of finite type; we have $\phi^{-1}(i_x) = i_y$, and the homomorphism $\phi_x : B_y \to A_x$ induced by $\phi$ is surjective. Let $(t_i)$ $(1 \leq i \leq n)$ be a system
of generators of the $B$-algebra $A$; the hypothesis on $\phi_x$ implies that there exist $b_i \in B$ and some $c \in B - 1$ such that, in the ring of fractions $A_x$, we have $t_i/1 = \phi(b_i)/\phi(c)$ for $1 \leq i \leq n$. Then (1.3.3) there exists some $a \in A - 1_x$ such that, if we let $g = a\phi(c)$, we also have $t_i/1 = a\phi(b_i)/g$ in the ring of fractions $A_g$. With this, there exists, by hypothesis, a polynomial $Q(X_1, \ldots, X_n)$, with coefficients in the ring $\phi(B)$, such that $a = Q(t_1, \ldots, t_n)$; let $Q(X_1/T, \ldots, X_n/T) = P(X_1, \ldots, X_n, T)/T^n$, where $P$ is homogeneous of degree $m$. In the ring $A_g$, we have

$$a/1 = a^mP(\phi(b_1), \ldots, \phi(b_n), \phi(c))/g^m = a^m\phi(d)/g^m$$

where $d \in B$. Since, in $A_g$, $g/1 = (a/1)(\phi(c)/1)$ is invertible by definition, so too are $a/1$ and $\phi(c)/1$, and we can thus write $a/1 = (\phi(d)/1)(\phi(c)/1)^{-m}$. From this we conclude that $\phi(d)/1$ is also invertible in $A_g$. So let $h = cd$; since $\phi(h)/1$ is invertible in $A_g$, the composite homomorphism $B \xrightarrow{\phi} A \to A_g$ factors as $B \to B_h \xrightarrow{\gamma} A_g$ (0.1.2.4). We will show that $\gamma$ is surjective; it suffices to show that the image of $B_h$ in $A_g$ contains the $t_i/1$ and $(g/1)^{-1}$. But we have $(g/1)^{-1} = (\phi(c)/1)^{m-1}(\phi(d)/1)^{-1} = \gamma(c^m/h)$, and $a/1 = \gamma(d^{m+1}/h^m)$, so $(a\phi(b_i))/1 = \gamma(b_id^{m+1}/h^m)$, and since $t_i/1 = (a\phi(b_i))/1(g/1)^{-1}$, our claim is proved. The choice of $h$ implies that $\psi(D(g)) \subset D(h)$, and we also know that the restriction of $f$ to $D(g)$ is equal to $(\alpha, \gamma)$; since $\gamma$ is surjective, this restriction is a closed immersion of $D(g)$ into $D(h)$ (4.2.3).

\[ \square \]

**Corollary (6.5.5).** — Let $f = (\psi, \theta) : X \to Y$ be a morphism of finite type. Assume that $X$ is irreducible, and denote by $x$ its generic point, and let $y = \psi(x)$.

(i) For $f$ to be a local immersion at any point of $X$, it is necessary and sufficient that $\theta_x^{\sharp} : \mathcal{O}_y \to \mathcal{O}_x$ be surjective.

(ii) Assume further that $Y$ is irreducible and locally Noetherian. For $f$ to be a local isomorphism at any point of $X$, it is necessary and sufficient that $y$ be the generic point of $Y$ (or, equivalently (0.2.1.4), that $f$ be a dominant morphism) and that $\theta_x^{\sharp}$ be an isomorphism (in other words, that $f$ be birational (2.2.9)).

**Proof.** It is clear that (i) follows from (6.5.4, i), taking into account the fact that every nonempty open subset of $X$ contains $x$; similarly, (ii) follows from (6.5.4, ii). \[ \square \]

### 6.6. Quasi-compact morphisms and morphisms locally of finite type

**Definition (6.6.1).** — We say that a morphism $f : X \to Y$ is quasi-compact if the inverse image of any quasi-compact open subset of $Y$ under $f$ is quasi-compact.

Let $\mathfrak{B}$ be a base of the topology of $Y$ consisting of quasi-compact open subsets (for example, affine open subsets); for $f$ to be quasi-compact, it is necessary and sufficient that the inverse image of every set of $\mathfrak{B}$ under $f$ be quasi-compact (or, equivalently, a finite union of affine open subsets), because every quasi-compact open subset of $Y$ is a finite union of sets of $\mathfrak{B}$. For example, if $X$ is quasi-compact and $Y$ affine, then every morphism $f : X \to Y$ is quasi-compact: indeed, $X$ is a finite union of affine open subsets $U_i$, and for every affine open subset $V$ of $Y$, $U_i \cap f^{-1}(V)$ is affine (5.5.10), and so quasi-compact.

If $f : X \to Y$ is a quasi-compact morphism, it is clear that, for every open subset $V$ of $Y$, the restriction of $f$ to $f^{-1}(V)$ is a quasi-compact morphism $f^{-1}(V) \to V$. Conversely, if $(U_\alpha)$ is an open cover of $Y$, and $f : X \to Y$ a morphism such that the restrictions $f^{-1}(U_\alpha) \to U_\alpha$ are quasi-compact, then $f$ is quasi-compact.

**Definition (6.6.2).** — We say that a morphism $f : X \to Y$ is locally of finite type if, for every $x \in X$, there exists an open neighbourhood $U$ of $x$ and an open neighbourhood $V \supset f(U)$ of $y$ such that the restriction of $f$ to $U$ is a morphism of finite type from $U$ to $V$. We then also say that $X$ is a prescheme locally of finite type over $Y$, or a $Y$-prescheme locally of finite type.

It follows immediately from (6.3.2) that, if $f$ is locally of finite type, then, for every open subset $W$ of $Y$, the restriction of $f$ to $f^{-1}(W)$ is a morphism $f^{-1}(W) \to W$ that is locally of finite type.
If $Y$ is locally Noetherian and $X$ locally of finite type over $Y$, then $X$ is locally Noetherian thanks to (6.3.7).

**Proposition (6.6.3).** — *For a morphism $f : X \to Y$ to be of finite type, it is necessary and sufficient that it be quasi-compact and locally of finite type.*

**Proof.** The necessity of the conditions is immediate, given (6.3.1) and the remark following (6.6.1). Conversely, suppose that the conditions are satisfied, and let $U$ be an affine open subset of $Y$, given by some ring $A$; for all $x \in f^{-1}(U)$, there is, by hypothesis, a neighbourhood $V(x) \subset f^{-1}(U)$ of $x$, and a neighbourhood $W(x) \subset U$ of $y = f(x)$ containing $f(V(x))$, and such that the restriction of $f$ to $V(x)$ is a morphism $V(x) \to W(x)$ of finite type. Replacing $W(x)$ with a neighbourhood $W_1(x) \subset W(x)$ of $x$ of the form $D(g)$ (with $g \in A$), and $V(x)$ with $V(x) \cap f^{-1}(W_1(x))$, we can assume that $W(x)$ is of the form $D(g)$, and thus of finite type over $U$ (because its ring can be written as $A[1/g]$); so $V(x)$ is of finite type over $U$. Further, $f^{-1}(U)$ is quasi-compact by hypothesis, and so the finite union of open subsets $V(x_i)$, which finishes the proof. □

**Proposition (6.6.4).** —

(i) *An immersion $X \to Y$ is quasi-compact if it is closed, or if the underlying space of $Y$ is locally Noetherian, or if the underlying space of $X$ is Noetherian.*

(ii) *The composition of any two quasi-compact morphisms is quasi-compact.*

(iii) If $f : X \to Y$ is a quasi-compact $S$-morphism, then so too is $f(s') : X(s') \to Y(s')$ for any extension $g : S \to S'$ of the base prescheme.

(iv) If $f : X \to X'$ and $g : Y \to Y'$ are two quasi-compact $S$-morphisms, then $f \times_S g$ is quasi-compact.

(v) *If the composition of any two morphisms $f : X \to Y$ and $g : Y \to Z$ is quasi-compact, and if either $g$ is separated or the underlying space of $X$ is locally Noetherian, then $f$ is quasi-compact.*

(vi) *For a morphism $f$ to be quasi-compact, it is necessary and sufficient that $f_{\text{red}}$ be quasi-compact.*

**Proof.** We note that (vi) is evident because the property of being quasi-compact, for a morphism, depends only on the corresponding continuous map of underlying spaces. We will similarly prove the part of (v) corresponding to the case where the underlying space of $X$ is locally Noetherian.

Set $h = g \circ f$, and let $U$ be a quasi-compact open subset of $Y$; $g(U)$ is quasi-compact (but not necessarily open) in $Z$, and so contained in a finite union of quasi-compact open subsets $V_j (2.1.3)$, and $f^{-1}(U)$ is thus contained in the union of the $h^{-1}(V_j)$, which are quasi-compact subspaces of $X$, and thus Noetherian subspaces. We thus conclude (0, 2.2.3) that $f^{-1}(U)$ is a Noetherian space, and *a fortiori* quasi-compact.

To prove the other claims, it suffices to prove (i), (ii), and (iii) (5.5.12). But (ii) is evident, and (i) follows from (6.3.5) whenever the space $Y$ is locally Noetherian or the space $X$ is Noetherian, and is evident for a closed immersion. To show (iii), we can restrict to the case where $S = Y$ (3.3.11); let $f' = f|_{S'}$, and let $U'$ be a quasi-compact open subset of $S'$. For every $s' \in U'$, let $T$ be an affine open neighbourhood of $g(s')$ in $S$, and let $W$ be an affine open neighbourhood of $s'$ contained in $U' \cap g^{-1}(T)$; it will suffice to show that $f'^{-1}(W)$ is quasi-compact; in other words, we can restrict to showing that, when $S$ and $S'$ are affine, the underlying space of $X \times_S S'$ is quasi-compact. But since $X$ is then, by hypothesis, a finite union of affine open subsets $V_j$, $X \times_S S'$ is a union of the underlying spaces of the affine schemes $V_j \times_S S'$ (3.2.2 and 3.2.7)), which proves the proposition. □

We note also that, if $X = X' \sqcup X''$ is the sum of two preschemes, a morphism $f : X \to Y$ is quasi-compact if and only if its restrictions to both $X'$ and $X''$ are quasi-compact.

**Proposition (6.6.5).** — *Let $f : X \to Y$ be a quasi-compact morphism. For $f$ to be dominant, it is necessary and sufficient that, for every generic point $y$ of an irreducible component of $Y$, $f^{-1}(y)$ contain the generic point of an irreducible component of $X$.*

**Proof.** It is immediate that the condition is sufficient (even without assuming that $f$ is quasi-compact). To see that it is necessary, consider an affine open neighbourhood $U$ of $y$; $f^{-1}(U)$ is quasi-compact, and so a finite union of affine open subsets $V_j$, and the hypothesis that $f$ be dominant implies that $y$ belongs to the closure in $U$ of one of the $f(V_j)$. We can clearly assume $X$ and $Y$ to be reduced; since the closure in $X$ of an irreducible component of $V_j$ is an irreducible component on $X$...
We define a rational $\mathcal X$ with a minimal prime ideal of $\mathcal A$ as its underlying space (5.2.1), and we are thus led to proving the proposition when $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$ are affine and reduced. Since $f$ is dominant, $B$ is a subring of $A$ (1.2.7), and the proposition then follows from the fact that every minimal prime ideal of $B$ is the intersection of $B$ with a minimal prime ideal of $A$ (0, 1.5.8).

Proposition (6.6.6). —

(i) Every local immersion is locally of finite type.

(ii) If two morphisms $f : X \to Y$ and $g : Y \to Z$ are locally of finite type, then so too is $g \circ f$.

(iii) If $f : X \to Y$ is an $S$-morphism locally of finite type, then $f(s') : X(s') \to Y(s')$ is locally of finite type for any extension $S' \to S$ of the base prescheme.

(iv) If $f : X \to X'$ and $g : Y \to Y'$ are $S$-morphisms locally of finite type, then $f \times_S g$ is locally of finite type.

(v) If the composition $g \circ f$ of two morphisms is locally of finite type, then $f$ is locally of finite type.

(vi) If a morphism $f$ is locally of finite type, then so too is $f_{\text{red}}$.

Proof. By (5.5.12), it suffices to prove (i), (ii), and (iii). If $j : X \to Y$ is a local immersion then, for every $x \in X$, there is an open neighbourhood $V$ of $j(x)$ in $Y$ and an open neighbourhood $U$ of $x$ in $X$ such that the restriction of $j$ to $U$ is a closed immersion $U \to V$ (4.5.1), and so this restriction is of finite type. To prove (ii), consider a point $x \in X$; by hypothesis, there is an open neighbourhood $W$ of $g(f(x))$ and an open neighbourhood $V$ of $f(x)$ such that $g(V) \subset W$ and such that $V$ is of finite type over $W$; furthermore, $f^{-1}(V)$ is locally of finite type over $V$ (6.6.2), so there is an open neighbourhood $U$ of $x$ that is contained in $f^{-1}(V)$ and of is finite type over $V$; thus we have $g(f(U)) \subset W$, and that $U$ is of finite type over $W$ (6.3.4, ii). Finally, to prove (iii), we can restrict to the case where $Y = S$ (3.3.11); for every $x' \in X' = X(s')$, let $x$ be the image of $x'$ in $X$, $s$ the image of $x$ in $S$, $T$ an open neighbourhood of $s$, $T'$ the inverse image of $T$ in $S'$, and $U$ an open neighbourhood of $x$ that is of finite type over $T$ and whose image is contained in $T$; then $U \times_S T' = U \times_T T'$ is an open neighbourhood of $x'$ (3.2.7) that is of finite type over $T'$ (6.3.4, iv).

Corollary (6.6.7). — Let $X$ and $Y$ be $S$-preschemes that are locally of finite type over $S$. If $S$ is locally Noetherian, then $X \times_S Y$ is locally Noetherian.

Proof. Indeed, $X$ being locally of finite type over $S$ means that it is locally Noetherian, and that $X \times_S Y$ is locally of finite type over $X$, and so $X \times_S Y$ is also locally Noetherian.

Remark (6.6.8). — Proposition (6.3.10) and its proof extend immediately to the case where we suppose only that the morphism $f$ is locally of finite type. Similarly, propositions (6.4.2) and (6.4.9) hold true when we suppose only that the preschemes $X$ and $Y$ in the claim are locally of finite type over the field $K$.

7. Rational maps and rational functions

(7.1.1). Let $X$ and $Y$ be preschemes, $U$ and $V$ dense open subsets of $X$, and $f$ (resp. $g$) a morphism from $U$ (resp. $V$) to $Y$; we say that $f$ and $g$ are equivalent if they agree on a dense open subset of $U \cap V$. Since a finite intersection of dense open subsets of $X$ is a dense open subset of $X$, it is clear that this relation is an equivalence relation.

Definition (7.1.2). — Given preschemes $X$ and $Y$, we define a rational map from $X$ to $Y$ to be an equivalence class of morphisms from a dense open subset of $X$ to a dense open subset of $Y$, under the equivalence relation defined in (7.1.1). If $X$ and $Y$ are $S$-preschemes, we say that such a class is a rational $S$-map if there exists a representative of the class that is also an $S$-morphism. We define a rational $S$-section of $X$ to be any rational $S$-map from $S$ to $X$. We define a rational function on a prescheme $X$ to be any rational $X$-section on the $X$-prescheme $X \otimes_Z \mathbb Z[T]$ (where $T$ is an indeterminate).

By an abuse of language, whenever we are discussing only $S$-preschemes, we will say “rational map” instead of “rational $S$-map” if no confusion may arise.
Let $f$ be a rational map from $X$ to $Y$, and $U$ an open subset of $X$; if $f_1$ and $f_2$ are two morphisms belonging to the class of $f$, defined (respectively) on dense open subsets $V$ and $W$ of $X$, then the restrictions $f_1|_{(U \cap V)}$ and $f_2|_{(U \cap W)}$ agree on $U \cap V \cap W$, which is dense in $U$; the class $f$ of morphisms thus defines a rational map from $U$ to $Y$, called the restriction of $f$ to $U$, and denoted by $f|_U$.

If, to every $S$-morphism $f : X \to Y$, we take the corresponding rational $S$-map to which $f$ belongs, we obtain a canonical map from $\text{Hom}_S(X, Y)$ to the set of rational $S$-maps from $X$ to $Y$. We denote by $\Gamma_{\text{rat}}(X/Y)$ the set of rational $Y$-sections on $X$, and we thus have a canonical map $\Gamma(X/Y) \to \Gamma_{\text{rat}}(X/Y)$. It is also clear that, if $X$ and $Y$ are $S$-preschemes, then the set of rational $S$-maps from $X$ to $Y$ is canonically identified with $\Gamma_{\text{rat}}((X \times_S Y)/X)$ (3.3.14). (7.1.3)

It also follows from (7.1.2) and (3.3.14) that the rational functions on $X$ are canonically identified with equivalence classes of sections of the structure sheaf $\mathcal{O}_X$ over dense open subsets of $X$, where two such sections are equivalent if the agree on some dense open subset of $X$. When the intersection of the subsets on which they are defined. In particular, it follows that the rational functions on $X$ form a ring $R(X)$.

(7.1.4). When $X$ is an irreducible prescheme, every nonempty open subset of $X$ is dense in $X$; so we can say that the nonempty open subsets of $X$ are the open neighbourhoods of the generic point $x$ of $X$. To say that two morphisms from nonempty open subsets of $X$ to $Y$ are equivalent thus means, in this case, that they have the same germ at the point $x$. In other words, the rational maps (resp. rational $S$-maps) $X \to Y$ are identified with the germs of morphisms (resp. $S$-morphisms) from nonempty open subsets of $X$ to $Y$ at the generic point $x$ of $X$. In particular:

**Proposition (7.1.5).** — If $X$ is an irreducible prescheme, then the ring $R(X)$ of rational maps on $X$ is canonically identified with the local ring $\mathcal{O}_x$ of the generic point $x$ of $X$. It is a local ring of dimension 0, and thus a local Artinian ring when $X$ is Noetherian; it is a field when $X$ is integral, and, when $X$ is further an affine scheme, it is identified with the field of fractions of $A(X)$.

**Proof.** Given the above, and the identification of rational functions with sections of $\mathcal{O}_X$ over a dense open subset of $X$, the first claim is nothing but the definition of the fibre of a sheaf above a point. For the other claims, we can reduce to the case where $X$ is affine, given by some ring $A$; then $\mathcal{O}_x$ is the nilradical of $A$, and $\mathcal{O}_x$ is thus of dimension 0; if $A$ is integral, then $\mathcal{O}_x = \{0\}$, and $\mathcal{O}_x$ is thus the field of fractions of $A$. Finally, if $A$ is Noetherian, we know ([Sam53b, p. 127, cor. 4]) that $\mathcal{O}_x$ is nilpotent, and $\mathcal{O}_x = A_x$ Artinian. □

If $X$ is integral, then the ring $\mathcal{O}_x$ is integral for all $z \in X$; every affine open subset $U$ containing $z$ also contains $x$, and $R(U)$, being equal to the field of fractions of $A(U)$, is identified with $R(X)$; we thus conclude that $R(X)$ can also be identified with the field of fractions of $\mathcal{O}_x$: the canonical identification of $\mathcal{O}_x$ to a subring of $R(X)$ consists of associating, to every germ of a section $s \in \mathcal{O}_x$, the unique rational function on $X$, class of a section of $\mathcal{O}_X$, (necessarily defined on a dense open subset of $X$) having $s$ as its germ at the point $z$.

(7.1.6). Now suppose that $X$ has a finite number of irreducible components $X_i$ ($1 \leq i \leq n$) (which will be the case whenever the underlying space of $X$ is Noetherian); let $X'_i$ be the open subset of $X$ given by the complement of the $X_i \cap X_j$ for $j \neq i$ inside $X_i$; $X'_i$ is irreducible, its generic point $x_i$ is the generic point of $X_i$, and the $X'_i$ are pairwise disjoint, with their union being dense in $X$ (0.2.1.6). For every dense open subset $U$ of $X$, $U_i = U \cap X'_i$ is a nonempty dense open subset of $X'_i$, with the $U_i$ being pairwise disjoint, and so $U' = \bigcup_i U_i$ is dense in $X$. Giving a morphism from $U'$ to $Y$ consists of giving (arbitrarily) a morphism from each of the $U_i$ to $Y$.

Thus:

**Proposition (7.1.7).** — Let $X$ and $Y$ be two preschemes (resp. $S$-preschemes) such that $X$ has a finite number of irreducible components $X_i$, with generic points $x_i$ ($1 \leq i \leq n$). If $R_i$ is the set of germs of morphisms (resp. $S$-morphisms) from open subsets of $X$ to $Y$ at the point $x_i$, then the set of rational maps (resp. rational $S$-maps) from $X$ to $Y$ can be identified with the product of the $R_i$ ($1 \leq i \leq n$).

**Corollary (7.1.8).** — Let $X$ be a Noetherian prescheme. The ring of rational functions on $X$ is an Artinian ring, whose local components are the rings $\mathcal{O}_{x_i}$ of the generic points $x_i$ of the irreducible components of $X$. 


Corollary (7.1.9). — Let $A$ be a Noetherian ring, and $X = \text{Spec}(A)$. If $Q$ is the complement of the union of the minimal prime ideals of $A$, then the ring of rational functions on $X$ can be canonically identified with the ring of fractions $Q^{-1}A$.

This will follow from the following lemma:

Lemma (7.1.9.1). — For an element $f \in A$ to be such that $D(f)$ is dense in $X$, it is necessary and sufficient that $f \in Q$: every dense open subset of $X$ contains an open subset of the form $D(f)$, where $f \in Q$.

Proof. To show (7.1.9.1), we again denote by $X_i$ ($1 \leq i \leq n$) the irreducible components of $X$; if $D(f)$ is dense in $X$ then $D(f) \cap X_i \neq \emptyset$ for $1 \leq i \leq n$, and vice versa; but this means that $f \notin p_i$ for $1 \leq i \leq n$, where we set $p_i = \mathfrak{(X_i)}$, and since the $p_i$ are the minimal prime ideals of $A$ (1.1.14), the conditions $f \notin p_i$ ($1 \leq i \leq n$) are equivalent to $f \in Q$, whence the first claim of the lemma. For the other claim, if $U$ is a dense open subset of $X$, the complement of $U$ is a set of the form $V(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal which is not contained in any of the $p_i$; it is thus not contained in their union ([Nor53, p. 13]), and there thus exists some $f \in \mathfrak{a}$ a belonging to $Q$; whence $D(f) \subset U$, which finishes the proof. □

(7.1.10). Suppose again that $X$ is irreducible, with generic point $x$. Since every nonempty open subset $U$ of $X$ contains $x$, and thus also contains every $z \in X$ such that $x \in \{z\}$, every morphism $U \to Y$ can be composed with the canonical morphism $\text{Spec}(\mathcal{O}_x) \to X$ (2.4.1); and any two morphisms into $Y$ from two nonempty open subsets of $X$ which agree on a nonempty open subset of $X$ give, by composition, the same morphism $\text{Spec}(\mathcal{O}_x) \to Y$. In other words, to every rational map from $X$ to $Y$ there is a corresponding well-defined morphism $\text{Spec}(\mathcal{O}_x) \to Y$.

Proposition (7.1.11). — Let $X$ and $Y$ be two $S$-preschemes; suppose that $X$ is irreducible with generic point $x$, and that $Y$ is of finite type over $S$. Any two rational $S$-maps from $X$ to $Y$ that correspond to the same $S$-morphism $\text{Spec}(\mathcal{O}_x) \to Y$ are then identical. If we further suppose $S$ to be locally Noetherian, then every $S$-morphism from $\text{Spec}(\mathcal{O}_x)$ to $Y$ corresponds to exactly one rational $S$-map from $X$ to $Y$.

Proof. Taking into account that every nonempty subset of $X$ is dense in $X$, this follows from (6.5.1). □

Corollary (7.1.12). — Suppose that $S$ is locally Noetherian, and that the other hypotheses of (7.1.11) are satisfied. The rational $S$-maps from $X$ to $Y$ can then be identified with points of the $S$-prescheme $Y$, with values in the $S$-prescheme $\text{Spec}(\mathcal{O}_x)$.

Proof. This is nothing but (7.1.11), with the terminology introduced in (3.4.1). □

Corollary (7.1.13). — Suppose that the conditions of (7.1.12) are satisfied. Let $s$ be the image of $x$ in $S$. The data of a rational $S$-map from $X$ to $Y$ is equivalent to the data of a point $y$ of $Y$ over $s$ along with a local $\mathcal{O}_x$-homomorphism $\mathcal{O}_y \to \mathcal{O}_x = R(X)$.

Proof. This follows from (7.1.11) and (2.4.4). □

In particular:

Corollary (7.1.14). — Under the conditions of (7.1.12), rational $S$-maps from $X$ to $Y$ depend only (for any given $Y$) on the $S$-prescheme $\text{Spec}(\mathcal{O}_x)$, and, in particular, remain the same whenever $X$ is replaced by $\text{Spec}(\mathcal{O}_z)$, for any $z \in X$.

Proof. Since $z \in \{x\}$, $x$ is the generic point of $Z = \text{Spec}(\mathcal{O}_z)$, and $\mathcal{O}_{x,x} = \mathcal{O}_{z,z}$. □

When $X$ is integral, $R(X) = \mathcal{O}_x = k(x)$ is a field (7.1.5); the preceding corollaries then specialize to the following:

Corollary (7.1.15). — Suppose that the conditions of (7.1.12) are satisfied, and further that $X$ is integral. Let $s$ be the image of $x$ in $S$. Then rational $S$-maps from $X$ to $Y$ can be identified with the geometric points of $Y \otimes_S k(s)$ with values in the extension $R(X)$ of $k(s)$, or, in other words, every such map is equivalent to the data of a point $y \in Y$ above $s$ along with a $k(s)$-monomorphism from $k(y)$ to $k(x) = R(X)$.

Proof. The points of $Y$ above $s$ are identified with the points of $Y \otimes_S k(s)$ (3.6.3), and the local $\mathcal{O}_x$-homomorphisms $\mathcal{O}_y \to R(X)$ with the $k(s)$-monomorphisms $k(y) \to R(X)$. □
More precisely:

**Corollary (7.1.16).** — Let $k$ be a field, and $X$ and $Y$ two algebraic preschemes over $k$ (6.4.1); suppose further that $X$ is integral. Then the rational $k$-maps from $X$ to $Y$ can be identified with the geometric points of $Y$ with values in the extension $R(X)$ of $k$ (3.4.4).

### 7.2. Domain of definition of a rational map

(7.2.1). Let $X$ and $Y$ be preschemes, and $f$ a rational map from $X$ to $Y$. We say that $f$ is defined at a point $x \in X$ if there exists a dense open subset $U$ of $X$ that contains $x$, and a morphism $U \to Y$ belonging to the equivalence class of $f$. The set of points $x \in X$ where $f$ is defined is called the domain of definition of $f$; it is clear that it is an open dense subset of $X$.

**Proposition (7.2.2).** — Let $X$ and $Y$ be $S$-preschemes such that $X$ is reduced and $Y$ is separated over $S$. Let $f$ be a rational $S$-map from $X$ to $Y$, with domain of definition $U_0$. Then there exists exactly one $S$-morphism $U_0 \to Y$ belonging to the class of $f$.

Since, for every morphism $U \to Y$ belonging to the class of $f$, we necessarily have $U \subset U_0$, it is clear that the proposition will be a consequence of the following:

**Lemma (7.2.2.1).** — Under the hypotheses of (7.2.2), let $U_1$ and $U_2$ be two dense open subsets of $X$, and $f_i : U_i \to Y$ $(i = 1, 2)$ two $S$-morphisms such that there exists an open subset $V \subset U_1 \cap U_2$, dense in $X$, and on which $f_1$ and $f_2$ agree. Then $f_1$ and $f_2$ agree on $U_1 \cap U_2$.

**Proof.** We can clearly restrict to the case where $X = U_1 = U_2$. Since $X$ (and thus $V$) is reduced, $X$ is the smallest closed subscheme of $X$ containing $V$ (5.2.2). Let $g = (f_1, f_2)_S : X \to Y \times_S Y$; since, by hypothesis, the diagonal $T = \Delta_Y(Y)$ is a closed subscheme of $Y \times_S Y$, $Z = g^{-1}(T)$ is a closed subscheme of $X$ (4.4.1). If $h : V \to Y$ is the common restriction of $f_1$ and $f_2$ to $V$, then the restriction of $g$ to $V$ is $g' = (h, h)_S$, which factors as $g' = \Delta_Y \circ h$; since $\Delta_Y^{-1}(T) = Y$, we have that $g'^{-1}(T) = V$, and so $Z$ is a closed subscheme of $X$ inducing $V$, thus containing $V$, which implies that $Z = X$. From the equation $g^{-1}(T) = X$, we deduce (4.4.1) that $g$ factors as $\Delta_Y \circ f$, where $f$ is a morphism $X \to Y$, which implies, by the definition of the diagonal morphism, that $f_1 = f_2 = f$. □

It is clear that the morphism $U_0 \to Y$ defined in (7.2.2) is the unique morphism of the class that cannot be extended to a morphism from an open subset of $X$ that strictly contains $U_0$. Under the hypotheses of (7.2.2), we can thus identify the rational maps from $X$ to $Y$ with the non-extendible (to strictly larger open subsets) morphisms from dense open subsets of $X$ to $Y$. With this identification, Proposition (7.2.2) implies:

**Corollary (7.2.3).** — With the hypotheses from (7.2.2) on $X$ and $Y$, let $U$ be a dense open subset of $X$. Then there exists a canonical bijective correspondence between $S$-morphisms from $U$ to $Y$ and rational $S$-maps from $X$ to $Y$ that are defined at all points of $U$.

**Proof.** By (7.2.2), for every $S$-morphism $f$ from $U$ to $Y$, there exists exactly one rational $S$-map $\overline{f}$ from $X$ to $Y$ which extends $f$. □

**Corollary (7.2.4).** — Let $S$ be a scheme, $X$ a reduced $S$-prescheme, $Y$ an $S$-scheme, and $f : U \to Y$ an $S$-morphism from a dense open subset $U$ of $X$ to $Y$. If $\overline{f}$ is the rational $S$-map from $X$ to $Y$ that extends $f$, then $\overline{f}$ is an $S$-morphism (and thus the rational $S$-map from $X$ to $Y$ that extends $f$).

**Proof.** Indeed, if $\phi : X \to S$ and $\psi : Y \to S$ are the structure morphisms, $U_0$ the domain of definition of $\overline{f}$, and $j$ the injection $U_0 \to X$, then it suffices to show that $\psi \circ \overline{f} = \phi \circ j$, but this follows from (7.2.2.1), since $f$ is an $S$-morphism. □

**Corollary (7.2.5).** — Let $X$ and $Y$ be two $S$-preschemes; suppose that $X$ is reduced, and that $X$ and $Y$ are separated over $S$. Let $p : Y \to X$ be an $S$-morphism (making $Y$ an $X$-prescheme), $U$ a dense open subset of $X$, and $f$ a $U$-section of $Y$; then the rational map $\overline{f}$ from $X$ to $Y$ extending $f$ is a rational $X$-section of $Y$.

**Proof.** We have to show that $p \circ \overline{f}$ is the identity on the domain of definition of $\overline{f}$; since $X$ is separated over $S$, this again follows from (7.2.2.1). □
Corollary (7.2.6). — Let $X$ be a reduced prescheme, and $U$ a dense open subset of $X$. Then there is a canonical bijective correspondence between sections of $\mathcal{O}_X$ over $U$ and rational functions on $X$ defined at every point of $U$.

Proof. Taking (7.2.3), (7.1.2), and (7.1.3) into account, it suffices to note that the $X$-prescheme $X \otimes_Z \mathcal{O}(T)$ is separated over $X$ (5.5.1, iv).

Corollary (7.2.7). — Let $Y$ be a reduced prescheme, $f : X \to Y$ a separated morphism, $U$ a dense open subset of $Y$, $g : U \to f^{-1}(U)$ a $U$-section of $f^{-1}(U)$, and $Z$ the reduced subscheme of $X$ that has $g(U)$ as its underlying space (5.2.1). For $g$ to be the restriction of a $Y$-section of $X$ (in other words (7.2.5), for the rational map from $Y$ to $X$ extending $g$ to be defined everywhere), it is necessary and sufficient for the restriction of $f$ to $Z$ to be an isomorphism from $Z$ to $Y$.

Proof. The restriction of $f$ to $f^{-1}(U)$ is a separated morphism (5.5.1, i), so $g$ is a closed immersion (5.4.6), and so $g(U) = Z \cap f^{-1}(U)$, and the subscheme induced by $Z$ on the open subset $g(U)$ of $Z$ is identical to the closed subscheme of $f^{-1}(U)$ associated to $g$ (5.2.1). It is then clear that the stated condition is sufficient, because, if satisfied, and if $f_Z : Z \to Y$ is the restriction of $f$ to $Z$, and $\overline{g} : Y \to Z$ is the inverse isomorphism, then $\overline{g}$ extends $g$. Conversely, if $g$ is the restriction to $U$ of a $Y$-section $h$ of $X$, then $h$ is a closed immersion (3.4.6), and so $h(Y)$ is closed, and, since it is contained in $Z$, is equal to $Z$, and it follows from (5.2.1) that $h$ is necessarily an isomorphism from $Y$ to the closed subscheme $Z$ of $X$.

#### 7.8. (7.2.8) Let $X$ and $Y$ be two $S$-preschemes, with $X$ reduced, and $Y$ separated over $S$. Let $f$ be a rational $S$-map from $X$ to $Y$, and let $x$ be a point of $X$; we can compose $f$ with the canonical $S$-morphism $\text{Spec}(\mathcal{O}_X) \to X$ (2.4.1) provided that the intersection of $\text{Spec}(\mathcal{O}_X)$ with the domain of definition of $f$ is dense in $\text{Spec}(\mathcal{O}_X)$ (identified with the set of $z \in X$ such that $x \in \{z\}$ (2.4.2)). This will happen in the follow cases:

1st. $X$ is irreducible (and thus integral), because then the generic point $\xi$ of $X$ is the generic point of $\text{Spec}(\mathcal{O}_X)$; since the domain of definition $U$ of $f$ contains $\xi$, $U \cap \text{Spec}(\mathcal{O}_X)$ contains $\xi$, and so is dense in $\text{Spec}(\mathcal{O}_X)$.

2nd. $X$ is locally Noetherian; our claim then follows from:

**Lemma (7.2.8.1).** — Let $X$ be a prescheme whose underlying space is locally Noetherian, and $x$ a point of $X$. The irreducible components of $\text{Spec}(\mathcal{O}_X)$ are the intersections of $\text{Spec}(\mathcal{O}_X)$ with the irreducible components of $X$ containing $x$. For an open subset $U \subset X$ to be such that $U \cap \text{Spec}(\mathcal{O}_X)$ is dense $\text{Spec}(\mathcal{O}_X)$, it is necessary and sufficient for it to have a nonempty intersection with the irreducible components of $X$ that contain $x$ (which will be the case whenever $U$ is dense in $X$).

Proof. It suffices to show just the first claim, since the second then follows. Since $\text{Spec}(\mathcal{O}_X)$ is contained in every affine open subset $U$ that contains $x$, and since the irreducible components of $U$ that contain $x$ are the intersections of $U$ with the irreducible components of $X$ containing $x$ (0.2.1.6), we can suppose that $X$ is affine, given by some ring $A$. Since the prime ideals of $A$ correspond bijectively to the prime ideals of $\mathcal{O}_X$ that are contained in $\text{Spec}(\mathcal{O}_X)$ (1.2.6), the minimal prime ideals of $\mathcal{O}_X$ correspond to the minimal prime ideals of $A$ that are contained in $\text{Spec}(\mathcal{O}_X)$, hence the lemma.

With this in mind, suppose that we are in one of the two cases mentioned in (7.2.8). If $U$ is the domain of definition of the rational $S$-map $f$, then we denote by $f'$ the rational map from $\text{Spec}(\mathcal{O}_X)$ to $Y$ which agrees (taking (2.4.2) into account) with $f$ on $U \cap \text{Spec}(\mathcal{O}_X)$; we say that this rational map is induced by $f$.

**Proposition (7.2.9).** — Let $S$ be a locally Noetherian prescheme, $X$ a reduced $S$-prescheme, and $Y$ an $S$-scheme of finite type. Suppose further that $X$ is either irreducible or locally Noetherian. Then let $f$ be a rational $S$-map from $X$ to $Y$, and $x$ a point of $X$. For $f$ to be defined at a point $x$, it is necessary and sufficient for the rational map $f'$ from $\text{Spec}(\mathcal{O}_X)$ to $Y$, induced by $f$ (7.2.8), to be a morphism.

Proof. The condition clearly being necessary (since $\text{Spec}(\mathcal{O}_X)$ is contained in every open subset containing $x$), we show that it is sufficient. By (6.5.1), there exists an open neighbourhood $U$ of $x$ in $X$, and an $S$-morphism $g$ from $U$ to $Y$ that induces $f'$ on $\text{Spec}(\mathcal{O}_X)$. If $X$ is irreducible, then $U$ is dense in $X$, and, by (7.2.3), we can suppose that $g$ is a rational $S$-map. Further, the generic point of
7. RATIONAL MAPS

X belongs to both Spec(\(\mathcal{O}_x\)) and the domain of definition of \(f\), and so \(s\) and \(g\) agree at this point, and thus on a nonempty open subset of \(X\) (6.5.1). But since \(f\) and \(g\) are rational \(S\)-maps, they are identical (7.2.3), and so \(f\) is defined at \(x\).

If we now suppose that \(X\) is locally Noetherian, then we can suppose that \(U\) is Noetherian; then there are only a finite number of irreducible components \(X_i\) of \(X\) that contain \(x\) (7.2.8.1), and we can suppose that they are the only ones that have a nonempty intersection with \(U\), by replacing, if needed, \(U\) with a smaller open subset (since there are only a finite number of irreducible components of \(X\) that have a nonempty intersection with \(U\), because \(U\) is Noetherian). We then have, as above, that \(f\) and \(g\) agree on a nonempty open subset of each of the \(X_i\). Taking into account the fact that each of the \(X_i\) is contained in \(\mathbb{U}\), we consider the morphism \(f_1\), defined on a dense open subset of \(U \cup (X - \mathbb{U})\), equal to \(g\) on \(U\), and to \(f\) on the intersection of \(X - \mathbb{U}\) with the domain of definition of \(f\). Since \(U \cup (X - \mathbb{U})\) is dense in \(X\), \(f_1\) and \(f\) agree on a dense open subset of \(X\), and since \(f\) is a rational map, \(f\) is an extension of \(f_1\) (7.2.3), and is thus defined at the point \(x\).

\[\square\]

7.3. Sheaf of rational functions

(7.3.1). Let \(X\) be a prescheme. For every open subset \(U \subset X\), we denote by \(R(U)\) the ring of rational functions on \(U\) (7.1.3); this is a \(\Gamma(U, \mathcal{O}_X)\)-algebra. Further, if \(V \subset U\) is a second open subset of \(X\), then every section of \(\mathcal{O}_X\) over a dense (in \(X\)) open subset of \(V\) gives, by restriction to \(V\), a section over a dense (in \(X\)) open subset of \(V\), and if two sections agree on a dense (in \(X\)) open subset of \(U\), then their restrictions to \(V\) agree on a dense (in \(X\)) open subset of \(V\). We can thus define a di-homomorphism of algebras \(\rho_{V,U} : R(U) \rightarrow R(V)\), and it is clear that, if \(U \subset V \subset W\) are open subsets of \(X\), then we have \(\rho_{W,V} = \rho_{W,U} \circ \rho_{V,U}\); the \(R(U)\) thus define a presheaf of algebras on \(X\).

**Definition (7.3.2).** — We define the sheaf of rational functions on a prescheme \(X\), denoted by \(\mathcal{R}(X)\), to be the \(\mathcal{O}_X\)-algebra associated to the presheaf defined by the \(R(U)\).

For every prescheme \(X\) and open subset \(U \subset X\), it is clear that the induced sheaf \(\mathcal{R}(X)|\!|U\) is exactly \(\mathcal{R}(U)\).

**Proposition (7.3.3).** — Let \(X\) be a prescheme such that the family \((X_\lambda)\) of its irreducible components is locally finite (which is the case whenever the underlying space of \(X\) is locally Noetherian). Then the \(\mathcal{O}_X\)-module \(\mathcal{R}(X)\) is quasi-coherent, and for every open subset \(U\) of \(X\) that has a nonempty intersection with only finitely many of the components \(X_\lambda\), \(R(U)\) is equal to \(\Gamma(U, \mathcal{R}(X))\), and can be canonically identified with the direct sum of the local rings of the generic points of the \(X_\lambda\) such that \(U \cap X_\lambda \neq \emptyset\).

**Proof.** We can evidently restrict to the case where \(X\) has only a finite number of irreducible components \(X_i\), with generic points \(x_i\) (1 ≤ \(i\) ≤ \(n\)). The fact that \(R(U)\) can be canonically identified with the direct sum of the \(\mathcal{O}_{x_i} = R(X_i)\) such that \(U \cap X_i \neq \emptyset\) then follows from (7.1.7). We will show that the presheaf \(U \rightarrow R(U)\) satisfies the sheaf axioms, which will prove that \(R(U) = \Gamma(U, \mathcal{R}(X))\). Indeed, it satisfies (F1) by what has already been discussed. To see that it satisfies (F2), consider a cover of an open subset \(U\) of \(X\) by open subsets \(V_a \subset U\); if the \(s_a \in R(V_a)\) are such that the restrictions of \(s_a\) and \(s_b\) to \(V_a \cap V_b\) agree for every pair of indices, then we can conclude that, for every index \(i\) such that \(U \cap X_i \neq \emptyset\), the components in \(R(X_i)\) of all the \(s_a\) such that \(V_a \cap X_i \neq \emptyset\) are all the same; denoting this component by \(t_i\), it is clear that the element of \(R(U)\) that has the \(t_i\) as its components has \(s_a\) as its restriction to each \(V_a\). Finally, to see that \(\mathcal{R}(X)\) is quasi-coherent, we can restrict to the case where \(X = \text{Spec}(A)\) is affine; by taking \(U\) to be an affine open subset of the form \(D(f)\), where \(f \in A\), it follows from the above and from Definition (1.3.4) that we have \(\mathcal{R}(X) = M\), where \(M\) is the direct sum of the \(A\)-modules \(A_{X_i}\).

**Corollary (7.3.4).** — Let \(X\) be a reduced prescheme that has only a finite number of irreducible components, and let \(X_i\) (1 ≤ \(i\) ≤ \(n\)) be the closed reduced preschemes of \(X\) that have the irreducible components of \(X\) as their underlying spaces (5.2.1). If \(h_i\) is the canonical injection \(X_i \rightarrow X\), then \(\mathcal{R}(X)\) is the direct sum of the \(\mathcal{O}_{X_i}\)-algebras \((h_i)_*(\mathcal{R}(X_i))\).

**Corollary (7.3.5).** — If \(X\) is irreducible, then every quasi-coherent \(\mathcal{R}(X)\)-module \(\mathcal{F}\) is a simple sheaf.

**Proof.** It suffices to show that every \(x \in X\) admits a neighbourhood \(U\) such that \(\mathcal{F}|\!|U\) is a simple sheaf (0, 3.6.2); in other words, we are led to considering the case where \(X\) is affine; we can
further suppose that \( \mathcal{F} \) is the cokernel of a homomorphism \((\mathcal{A}(X))^I \to (\mathcal{A}(X))^I (0, 5.1.3)\), and everything then follows from showing that \( \mathcal{A}(X) \) is a simple sheaf; but this is evident, because \( \Gamma(U, \mathcal{A}(X)) = R(X) \) for every nonempty open subset \( U \), where \( U \) contains the generic point of \( X \).

\( \square \)

**Corollary (7.3.6).** — If \( X \) is irreducible, then, for every quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \), \( \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}(X) \) is a simple sheaf; if, further, \( X \) is reduced (and thus integral), then \( \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}(X) \) is isomorphic to a sheaf of the form \((\mathcal{A}(X))U^{(1)}\).

**Proof.** The second claim follows from the fact that \( R(X) \) is a field. \( \square \)

**Proposition (7.3.7).** — Suppose that the prescheme \( X \) is locally integral or locally Noetherian. Then \( \mathcal{A}(X) \) is a quasi-coherent \( \mathcal{O}_X \)-algebra; if, further, \( X \) is reduced (which will be the case whenever \( X \) is locally integral), then the canonical homomorphism \( \mathcal{O}_X \to \mathcal{A}(X) \) is injective.

**Proof.** Since the questions is local, the first claim follows from (7.3.3); the second follows from (7.2.3). \( \square \)

(7.3.8).\(^{10}\) Let \( X \) and \( Y \) be two integral preschemes, which implies that \( \mathcal{A}(X) \) (resp. \( \mathcal{A}(Y) \)) is a quasi-coherent \( \mathcal{O}_X \)-module (resp. \( \mathcal{O}_Y \)-module) (7.3.3). Let \( f : X \to Y \) be a dominant morphism; then there exists a canonical homomorphism of \( \mathcal{O}_X \)-modules

\[ \tau : f^*(\mathcal{A}(Y)) \to \mathcal{A}(X). \]

**Proof.** Suppose first that \( X = \text{Spec}(A) \) and \( Y = \text{Spec}(B) \) are affine, given by integral rings \( A \) and \( B \), with \( f \) thus corresponding to an injective homomorphism \( B \to A \) which extends to a monomorphism \( L \to K \) from the field of fractions \( L \) of \( B \) to the field of fractions \( K \) of \( A \). The homomorphism (7.3.8.1) then corresponds to the canonical homomorphism \( L \otimes_B A \to K \) (1.6.5). In the general case, for each pair of nonempty affine open sets \( U \subset X \) and \( V \subset Y \) such that \( f(U) \subset V \), we define, as above, a homomorphism \( \tau_{U,V} \) and we immediately have that, if \( U' \subset U \), \( V' \subset V \), \( f(U') \subset V' \), then \( \tau_{U',V'} \) extends \( \tau_{U,V} \), and hence our assertion. If \( x \) and \( y \) are the generic points of \( X \) and \( Y \) respectively, then we have \( f(x) = y \),

\[ (f^*(\mathcal{A}(Y)))_x = \mathcal{O}_y \otimes_{\mathcal{O}_y} \mathcal{O}_x = \mathcal{O}_x \]

(0, 4.3.1) and \( \tau_\chi \) is thus an isomorphism. \( \square \)

7.4. Torsion sheaves and torsion-free sheaves

(7.4.1). Let \( X \) be an integral scheme. For every \( \mathcal{O}_X \)-module \( \mathcal{F} \), the canonical homomorphism \( \mathcal{O}_X \to \mathcal{A}(X) \) defines, by tensoring, a homomorphism (again said to be canonical) \( \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}(X) \), which, on each fibre, is exactly the homomorphism \( z \to z \otimes 1 \) from \( \mathcal{F}_x \) to \( \mathcal{F}_x \otimes_{\mathcal{O}_x} R(X) \). The kernel \( \mathcal{I} \) of this homomorphism is a \( \mathcal{O}_X \)-submodule of \( \mathcal{F} \), called the torsion sheaf of \( \mathcal{F} \); it is quasi-coherent if \( \mathcal{F} \) is quasi-coherent ((4.1.1) and (7.3.6)). We say that \( \mathcal{F} \) is torsion free if \( \mathcal{I} = 0 \), and that \( \mathcal{F} \) is a torsion sheaf if \( \mathcal{I} = \mathcal{F} \). For every \( \mathcal{O}_X \)-module \( \mathcal{F} \), \( \mathcal{F} / \mathcal{I} \) is torsion free. We deduce from (7.3.5) that:

**Proposition (7.4.2).** — If \( X \) is an integral prescheme, then every torsion-free quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \) is isomorphic to a subsheaf \( \mathcal{G} \) of a simple sheaf of the form \((\mathcal{A}(X))U^{(1)}\), generated (as a \( \mathcal{A}(X) \)-module) by \( \mathcal{G} \).

The cardinality of \( I \) is called the rank of \( \mathcal{F} \); for every nonempty affine open subset \( U \) of \( X \), the rank of \( \mathcal{F} \) is equal to the rank of \( \Gamma(U, \mathcal{F}) \) as a \( \Gamma(U, \mathcal{O}_X) \)-module, as we see by considering the generic point of \( X \), contained in \( U \). In particular:

**Corollary (7.4.3).** — On an integral prescheme \( X \), every torsion-free quasi-coherent \( \mathcal{O}_X \)-module of rank 1 (in particular, every invertible \( \mathcal{O}_X \)-module) is isomorphic to a \( \mathcal{O}_X \)-submodule of \( \mathcal{A}(X) \), and vice versa.

**Corollary (7.4.4).** — Let \( X \) be an integral prescheme, \( \mathcal{L} \) and \( \mathcal{L}' \) torsion-free \( \mathcal{O}_X \)-modules, and \( f \) (resp. \( f' \)) a section of \( \mathcal{L} \) (resp. \( \mathcal{L}' \)) over \( X \). In order to have \( f \otimes f' = 0 \), it is necessary and sufficient for one of the sections \( f \) and \( f' \) to be zero.

\(^{10}\) Trans. This paragraph was changed entirely in the Errata of EGA II.
8. CHEVALLEY SCHEMES

8.1. Allied local rings

For each local ring \( A \), we denote by \( m(A) \) the maximal ideal of \( A \).

Lemma (8.1.1). — Let \( A \) and \( B \) be two local rings such that \( A \subset B \); then the following conditions are equivalent.

(i) \( m(B) \cap A = m(A) \).
(ii) \( m(A) \subset m(B) \).
(iii) \( 1 \) is not an element of the ideal of \( B \) generated by \( m(A) \).

Proof. It is evident that (i) implies (ii), and (ii) implies (iii); lastly, if (iii) is true, then \( m(B) \cap A \) contains \( m(A) \), and does not contain 1, and is thus equal to \( m(A) \).

When the equivalent conditions of (8.1.1) are satisfied, we say that \( B \) dominates \( A \); this is equivalent to saying that the injection \( A \to B \) is a local homomorphism. It is clear that, in the set of local subrings of a ring \( R \), the relation given by domination is an order.

(8.1.2). Now consider a field \( R \). For all subrings \( A \) of \( R \), we denote by \( L(A) \) the set of local rings \( A_p \), where \( p \) ranges over the prime spectrum of \( A \); such local rings are identified with the subrings of \( R \) containing \( A \). Since \( p = (pA_p) \cap A \), the map \( p \to A_p \) from Spec(\( A \)) to \( L(A) \) is bijective.

Lemma (8.1.3). — Let \( R \) be a field, and \( A \) a subring of \( R \). For a local subring \( M \) of \( R \) to dominate a ring \( A_p \in L(A) \), it is necessary and sufficient that \( A \subset M \); the local ring \( A_p \) dominated by \( M \) is then unique, and corresponds to \( p = (m(M)) \cap A \).

Proof. If \( M \) dominates \( A_p \), then \( m(M) \cap A = pA_p \), by (8.1.1), whence the uniqueness of \( p \); on the other hand, if \( A \subset M \), then \( m(M) \cap A = p \) is prime in \( A \), and since \( A - p \subset M \), we have that \( A_p \subset M \) and \( pA_p \subset m(M) \), so \( M \) dominates \( A_p \).

Lemma (8.1.4). — Let \( R \) be a field, \( M \) and \( N \) local subrings of \( R \), and \( P \) the subring of \( R \) generated by \( M \cup N \). Then the following conditions are equivalent.

(i) There exists a prime ideal \( p \) of \( P \) such that \( m(M) = p \cap M \) and \( m(N) = p \cap N \).
(ii) The ideal \( a \) generated in \( P \) by \( m(M) \cup m(N) \) is distinct from \( p \).
(iii) There exists a local subring \( Q \) of \( R \) simultaneously dominating both \( M \) and \( N \).
Then the following conditions are equivalent.

Proof. It is clear that (i) implies (ii); conversely, if \( a \neq P \), then \( a \) is contained in a maximal ideal \( n \) of \( P \), and since \( 1 \notin n \), \( n \cap M \) contains \( m(M) \) and is distinct from \( M \), so \( n \cap M = m(M) \), and similarly \( n \cap N = m(N) \). It is clear that, if \( Q \) dominates both \( M \) and \( N \), then \( P \subset Q \) and \( m(M) = m(Q) \cap M = (m(Q) \cap P) \cap M \), and \( m(N) = (m(Q) \cap P) \cap N \), so (iii) implies (i); the converse is evident when we take \( Q = P_p \). □

When the conditions of (8.1.4) are satisfied, we say, with C. Chevalley, that the local rings \( M \) and \( N \) are allied.

Proposition (8.1.5). — Let \( A \) and \( B \) be subrings of a field \( R \), and \( C \) the subring of \( R \) generated by \( A \cup B \). Then the following conditions are equivalent.

(i) For every local ring \( Q \) containing \( A \) and \( B \), we have that \( A_p = B_q \), where \( p = m(Q) \cap A \) and \( q = m(Q) \cap B \).

(ii) For all prime ideals \( r \) of \( C \), we have that \( A_p = B_q \), where \( p = r \cap A \) and \( q = r \cap B \).

(iii) If \( M \in L(A) \) and \( N \in L(B) \) are allied, then they are identical.

(iv) \( L(A) \cap L(B) = L(C) \).

Proof. Lemmas (8.1.3) and (8.1.4) prove that (i) and (iii) are equivalent; it is clear that (i) implies (ii) by taking \( Q = C \); conversely, (ii) implies (i), because if \( Q \) contains \( A \cup B \) then it contains \( C \), and if \( r = m(Q) \cap C \), then \( p = r \cap A \) and \( q = r \cap B \), by (8.1.3). It is immediate that (iv) implies (i), because if \( Q \) contains \( A \cup B \) then it dominates a local ring \( C \in L(C) \) by (8.1.3); by hypothesis we have that \( C = L(A) \cap L(B) \), and (8.1.1) and (8.1.3) prove that \( C = A_p = B_q \). We prove finally that (iii) implies (iv). Let \( Q \in L(C) \); \( Q \) dominates some \( M \in L(A) \) and some \( N \in L(B) \) (8.1.3), so \( M \) and \( N \), being allied, are identical by hypothesis. As we then have that \( C \subset M \), we know that \( M \) dominates some \( Q' \in L(C) \) (8.1.3), so \( Q \) dominates \( Q' \), whence necessarily (8.1.3) \( Q = Q' = M \), so \( Q \in L(A) \cap L(B) \). Conversely, if \( Q \in L(A) \cap L(B) \), then \( C \subset Q \), so (8.1.3) \( Q \) dominates some \( Q'' \in L(C) \subset L(A) \cap L(B) \); \( Q \) and \( Q'' \), being allied, are identical, so \( Q'' = Q \in L(C) \), which completes the proof. □

8. Local rings of an integral scheme

(8.2.1). Let \( X \) be an integral prescheme, and \( R \) its field of rational functions, identical to the local ring of the generic point \( a \) of \( X \); for all \( x \in X \), we know that \( \mathcal{O}_x \) can be canonically identified with a subring of \( R \) (7.1.5), and for every rational function \( f \in R \), the domain of definition \( \delta(f) \) of \( f \) is the open set of \( x \in X \) such that \( f \in \mathcal{O}_x \). It thus follows, from (7.2.6), that, for every open \( U \subset X \), we have

\[
\Gamma(U, \mathcal{O}_X) = \bigcap_{x \in U} \mathcal{O}_x.
\]

Proposition (8.2.2). — Let \( X \) be an integral prescheme, and \( R \) its field of rational fractions. For \( X \) to be a scheme, it is necessary and sufficient for the relation “\( \mathcal{O}_x \) and \( \mathcal{O}_y \) are allied” (8.1.4), for points \( x \) and \( y \) of \( X \), to imply that \( x = y \).

Proof. We suppose that this condition is satisfied, and aim to show that \( X \) is separated. Let \( U \) and \( V \) be two distinct affine open subsets of \( X \), given by rings \( A \) and \( B \) (respectively), identified with subrings of \( R \); \( U \) (resp. \( V \)) is thus identified (8.1.2) with \( L(A) \) (resp. \( L(B) \)), and the hypotheses tell us (8.1.5) that \( C \) is the subring of \( R \) generated by \( A \cup B \), and \( W = U \cap V \) is identified with \( L(A) \cap L(B) = L(C) \). Furthermore, we know ([CC], p. 5-03, 4 bis) that every subring \( E \) of \( R \) is equal to the intersection of the local rings belonging to \( L(E) \); \( C \) is thus identified with the intersection of the rings \( \mathcal{O}_{z} \) for \( z \in W \), or, equivalently (8.2.1.1), with \( \Gamma(W, \mathcal{O}_X) \). So consider the subprescheme induced by \( X \) on \( W \); to the identity morphism \( \phi : C \to \Gamma(W, \mathcal{O}_X) \) there corresponds (2.2.4) a morphism \( \Phi = (\phi, \theta) : W \to \text{Spec}(C) \); we will see that \( \Phi \) is an isomorphism of preschemes, whence \( W \) is an affine open subset. The identification of \( W \) with \( L(C) = \text{Spec}(C) \) shows that \( \psi \) is bijective. On the other hand, for all \( x \in W \), \( \theta_x^1 \) is the injection \( C_r \to \mathcal{O}_x \), where \( r = m_x \cap C \), and, by definition, \( C_r \) is identified with \( \mathcal{O}_x \), so \( \theta_x^1 \) is bijective. It thus remains to show that \( \psi \) is a homeomorphism, or, in other words, that for every closed subset \( F \subset W \), \( \psi(F) \) is closed in \( \text{Spec}(C) \). But \( F \) is the intersection of \( W \) with a closed subspace of \( U \) of the form \( V(a) \), where \( a \) is an ideal of \( A \); we will show that \( \psi(F) = V(aC) \), which proves our claim. In fact, the prime ideals of \( C \) containing \( aC \) are the prime
Ideals of $C$ containing $a$, and so are the ideals of the form $\psi(x) = m_y \cap C$, where $a \subset m_x$ and $x \in W$; since $a \subset m_x$ is equivalent to $x \in V(a) = W \cap F$ for $x \in U$, we do indeed have that $\psi(F) = V(aC)$.

It follows that $X$ is separated, because $U \cap V$ is affine and its ring $C$ is generated by the union $A \cup B$ of the rings of $U$ and $V$ (5.5.6).

Conversely, suppose that $X$ is separated, and let $x$ and $y$ be points of $X$ such that $\mathcal{O}_x$ and $\mathcal{O}_y$ are allied. Let $U$ (resp. $V$) be an affine open subset containing $x$ (resp. $y$), of ring $A$ (resp. $B$); we then know that $U \cap V$ is affine and that its ring $C$ is generated by $A \cup B$ (5.5.6). If $p = m_x \cap A$ and $q = m_y \cap B$, then $A_p = \mathcal{O}_x$ and $B_q = \mathcal{O}_y$, and since $A_p$ and $B_q$ are allied, there exists a prime ideal $\pi$ of $C$ such that $p = \pi \cap A$ and $q = \pi \cap B$ (8.1.4). But then there exists a point $z \in U \cap V$ such that $r = m_z \cap C$, since $U \cap V$ is affine, and so evidently $x = z$ and $y = z$, whence $x = y$.

**Corollary (8.2.3).** — Let $X$ be an integral scheme, and $x$ and $y$ points of $X$. In order for $x \in \{y\}$, it is necessary and sufficient for $\mathcal{O}_x \subset \mathcal{O}_y$, or, equivalently, for every rational function defined at $x$ to also be defined at $y$.

**Proof.** The condition is evidently necessary because the domain of definition $\delta(f)$ of a rational function $f \in R$ is open; we now show that it is sufficient. If $\mathcal{O}_x \subset \mathcal{O}_y$, then there exists a prime ideal $p$ of $\mathcal{O}_x$ such that $\mathcal{O}_y$ dominates $(\mathcal{O}_x)_p$ (8.1.3); but (2.4.2) there exists some $z \in X$ such that $x \in \{z\}$ and $\mathcal{O}_z = (\mathcal{O}_x)_p$; since $\mathcal{O}_z$ and $\mathcal{O}_y$ are allied, we have that $z = y$ by (8.2.2), whence the corollary. □

**Corollary (8.2.4).** — If $X$ is an integral scheme then the map $x \rightarrow \mathcal{O}_x$ is injective; equivalently, if $x$ and $y$ are two distinct points of $X$, then there exists a rational function defined at one of these points but not the other.

**Proof.** This follows from (8.2.3) and the axiom $(T_0)$ (2.1.4). □

**Corollary (8.2.5).** — Let $X$ be an integral scheme whose underlying space is Noetherian; letting $f$ range over the field $R$ of rational functions on $X$, the sets $\delta(f)$ generate the topology of $X$.

In fact, every closed subset of $X$ is thus a finite union of irreducible closed subsets, or, in other words, of the form $\{y\}$ (2.1.5). But, if $x \not\in \{y\}$, then there exists a rational function $f$ defined at $x$ but not at $y$ (8.2.3), or, equivalently, we have that $x \in \delta(f)$ and that $\delta(f)$ is not contained in $\{y\}$. The complement of $\{y\}$ is thus a union of sets of the form $\delta(f)$, and, by virtue of the first remark, every open subset of $X$ is the union of finite intersections of open sets of the form $\delta(f)$.

(8.2.6). Corollary (8.2.5) shows that the topology of $X$ is entirely characterised by the data of the local rings $(\mathcal{O}_x)_{x \in X}$ that have $R$ as their field of fractions. It is equivalent to say that the closed subsets of $X$ are defined in the following manner: given a finite subset $\{x_1, \ldots, x_n\}$ of $X$, consider the set of $y \in X$ such that $\mathcal{O}_y \subset \mathcal{O}_{x_i}$ for at least one index $i$, and these sets (over all choices of $\{x_1, \ldots, x_n\}$) are the closed subsets of $X$. Further, once the topology on $X$ is known, the structure sheaf $\mathcal{O}_X$ is also determined by the family of the $\mathcal{O}_x$, since $\Gamma(U, \mathcal{O}_X) = \bigcap_{x \in U} \mathcal{O}_x$ by (2.1.1.1). The family $(\mathcal{O}_X)_{x \in X}$ thus completely determines the prescheme $X$ when $X$ is an integral scheme whose underlying space is Noetherian.

**Proposition (8.2.7).** — Let $X$ and $Y$ be integral schemes, $f : X \rightarrow Y$ a dominant morphism (2.2.6), and $K$ (resp. $L$) the field of rational functions on $X$ (resp. $Y$). Then $L$ can be identified with a subfield of $K$, and, for all $x \in X$, $\mathcal{O}_{f(x)}$ is the unique local ring of $Y$ dominated by $\mathcal{O}_x$.

**Proof.** If $f = (\psi, \theta)$ and $a$ is the generic point of $X$, then $\psi(a)$ is the generic point of $Y$ (0, 2.1.5); $\theta_a$ is then a monomorphism of fields, from $L = \mathcal{O}_{\psi(a)}$ to $K = \mathcal{O}_a$. Since every nonempty affine open subset $U$ of $Y$ contains $\psi(a)$, it follows from (2.2.4) that the homomorphism $\Gamma(U, \mathcal{O}_Y) \rightarrow \Gamma(\psi^{-1}(U), \mathcal{O}_X)$ corresponding to $f$ is the restriction of $\theta_a$ to $\Gamma(U, \mathcal{O}_Y)$. So, for every $x \in X$, $\theta_x^\sharp$ is the restriction to $\mathcal{O}_{\psi(x)}$ of $\theta_a$, and is thus a monomorphism. We also know that $\theta_a^\sharp$ is a local homomorphism, so, if we identify $L$ with a subfield of $K$ by $\theta_a^\sharp$, $\mathcal{O}_{\psi(x)}$ is dominated by $\mathcal{O}_x$ (8.1.1); it is also the only local ring of $Y$ dominated by $\mathcal{O}_x$, since two local rings of $Y$ that are allied are identical (8.2.2). □

**Proposition (8.2.8).** — Let $X$ be an irreducible prescheme, $f : X \rightarrow Y$ a local immersion (resp. local isomorphism), and suppose further that $f$ is separated. Then $f$ is an immersion (resp. an open immersion).
9. SUPPLEMENT ON QUASI-COHERENT SHEAVES

9.1. Tensor product of quasi-coherent sheaves

Proposition (9.1.1). — Let \( X \) be a prescheme (resp. a locally Noetherian prescheme). Let \( \mathcal{F} \) and \( \mathcal{G} \) be quasi-coherent (resp. coherent) \( \mathcal{O}_X \)-modules; then \( \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \) is quasi-coherent (resp. coherent); it is further of finite type if both \( \mathcal{F} \) and \( \mathcal{G} \) are of finite type. If \( \mathcal{F} \) admits a finite presentation and if \( \mathcal{G} \) is quasi-coherent (resp. coherent), then \( \mathcal{H}om(\mathcal{F}, \mathcal{G}) \) is quasi-coherent (resp. coherent).
Proof. Being a local proposition, we can suppose that $X$ is affine (resp. Noetherian affine); further, if $\mathscr{F}$ is coherent, then we can assume that it is the cokernel of a homomorphism $\mathcal{O}_X^n \to \mathcal{O}_X^n$. The claims pertaining to quasi-coherent sheaves then follow from Corollaries (1.3.12) and (1.3.9); the claims pertaining to coherent sheaves follow from Theorem (1.5.1) and from the fact that if $M$ and $N$ are modules of finite type over a Noetherian ring $A$ then $M \otimes_A N$ and $	ext{Hom}_A(M, N)$ are both $A$-modules of finite type. □

Definition (9.1.2). — Let $X$ and $Y$ be $S$-schemes, $p$ and $q$ the projections of $X \times_S Y$, and $\mathscr{F}$ (resp. $\mathscr{G}$) a quasi-coherent $\mathcal{O}_X$-module (resp. quasi-coherent $\mathcal{O}_Y$-module). We define the tensor product of $\mathscr{F}$ and $\mathscr{G}$ over $\mathcal{O}_S$ (or over $S$), denoted by $\mathscr{F} \otimes_{\mathcal{O}_S} \mathscr{G}$ (or $\mathscr{F} \otimes_S \mathscr{G}$) to be the tensor product $p^*(\mathscr{F}) \otimes q^*(\mathscr{G})$ over the prescheme $X \times_S Y$.

If $X_i$ $(1 \leq i \leq n)$ are $S$-schemes, and $\mathscr{F}_i$ $(1 \leq i \leq n)$ are quasi-coherent $\mathcal{O}_{X_i}$-modules, then we similarly define the tensor product $\mathscr{F}_1 \otimes_{\mathcal{O}_S} \mathscr{F}_2 \otimes_{\mathcal{O}_S} \cdots \otimes_{\mathcal{O}_S} \mathscr{F}_n$ over the prescheme $Z = X_1 \times_S X_2 \times_S \cdots \times_S X_n$; it is a quasi-coherent $\mathcal{O}_Z$-module by virtue of (9.1.1) and (0, 5.1.4); it is further coherent if all the $\mathscr{F}_i$ are coherent and $Z$ is locally Noetherian, by virtue of (9.1.1), (0, 5.3.11), and (6.1.1).

Note that, if we take $X = Y = S$, then Definition (9.1.2) gives us back the tensor product of $\mathcal{O}_S$-modules. Furthermore, since $q^*(\mathcal{O}_Y) = \mathcal{O}_{X \times_S Y}$ (0. 4.3.4), the product $\mathscr{F} \otimes_S \mathcal{O}_Y$ is canonically identified with $p^*(\mathscr{F})$, and, in the same way, $\mathcal{O}_X \otimes_S \mathscr{G}$ is canonically identified with $q^*(\mathscr{G})$. In particular, if we take $Y = S$ and denote by $f$ the structure morphism $X \to Y$, then we have that $\mathcal{O}_X \otimes_Y \mathscr{G} = f^*(\mathscr{G})$: the ordinary tensor product and the inverse image thus appear as particular cases of the general tensor product.

Definition (9.1.2) leads immediately to the fact that, for fixed $X$ and $Y$, $\mathscr{F} \otimes_S \mathscr{G}$ is a right-exact additive covariant bifunctor in $\mathscr{F}$ and $\mathscr{G}$.

Proposition (9.1.3). — Let $S$, $X$, and $Y$ be affine schemes of rings $A$, $B$, and $C$ (respectively), with $B$ and $C$ being $A$-algebras. Let $M$ (resp. $N$) be a $B$-module (resp. $C$-module), and $\mathscr{F} = \tilde{M}$ (resp. $\mathscr{G} = \tilde{N}$) the associated quasi-coherent sheaf; then $\mathscr{F} \otimes_S \mathscr{G}$ is canonically isomorphic to the sheaf associated to the $(B \otimes_A C)$-module $M \otimes_A N$.

Proof. According to Proposition (1.6.5), $\mathscr{F} \otimes_S \mathscr{G}$ is canonically isomorphic to the sheaf associated to the $(B \otimes_A C)$-module

\[
(M \otimes_B (B \otimes_A C)) \otimes_{B \otimes_A C} ((B \otimes_A C) \otimes_C N)
\]

and, by the canonical isomorphisms between tensor products, this latter module is isomorphic to

\[
M \otimes_B (B \otimes_A C) \otimes_C N = (M \otimes_B B) \otimes_A (C \otimes_C N) = M \otimes_A N.
\]

□

Proposition (9.1.4). — Let $f : T \to X$ and $g : T \to Y$ be $S$-morphisms, and $\mathscr{F}$ (resp. $\mathscr{G}$) a quasi-coherent $\mathcal{O}_X$-module (resp. quasi-coherent $\mathcal{O}_Y$-module). Then

\[
(f \circ g)^*(\mathscr{F} \otimes_S \mathscr{G}) = f^*(\mathscr{F}) \otimes_{q^*} g^*(\mathscr{G}).
\]

Proof. If $p$, $q$ are the projections of $X \times_S Y$, then the formula follows from the equalities $(f \circ g)^* = f^* \circ p^*$ and $(f \circ g)^* = q^*$ (0, 3.5.5), and the fact that the inverse image of a tensor product of algebraic sheaves is the tensor product of their inverse images (0, 4.3.3). □

Corollary (9.1.5). — Let $f : X \to X'$ and $g : Y \to Y'$ be $S$-morphisms, and $\mathscr{F}'$ (resp. $\mathscr{G}'$) a quasi-coherent $\mathcal{O}_{X'}$-module (resp. quasi-coherent $\mathcal{O}_{Y'}$-module). Then

\[
(f \circ g)^*(\mathscr{F}' \otimes_S \mathscr{G}') = f^*(\mathscr{F}') \otimes_{q^*} g^*(\mathscr{G}').
\]

Proof. This follows from (9.1.4) and the fact that $f \times_S g = (f \circ p \circ g \circ q)\vert_S$, where $p$ and $q$ are the projections of $X \times_S Y$. □

Corollary (9.1.6). — Let $X$, $Y$, and $Z$ be $S$-schemes, and $\mathscr{F}$ (resp. $\mathscr{G}$, $\mathscr{H}$) a quasi-coherent $\mathcal{O}_X$-module (resp. quasi-coherent $\mathcal{O}_Y$-module, quasi-coherent $\mathcal{O}_Z$-module); then the sheaf $\mathscr{F} \otimes_S \mathscr{G} \otimes_S \mathscr{H}$ is the inverse image of $(\mathscr{F} \otimes_S \mathscr{G}) \otimes_S \mathscr{H}$ by the canonical isomorphism from $X \times_S Y \times_S Z$ to $(X \times_S Y) \times_S Z$. □
Proof. This isomorphism is given by \((p_1, p_2)_S \times_p p_3\), where \(p_1, p_2,\) and \(p_3\) are the projections of \(X \times_S Y \times_S Z\).

Similarly, the inverse image of \(\mathcal{G} \otimes_S \mathcal{F}\) under the canonical isomorphism from \(X \times_S Y\) to \(Y \times_S X\) is \(\mathcal{F} \otimes_S \mathcal{G}\).

\[\square\]

**Corollary (9.1.7).** — If \(X\) is an \(S\)-prescheme, then every quasi-coherent \(\mathcal{O}_X\)-module \(\mathcal{F}\) is the inverse image of \(\mathcal{F} \otimes_S \mathcal{O}_S\) by the canonical isomorphism from \(X \times_S S\) to \(X \times_S S\) (3.3.3).

Proof. This isomorphism is \((1_X, \phi)_S\), where \(\phi\) is the structure morphism \(X \to S\), and the corollary follows from (9.1.4) and the fact that \(\phi^*(\mathcal{O}_S) = \mathcal{O}_X\).

\[\square\]

**Proposition (9.1.8).** Let \(S\) be an \(S\)-prescheme, \(\mathcal{F}\) a quasi-coherent \(\mathcal{O}_S\)-module, and \(\phi : S' \to S\) a morphism; we denote by \(\mathcal{F}(\phi)\) or \(\mathcal{F}(S')\) the quasi-coherent sheaf \(\mathcal{F} \otimes_S \mathcal{O}_{S'}\) over \(X \times_S S' = \mathcal{F}(\phi) = \mathcal{F}(S')\), so \(\mathcal{F}(S') = p^*(\mathcal{F})\), where \(p\) is the projection \(X(S') \to X\).

**Proposition (9.1.9).** — Let \(\psi : S'' \to S'\) be a morphism. For every quasi-coherent \(\mathcal{O}_X\)-module \(\mathcal{F}\) on the \(S\)-prescheme \(X\), \((\mathcal{F}(\phi))(\psi')\) is the inverse image of \(\mathcal{F}(\phi')(\psi')\) by the canonical isomorphism \((X(\phi))(\psi') \simeq X(\phi'(\psi'))\) (3.3.9).

Proof. This follows immediately from the definitions and from (3.3.9), and is written

\[(\mathcal{F} \otimes_S \mathcal{O}_{S'}) \otimes_{S'} \mathcal{O}_{S''} = \mathcal{F} \otimes_S \mathcal{O}_{S''}.\]

\[\square\]

**Proposition (9.1.10).** — Let \(Y\) be an \(S\)-prescheme, and \(f : X \to Y\) an \(S\)-morphism. For every quasi-coherent \(\mathcal{O}_Y\)-module \(\mathcal{G}\) and every morphism \(S' \to S\), we have that \((f(S'))^*(\mathcal{G}(S')) = (f^*(\mathcal{G}))(S')\).

Proof. This follows immediately from the commutativity of the diagram

\[
\begin{array}{ccc}
X(S') & \xrightarrow{f(S')} & Y(S') \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y.
\end{array}
\]

\[\square\]

**Corollary (9.1.11).** — Let \(X\) and \(Y\) be \(S\)-preschemes, and \(\mathcal{F}\) (resp. \(\mathcal{G}\)) a quasi-coherent \(\mathcal{O}_X\)-module (resp. quasi-coherent \(\mathcal{O}_Y\)-module). Then the inverse image of the sheaf \((\mathcal{F}(S')) \otimes_{(S')} (\mathcal{G}(S'))\) by the canonical isomorphism \((X(S')) \times_{(S')} (Y(S')) \simeq (X(S')) \times_S (Y(S'))\) (3.3.10) is equal to \((\mathcal{F} \otimes_S \mathcal{G}((S'))\).

Proof. If \(p\) and \(q\) are the projections of \(X \times_S Y\), then the isomorphism in question is nothing but \((p(S'), q(S'))\); the corollary then follows from Propositions (9.1.4) and (9.1.10).

\[\square\]

**Proposition (9.1.12).** — With the notation from Definition (9.1.2), let \(z\) be a point of \(X \times_S Y\), and let \(x = p(z)\), and \(y = q(z)\); the stalk \((\mathcal{F} \otimes_S \mathcal{G})_z\) is isomorphic to \((\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_z) \otimes_{\mathcal{O}_z} (\mathcal{G}_y \otimes_{\mathcal{O}_y} \mathcal{O}_z) = \mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_z \otimes_{\mathcal{O}_z} \mathcal{G}_y \otimes_{\mathcal{O}_y} \mathcal{O}_z\).

Proof. Since we can reduce to the affine case, the proposition follows from Equation (1.6.5.1).

\[\square\]

**Corollary (9.1.13).** — If \(\mathcal{F}\) and \(\mathcal{G}\) are of finite type, then

\[
\text{Supp}(\mathcal{F} \otimes_S \mathcal{G}) = p^{-1}(\text{Supp}(\mathcal{F})) \cap q^{-1}(\text{Supp}(\mathcal{G})).
\]

Proof. Since \(p^*(\mathcal{F})\) and \(q^*(\mathcal{G})\) are both of finite type over \(\mathcal{O}_{X \times_S Y}\), we reduce, by Proposition (9.1.12) and by (0, 1.7.5), to the case where \(\mathcal{G} = \mathcal{O}_Y\), that is, it remains to prove the following equation:

\[(9.1.13.1) \quad \text{Supp}(p^*(\mathcal{F})) = p^{-1}(\text{Supp}(\mathcal{F})).\]

The same reasoning as in (0, 1.7.5) leads us to prove that, for all \(z \in X \times_S Y\), we have \(\mathcal{O}_z/m_z \mathcal{O}_z \neq 0\) (with \(x = p(z)\)), which follows from the fact that the homomorphism \(\mathcal{O}_x \to \mathcal{O}_z\) is local, by hypothesis.

\[\square\]
We leave it to the reader to extend the results in this section to the more general case of an arbitrary (but finite) number of factors, instead of just two.

9.2. Direct image of a quasi-coherent sheaf

**Proposition (9.2.1).** — Let \( f : X \to Y \) be a morphism of preschemes. We suppose that there exists a cover \((Y_{\alpha})\) of \( Y \) by affine opens having the following property: every \( f^{-1}(Y_{\alpha}) \) admits a finite cover \((X_{\alpha_i})\) by affine opens that are contained in \( f^{-1}(Y_{\alpha}) \) and that are such that every intersection \( X_{\alpha_i} \cap X_{\alpha_j} \) is itself a finite union of affine opens. With these hypotheses, for every quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \), \( f_* (\mathcal{F}) \) is a quasi-coherent \( \mathcal{O}_Y \)-module.

**Proof.** Since this is a local condition on \( Y \), we can assume that \( Y \) is equal to one of the \( Y_{\alpha} \), and thus omit the indices \( \alpha \).

(a) First, suppose that the \( X_i \cap X_j \) are themselves affine opens. We set \( \mathcal{F}_i = \mathcal{F}|_{X_i} \) and \( \mathcal{F}_{ij} = \mathcal{F}|_{(X_i \cap X_j)} \), and let \( \mathcal{F}_i' \) and \( \mathcal{F}_{ij}' \) be the images of \( \mathcal{F}_i \) and \( \mathcal{F}_{ij} \) (respectively) by the restriction of \( f \) to \( X_i \) and to \( X_i \cap X_j \) (respectively); we know that the \( \mathcal{F}_i' \) and \( \mathcal{F}_{ij}' \) are quasi-coherent (1.6.3). Set \( \mathcal{G} = \bigoplus_i \mathcal{F}_i' \) and \( \mathcal{H} = \bigoplus_{ij} \mathcal{F}_{ij}' \); \( \mathcal{G} \) and \( \mathcal{H} \) are quasi-coherent \( \mathcal{O}_Y \)-modules; we will define a homomorphism \( u : \mathcal{G} \to \mathcal{H} \) such that \( f_*(\mathcal{F}) \) is the kernel of \( u \); it will follow from this that \( f_*(\mathcal{F}) \) is quasi-coherent (1.3.9). It suffices to define \( u \) as a homomorphism of presheaves; taking into account the definitions of \( \mathcal{G} \) and \( \mathcal{H} \), it thus suffices, for every open subset \( W \subset Y \), to define a homomorphism

\[
u_W : \bigoplus_i \Gamma(f^{-1}(W) \cap X_i, \mathcal{F}) \longrightarrow \bigoplus_{ij} \Gamma(f^{-1}(W) \cap X_i \cap X_j, \mathcal{F})
\]

that satisfies the usual compatibility conditions when we let \( W \) vary. If, for every section \( s_i \in \Gamma(f^{-1}(W) \cap X_i, \mathcal{F}) \), we denote by \( s_{ij} \) its restriction to \( f^{-1}(W) \cap X_i \cap X_j \), then we set

\[
u_W((s_i)) = (s_{ij} - s_{ji})
\]

and the compatibility conditions are clearly satisfied. To prove that the kernel \( \mathcal{R} \) of \( u \) is \( f_*(\mathcal{F}) \), we define a homomorphism from \( f_*(\mathcal{F}) \) to \( \mathcal{R} \) by sending each section \( s \in \Gamma(f^{-1}(W), \mathcal{F}) \) to the family \( (s_i) \), where \( s_i \) is the restriction of \( s \) to \( f^{-1}(W) \cap X_i \); axioms (F1) and (F2) of sheaves (G, II, 1.1) tell us that this homomorphism is bijective, which finishes the proof in this case.

(b) In the general case, the same reasoning applies once we have established that the \( \mathcal{F}_{ij} \) are quasi-coherent. But, by hypothesis, \( X_i \cap X_j \) is a finite union of affine opens \( X_{\alpha_i} \cap X_{\alpha_j} \); and since the \( X_{\alpha_i} \cap X_{\alpha_j} \) are affine opens in a scheme, the intersection of any two of them is again an affine open (5.5.6). We are thus led to the first case, and so we have proved Proposition (9.2.1).

\[\square\]

**Corollary (9.2.2).** — The conclusion of Proposition (9.2.1) holds true in each of the following cases:

(a) \( f \) is separated and quasi-compact;
(b) \( f \) is separated and of finite type;
(c) \( f \) is quasi-compact, and the underlying space of \( X \) is locally Noetherian.

**Proof.** In case (a), the \( X_{\alpha_i} \cap X_{\alpha_j} \) are affine (5.5.6). Case (b) is a particular example of case (a) (6.6.3). Finally, in case (c), we can reduce to the case where \( Y \) is affine and the underlying space of \( X \) is Noetherian; then \( X \) admits a finite cover of affine opens \( (X_i) \), and the \( X_i \cap X_j \) being quasi-compact, are finite unions of affine opens (2.1.3).

\[\square\]
9.3. Extension of sections of quasi-coherent sheaves

**Theorem (9.3.1).** — Let $X$ be a prescheme whose underlying space is Noetherian, or a scheme whose underlying space is quasi-compact. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, $f$ a section of $\mathcal{L}$ over $X$, and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module.

(i) If $s \in \Gamma(X, \mathcal{F})$ is such that $s|_{X_f} = 0$, then there exists an integer $n > 0$ such that $s \otimes f^{\otimes n} = 0$.

(ii) For every section $s \in \Gamma(X, \mathcal{F})$, there exists an integer $n > 0$ such that $s \otimes f^{\otimes n}$ extends to a section of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ over $X$.

**Proof.**

(i) Since the underlying space of $X$ is quasi-compact, and thus the union of finitely-many affine opens $U_i$ with $\mathcal{L}|_{U_i}$ isomorphic to $\mathcal{O}_X|_{U_i}$, we can reduce to the case where $X$ is affine and $\mathcal{L} = \mathcal{O}_X$. In this case, $f$ can be identified with an element of $A(X)$, and we have that $X_f = D(f)$; $s$ can be identified with an element of an $A(X)$-module $M$, and $s|_{X_f}$ to the corresponding element of $M_f$, and the result is then trivial, recalling the definition of a module of fractions.

(ii) Again, $X$ is a finite union of affine opens $U_i$ $(1 \leq i \leq r)$ such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_X|_{U_i}$, and for every $i$, $(s \otimes f^{\otimes n})|_{U_i \cap X_f}$ can be identified (by the aforementioned isomorphism) with $(f(U_i \cap X_f))^{\otimes n}(s(U_i \cap X_f))$. We then know (1.4.1) that there exists an integer $n > 0$ such that, for all $i$, $(s \otimes f^{\otimes n})|_{U_i \cap X_f}$ extends to a section $s_i$ of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ over $U_i$. Let $s_{ij}$ be the restriction of $s_i$ to $U_i \cap U_j$; by definition we have that $s_{ij} - s_{ji} = 0$ on $X_f \cap U_i \cap U_j$. But, if $X$ is a Noetherian space, then $U_i \cap U_j$ is quasi-compact; if $X$ is a scheme, then $U_i \cap U_j$ is an affine open (5.5.6), and so again quasi-compact. By virtue of (i), there thus exists an integer $m$ (independent of $i$ and $j$) such that $(s_{ij} - s_{ji}) \otimes f^{\otimes m} = 0$. It immediately follows that there exists a section $s'$ of $\mathcal{F} \otimes \mathcal{L}^{\otimes (n+m)}$ over $X$ that restricts to $s_i \otimes f^{\otimes m}$ over each $U_i$, and restricts to $s \otimes f^{\otimes (n+m)}$ over $X_f$.

The following corollaries give an interpretation of Theorem (9.3.1) in a more algebraic language:

**Corollary (9.3.2).** — With the hypotheses of (9.3.1), consider the graded ring $A_\ast = \Gamma_\ast(\mathcal{L})$ and the graded $A_\ast$-module $M_\ast = \Gamma_\ast(\mathcal{L}, \mathcal{F})$ (0, 5.4.6). If $f \in A_\ast$, where $n \in \mathbb{Z}$, then there is a canonical isomorphism $\Gamma(X_f, \mathcal{F}) \cong ((M_\ast)_f)_0$ (the subgroup of the module of fractions $(M_\ast)_f$ consisting of elements of degree 0).

**Corollary (9.3.3).** — Suppose that the hypotheses of (9.3.1) are satisfied, and suppose further that $\mathcal{L} = \mathcal{O}_X$. Then, setting $A = \Gamma(X, \mathcal{O}_X)$ and $M = \Gamma(X, \mathcal{F})$, the $A_\ast$-module $\Gamma(X_f, \mathcal{F})$ is canonically isomorphic to $M_f$.

**Proposition (9.3.4).** — Let $X$ be a Noetherian prescheme, $\mathcal{F}$ a coherent $\mathcal{O}_X$-module, and $\mathcal{J}$ a coherent sheaf of ideals in $\mathcal{O}_X$, such that the support of $\mathcal{F}$ is contained in that of $\mathcal{O}_X|_{\mathcal{J}}$. Then there exists an integer $n > 0$ such that $\mathcal{F}^n \mathcal{J} = 0$.

**Proof.** Since $X$ is a union of finitely-many affine opens whose rings are Noetherian, we can suppose that $X$ is affine, given by some Noetherian ring $A$; then $\mathcal{F} = \mathcal{M}$, where $M = \Gamma(X, \mathcal{F})$ is an $A$-module of finite type, and $\mathcal{J} = \mathcal{J}$, where $\mathcal{J} = \Gamma(X, \mathcal{J})$ is an ideal of $A$ ((1.4.1) and (1.5.1)). Since $A$ is Noetherian, $\mathcal{J}$ admits a finite system of generators $f_i$ $(1 \leq i \leq m)$. By hypothesis, every section of $\mathcal{F}$ over $X$ is zero on each of the $D(f_j)$; if $s_j (1 \leq j \leq q)$ are sections of $\mathcal{F}$ that generate $M$, then there exists an integer $h_i$ independent of $i$ and $j$, such that $f_i^h s_j = 0$ (1.4.1), whence $f_i^h s_j = 0$ for all $s \in M$. We thus conclude that, if $n = mh$, then $\mathcal{J}^n M = 0$, and so the corresponding $\mathcal{O}_X$-module $\mathcal{F}^n \mathcal{J} = \mathcal{J}^n M (1.3.13)$ is zero.

**Corollary (9.3.5).** — With the hypotheses of (9.3.4), there exists a closed subscheme $Y$ of $X$, whose underlying space is the support of $\mathcal{O}_X|_{\mathcal{J}}$, such that, if $j : Y \to X$ is the canonical injection, then $\mathcal{F} = j_\ast (j^\ast(\mathcal{F}))$. 

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9. Supplementary On Quasi-Coherent Sheaves

(9.4.1). Let $X$ be a topological space, $\mathcal{F}$ a sheaf of sets (resp. of groups, of rings) on $X$, $U$ an open subset of $X$, $\psi : U \to X$ the canonical injection, and $\mathcal{I}$ a subsheaf of $\mathcal{F}|U = \psi^*(\mathcal{F})$. Since $\psi_*$ is left exact, $\psi_*(\mathcal{I})$ is a subsheaf of $\psi_*\psi^*(\mathcal{F})$; we denote by $\rho$ the canonical homomorphism $\mathcal{F} \to \psi_*\psi^*(\mathcal{F})$ (0, 3.5.3), and we denote by $\mathcal{H}$ the subsheaf $\rho^{-1}(\psi_*\psi^*(\mathcal{I}))$ of $\mathcal{F}$. It follows immediately from the definitions that, for every open subset $V$ of $X$, $\Gamma(V, \mathcal{F})$ consists of sections $s \in \Gamma(V, \mathcal{F})$ whose restriction to $V \cap U$ is a section of $\mathcal{I}$ over $V \cap U$. We thus have that $\mathcal{H}|U = \psi^*(\mathcal{H}) = \mathcal{I}$, and that $\mathcal{H}$ is the largest subsheaf of $\mathcal{F}$ that restricts to $\mathcal{I}$ over $U$; we say that $\mathcal{H}$ is the canonical extension of the subsheaf $\mathcal{I}$ of $\mathcal{F}|U$ to a subsheaf of $\mathcal{F}$.

Proposition (9.4.2). — Let $X$ be a prescheme, and $U$ an open subset of $X$ such that the canonical injection $j : U \to X$ is a quasi-compact morphism (which will be the case for all $U$ if the underlying space of $X$ is locally Noetherian (6.6.4, i)). Then:

(i) for every quasi-coherent $(\mathcal{O}_X|U)$-module $\mathcal{I}$, $j_*(\mathcal{I})$ is a quasi-coherent $\mathcal{O}_X$-module, and $j_*(\mathcal{I})|U = j^*(j_*(\mathcal{I})) = \mathcal{I}$;
(ii) for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and every quasi-coherent $(\mathcal{O}_X|U)$-submodule $\mathcal{I}$, the canonical extension $\mathcal{H}$ of $\mathcal{I}$ (9.4.1) is a quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{F}$.

Proof. If $j = (\psi, \theta)$ ($\psi$ being the injection $U \to X$ of underlying spaces), then, by definition, we have that $j_*(\mathcal{I}) = \psi_*(\mathcal{I})$ for every $(\mathcal{O}_X|U)$-module $\mathcal{I}$, and, further, that $j^*(\mathcal{H}) = \psi^*(\mathcal{H}) = \mathcal{H}|U$ for every $\mathcal{O}_X$-module $\mathcal{H}$, by definition of the prescheme induced over an open subset. So (i) is thus a particular case of (9.2.2, a); for the same reason, $j_*(j^*(\mathcal{F}))$ is quasi-coherent, and since $\mathcal{H}$ is the inverse image of $j_*(\mathcal{I})$ by the homomorphism $\rho : \mathcal{F} \to j_*(j^*(\mathcal{F}))$, (ii) follows from (4.1.1).

Note that the hypothesis that the morphism $j : U \to X$ is quasi-compact holds whenever the open subset $U$ is quasi-compact and $X$ is a scheme: indeed, $U$ is then a union of finitely-many affine opens $U_i$, and, for every affine open $V$ of $X$, $V \cap U_i$ is an affine open (5.5.6), and thus quasi-compact.

Corollary (9.4.3). — Let $X$ be a prescheme, and $U$ a quasi-compact open subset of $X$ such that the injection morphism $j : U \to X$ is quasi-compact. Suppose as well that every quasi-coherent $\mathcal{O}_X$-module is the inductive limit of its quasi-coherent $\mathcal{O}_X$-submodules of finite type (which will be the case if $X$ is an affine scheme). Then let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module, and $\mathcal{I}$ a quasi-coherent $(\mathcal{O}_X|U)$-submodule of $\mathcal{F}|U$ of finite type. Then there exists a quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{H}$ of $\mathcal{F}$ of finite type such that $\mathcal{H}|U = \mathcal{I}$.

Proof. We have $\mathcal{I} = \mathcal{H}|U$, and $\mathcal{H}$ is quasi-coherent, from (9.4.2), so the inductive limit of its quasi-coherent $\mathcal{O}_X$-submodules $\mathcal{H}_n$ of finite type. It follows that $\mathcal{H}$ is the inductive limit of the $\mathcal{H}_n|U$, and thus equal to one of the $\mathcal{H}_n|U$, since it is of finite type (0, 5.2.3).

Remark (9.4.4). — Suppose that for every affine open $U \subset X$, the injection morphism $U \to X$ is quasi-compact. Then, if the conclusion of (9.4.3) holds for every affine open $U$ and for every quasi-coherent $(\mathcal{O}_X|U)$-submodule $\mathcal{I}$ of $\mathcal{F}|U$ of finite type, it follows that $\mathcal{F}$ is the inductive limit of its quasi-coherent $\mathcal{O}_X$-submodules of finite type. Indeed, for every affine open $U \subset X$, we have that $\mathcal{F}|U = \mathcal{M}$, where $\mathcal{M}$ is an $A(U)$-module, and since the latter is the inductive limit of its quasi-coherent submodules of finite type, $\mathcal{F}|U$ is the inductive limit of its $(\mathcal{O}_X|U)$-submodules of finite type (1.3.9). But, by hypothesis, each of these submodules is induced on $U$ by a quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{G}_{\mathcal{M},U}$ of $\mathcal{F}$ of finite type. The finite sums of the $\mathcal{G}_{\mathcal{M},U}$ are again quasi-coherent $\mathcal{O}_X$-modules of finite type, because this is a local property, and the case where $X$ is affine was covered in (1.3.10); it is clear then that $\mathcal{F}$ is the inductive limit of these finite sums, whence our claim.

Corollary (9.4.5). — Under the hypotheses of Corollary (9.4.3), for every quasi-coherent $(\mathcal{O}_X|U)$-module $\mathcal{I}$ of finite type, there exists a quasi-coherent $\mathcal{O}_X$-module $\mathcal{H}$ of finite type such that $\mathcal{H}|U = \mathcal{I}$.

Proof. Since $\mathcal{F} = j_*(\mathcal{I})$ is quasi-coherent (9.4.2) and $\mathcal{F}|U = \mathcal{I}$, it suffices to apply Corollary (9.4.3) to $\mathcal{F}$. 

□
Lemma (9.4.6). — Let $X$ be a prescheme, $L$ a well-ordered set, $(V_{\lambda})_{\lambda \in L}$ a cover of $X$ by affine opens, and $U$ an open subset of $X$; for all $\lambda \in L$, we set $W_{\lambda} = \bigcup_{\mu < \lambda} V_{\mu}$. Suppose that: (1) for every $\lambda \in L$, $V_{\lambda} \cap W_{\lambda}$ is quasi-compact; and (2) the immersion morphism $U \to X$ is quasi-compact. Then, for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and every quasi-coherent $(\mathcal{O}_X|U)$-submodule $\mathcal{G}$ of $\mathcal{F}|U$ of finite type, there exists a quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{G}'$ of $\mathcal{F}$ of finite type such that $\mathcal{G}'|U = \mathcal{G}$.

Proof. Let $U_{\lambda} = U \cup W_{\lambda}$, we will define a family $(\mathcal{G}'_{\lambda})$ by induction, where $\mathcal{G}'_{\lambda}$ is a quasi-coherent $(\mathcal{O}_X|U_{\lambda})$-submodule of $\mathcal{F}|U_{\lambda}$ of finite type, such that $\mathcal{G}'_{\lambda}|U_{\mu} = \mathcal{G}'_{\mu}$ for $\mu < \lambda$ and $\mathcal{G}'_{\lambda}|U = \mathcal{G}$. The unique $\mathcal{O}_X$-submodule $\mathcal{G}'$ of $\mathcal{F}$ such that $\mathcal{G}'|U_{\lambda} = \mathcal{G}'_{\lambda}$ for all $\lambda \in L$ (0, 3.3.1) will then give us what we want. So suppose that the $\mathcal{G}'_{\lambda}$ are defined and have the preceding properties for $\mu < \lambda$; if $\lambda$ does not have a predecessor then we take $\mathcal{G}'_{\lambda}$ to be the unique $(\mathcal{O}_X|U_{\lambda})$-submodule of $\mathcal{F}|U_{\lambda}$ such that $\mathcal{G}'_{\lambda}|U_{\mu} = \mathcal{G}'_{\mu}$ for all $\mu < \lambda$, which is allowed since the $U_{\mu}$ with $\mu < \lambda$ then form a cover of $U_{\lambda}$. If, conversely, $\lambda = \mu + 1$, then $U_{\lambda} = U_{\mu} \cup V_{\mu}$, and it suffices to define a quasi-coherent $((\mathcal{O}_X|V_{\mu})$-submodule $\mathcal{G}''_{\mu}$ of $\mathcal{F}|V_{\mu}$ of finite type such that

\[
\mathcal{G}''_{\mu}(U_{\mu} \cap V_{\mu}) = \mathcal{G}'_{\mu}(U_{\mu} \cap V_{\mu});
\]

and then to take $\mathcal{G}'_{\lambda}$ to be the $(\mathcal{O}_X|U_{\lambda})$-submodule of $\mathcal{F}|U_{\lambda}$ such that $\mathcal{G}'_{\lambda}|U_{\mu} = \mathcal{G}'_{\mu}$ and $\mathcal{G}'_{\lambda}|V_{\mu} = \mathcal{G}''_{\mu}$ (0, 3.3.1). But, since $V_{\mu}$ is affine, the existence of $\mathcal{G}''_{\mu}$ is guaranteed by (9.4.3) as soon as we show that $U_{\mu} \cap V_{\mu}$ is quasi-compact; but $U_{\mu} \cap V_{\mu}$ is the union of $U \cap V_{\mu}$ and $W_{\mu} \cap V_{\mu}$, which are both quasi-compact by virtue of the hypotheses. \[\square\]

Theorem (9.4.7). — Let $X$ be a prescheme, and $U$ an open subset of $X$. Suppose that one of the following conditions is verified:

(a) the underlying space of $X$ is locally Noetherian;
(b) $X$ is a quasi-compact scheme and $U$ is a quasi-compact open.

Then, for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and every quasi-coherent $(\mathcal{O}_X|U)$-submodule $\mathcal{G}$ of $\mathcal{F}|U$ of finite type, there exists a quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{G}'$ of $\mathcal{F}$ of finite type such that $\mathcal{G}'|U = \mathcal{G}$.

Proof. Let $(V_{\lambda})_{\lambda \in L}$ be a cover of $X$ by affine opens, with $L$ assumed to be finite in case (b); since $L$ is equipped with the structure of a well-ordered set, it suffices to check that the conditions of (9.4.6) are satisfied. It is clear in the case of (a), since the spaces $V_{\lambda}$ are Noetherian. For case (b), the $V_{\lambda} \cap W_{\lambda}$ are affine (5.5.6), and thus quasi-compact, and, since $L$ is finite, $V_{\lambda} \cap W_{\lambda}$ is quasi-compact. Whence the theorem. \[\square\]

Corollary (9.4.8). — Under the hypotheses of (9.4.7), for every quasi-coherent $(\mathcal{O}_X|U)$-module $\mathcal{G}$ of finite type, there exists a quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}'$ of finite type such that $\mathcal{G}'|U = \mathcal{G}$.

Proof. It suffices to apply (9.4.7) to $\mathcal{F} = j_*(\mathcal{G})$, which is quasi-coherent (9.4.2) and such that $\mathcal{F}|U = \mathcal{G}$. \[\square\]

Corollary (9.4.9). — Let $X$ be a prescheme whose underlying space is locally Noetherian, or a quasi-compact scheme. Then every quasi-coherent $\mathcal{O}_X$-module is the inductive limit of its quasi-coherent $\mathcal{O}_X$-submodules of finite type.

Proof. This follows from Theorem (9.4.7) and Remark (9.4.4). \[\square\]

Corollary (9.4.10). — Under the hypotheses of (9.4.9), if a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is such that every quasi-coherent $\mathcal{O}_X$-submodule of finite type of $\mathcal{F}$ is generated by its sections over $X$, then $\mathcal{F}$ is generated by its sections over $X$.

Proof. Let $U$ be an affine open neighborhood of a point $x \in X$, and let $s$ be a section of $\mathcal{F}$ over $U$; the $\mathcal{O}_X$-submodule $\mathcal{G}$ of $\mathcal{F}|U$ generated by $s$ is quasi-coherent and of finite type, so there exists a quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{G}'$ of $\mathcal{F}$ of finite type such that $\mathcal{G}'|U = \mathcal{G}$ (9.4.7). By hypothesis, there thus exists a finite number of sections $t_i$ of $\mathcal{G}'$ over $X$ and of sections $a_i$ of $\mathcal{O}_X$ over a neighborhood $V \subset U$ of $x$ such that $s|V = \sum a_i(t_i|V)$, which proves the corollary. \[\square\]
9.5. Closed image of a prescheme; closure of a subscheme

Proposition (9.5.1). — Let \( f : X \to Y \) be a morphism of preschemes such that \( f_* (\mathcal{O}_X) \) is a quasi-coherent \( \mathcal{O}_Y \)-module (which will be the case if \( f \) is quasi-compact and, in addition, either \( f \) is separated or \( X \) is locally Noetherian (9.2.2)). Then there exists a smaller subscheme \( Y' \) of \( Y \) such that \( f \) factors through the canonical injection \( j : Y' \to Y \) (or, equivalently (4.4.1), such that the subscheme \( f^{-1}(Y') \) of \( X \) is identical to \( X \)).

More precisely:

Corollary (9.5.2). — Under the conditions of (9.5.1), let \( f = (\psi, \theta) \), and let \( \mathcal{J} \) be the (quasi-coherent) kernel of the homomorphism \( \theta : \mathcal{O}_Y \to f_* (\mathcal{O}_X) \). Then the closed subscheme \( Y' \) of \( Y \) defined by \( \mathcal{J} \) satisfies the conditions of (9.5.1).

Proof. Since the functor \( \psi^* \) is exact, the canonical factorization \( \theta : \mathcal{O}_Y \to \mathcal{O}_X / \mathcal{J} \to \psi_* (\mathcal{O}_X) \)
gives (0, 3.5.4.3) a factorization \( \theta^\flat : \psi^*(\mathcal{O}_Y) \to \psi^*(\mathcal{O}_X) / \psi^*(\mathcal{J}) \to \mathcal{O}_X ; \) since \( \theta^\flat \) is a local homomorphism for every \( x \in X \), the same is true of \( \theta^\flat_x \); if we denote by \( \psi_0 \) the continuous map \( \psi \) considered as a map from \( X \) to \( Y' \), and by \( \theta_0 \) the restriction \( \theta^\flat | X : (\mathcal{O}_X / \mathcal{J})(Y) \to \psi_* (\mathcal{O}_X;j)^{\flat} = (\psi_0)_*(\mathcal{O}_X) \),
then we see that \( f = (\psi_0, \theta_0) \) is a morphism of preschemes \( X \to Y' \) (2.2.1) such that \( f = f \circ f_0 \). Now, if \( Y'' \) is a second closed subscheme of \( Y \), defined by a quasi-coherent sheaf of ideals \( \mathcal{J}' \)
of \( \mathcal{O}_Y \), such that \( f \) factors through the injection \( j' : Y'' \to Y \), then we should immediately have that \( \psi' (X) \subseteq Y'' \), and thus that \( \psi' \subseteq Y'' \), since \( Y'' \) is closed. Furthermore, for all \( y \in Y'' \), \( \mathcal{J} \) should factor as \( \mathcal{O}_y \to \mathcal{O}_y / \mathcal{J}_y \to (\psi_* (\mathcal{O}_X))_y \), which, by definition, leads to \( \mathcal{J}_y \subseteq \mathcal{J}_y \), and thus \( X' \) is a closed subscheme of \( Y'' \) (4.1.10).

Definition (9.5.3). — Whenever there exists a smaller subscheme \( Y' \) of \( Y \) such that \( f \) factors through the canonical injection \( j : Y' \to Y \), we say that \( Y' \) is the closed image prescheme of \( X \) under the morphism \( f \).

Proposition (9.5.4). — If \( f_* (\mathcal{O}_X) \) is a quasi-coherent \( \mathcal{O}_Y \)-module, then the underlying space of the closed image of \( X \) under \( f \) is the closure \( \overline{f(X)} \) in \( Y \).

Proof. As the support of \( f_* (\mathcal{O}_X) \) is contained in \( \overline{f(X)} \), we have (with the notation of (9.5.2)) \( \mathcal{J}_y = \mathcal{O}_y \) for \( y \notin f(X) \), thus the support of \( \mathcal{O}_Y / \mathcal{J} \) is contained in \( f(X) \). In addition, this support is closed and contains \( f(X) \); indeed, if \( y \in f(X) \), the unit element of the ring \( (\psi_* (\mathcal{O}_X))_y \) is not zero, being the germ at \( y \) of the section

\[
1 \in \Gamma(X, \mathcal{O}_X) = \Gamma(Y, \psi_* (\mathcal{O}_X));
\]

since it is the image under \( \theta \) of the unit element of \( \mathcal{O}_y \), the latter does not belong to \( \mathcal{J}_y \), hence \( \mathcal{O}_y / \mathcal{J}_y \neq 0 \); this finishes the proof.

Proposition (9.5.5). — (Transitivity of closed images). Let \( f : X \to Y \) and \( g : Y \to Z \) be two morphisms of preschemes; we suppose that the closed image \( Y' \) of \( X \) under \( f \) exists, and that, if \( g' \) is the restriction of \( g \) to \( Y' \), then the closed image \( Z' \) of \( Y' \) under \( g' \) exists. Then the closed image of \( X \) under \( g \circ f \) exists and is equal to \( Z' \).

Proof. It suffices (9.5.1) to show that \( Z' \) is the smallest closed subscheme \( Z_1 \) of \( Z \) such that the closed subscheme \( (g \circ f)^{-1}(Z_1) \) of \( X \) (equal to \( f^{-1}(g^{-1}(Z_1)) \) by Corollary (4.4.2)) is equal to \( X \); it is equivalent to say that \( Z' \) is the smallest closed subscheme of \( Z \) such that \( f \) factors through the injection \( g^{-1}(Z_1) \to Y \) (4.4.1). By virtue of the existence of the closed image \( Y' \), every \( Z_1 \) with this property is such that \( g^{-1}(Z_1) \) factors through \( Y' \), which is equivalent to saying that \( j^{-1}(g^{-1}(Z_1)) = g'^{-1}(Z_1) = Y' \), denoting by \( j \) the injection \( Y' \to Y \). By the definition of \( Z' \), we indeed conclude that \( Z' \) is the smallest closed subscheme of \( Z \) satisfying the preceding condition.

Corollary (9.5.6). — Let \( f : X \to Y \) be an S-morphism such that \( Y \) is the closed image of \( X \) under \( f \). Let \( Z \) be an S-scheme; if two S-morphisms \( g_1, g_2 \) from \( Y \) to \( Z \) are such that \( g_1 \circ f = g_2 \circ f \), then \( g_1 = g_2 \).
Proof. Let \( h = (g_1, g_2) : Y \to Z \times Z \); since the diagonal \( T = \Delta_Z(Z) \) is a closed subprescheme of \( Z \times Z \), \( Y' = h^{-1}(T) \) is a closed subprescheme of \( Y \) \((4.4.1)\). Let \( u = g_1 \circ f = g_2 \circ f \); we then have, by definition of the product, \( h' = h \circ f = (u, u) \circ f \); \( h \circ f = \Delta_Z \circ u \) since \( \Delta_Z^{-1}(T) = Z \), we have \( h^{-1}(T) = u^{-1}(Z) = X \), so \( f^{-1}(Y') = X \). From this, we conclude \((4.4.1)\) that the \( f \) factors through the canonical injection \( Y' \to Y \), so \( Y' = Y \) by hypothesis; it then follows \((4.4.1)\) that \( h \) factors as \( \Delta_Z \circ v \), where \( v \) is a morphism \( Y \to Z \), which implies that \( g_1 = g_2 = v \).

Remark (9.5.7). — If \( X \) and \( Y \) are \( S \)-schemes, Proposition (9.5.6) implies that, when \( Y \) is the closed image of \( X \) under \( f \), \( f \) is an epimorphism in the category of \( S \)-schemes \((T, 1.1)\). We will show in Chapter V that, conversely, if the closed image \( Y' \) of \( X \) under \( f \) exists and if \( f \) is an epimorphism of \( S \)-schemes, then we necessarily have that \( Y' = Y \).

Proposition (9.5.8). — Suppose that the hypotheses of \((9.5.1)\) are satisfied, and let \( Y' \) be the closed image of \( X \) under \( f \). For every open \( V \) of \( Y \), let \( f_V : f^{-1}(V) \to V \) be the restriction of \( f \); then the closed image of \( f^{-1}(V) \) under \( f_V \) in \( V \) exists and is equal to the prescheme induced by \( Y' \) on the open \( V \cap Y' \) of \( Y' \) (in other words, to the subprescheme \( \text{inf}(V, Y) \) of \( Y \) \((4.4.3)\)).

Proof. Let \( X' = f^{-1}(V) \); since the direct image of \( \mathcal{O}_{X'} \) by \( f_V \) is exactly the restriction of \( f_* (\mathcal{O}_X) \) to \( V \), it is clear that the kernel \( \mathcal{J}' \) of the homomorphism \( \mathcal{O}_V \to (f_V)_* (\mathcal{O}_{X'}) \) is the restriction of \( \mathcal{J} \) to \( V \), from which the proposition quickly follows.

We will see that this result can be understood as saying that taking the closed image commutes with an extension \( Y_1 \to Y \) of the base prescheme, which is an open immersion. We will see in Chapter IV that it is the same for an extension \( Y_1 \to Y \) which is a flat morphism, provided that \( f \) is separated and quasi-compact.

Proposition (9.5.9). — Let \( f : X \to Y \) be a morphism such that the closed image \( Y' \) of \( X \) under \( f \) exists.

(i) If \( X \) is reduced, then so too is \( Y' \).

(ii) If the hypotheses of Proposition (9.5.1) are satisfied and \( X \) is irreducible \((\text{resp. integral})\), then so too is \( Y' \).

Proof. By hypothesis, the morphism \( f \) factors as \( X \xrightarrow{g} Y' \xrightarrow{j} Y \), where \( j \) is the canonical injection. Since \( X \) is reduced, \( g \) factors as \( X \xrightarrow{h} Y'_\text{red} \xrightarrow{j'} Y' \), where \( j' \) is the canonical injection \((5.2.2)\), and it then follows from the definition of \( Y' \) that \( Y'_\text{red} = Y' \). If, moreover, the conditions of Proposition (9.5.1) are satisfied, then it follows from \((9.5.4)\) that \( f(X) \) is dense in \( Y' \); if \( X \) is irreducible, then so is \( Y' \) \((0, 2.1.5)\). The claim about integral preschemes follows from the conjunction of the two others.

Proposition (9.5.10). — Let \( Y \) be a subprescheme of a prescheme \( X \), such that the canonical injection \( i : Y \to X \) is a quasi-compact morphism. Then there exists a smaller closed subprescheme \( \overline{Y} \) of \( X \) containing \( Y \); its underlying space is the closure of that of \( Y \); the latter is open in its closure, and the prescheme \( Y \) is induced on this open by \( \overline{Y} \).

Proof. It suffices to apply Proposition (9.5.1) to the injection \( j \), which is separated \((5.5.1)\) and quasi-compact by hypothesis; \((9.5.1)\) thus proves the existence of \( \overline{Y} \), and \((9.5.4)\) shows that its underlying space is the closure of \( Y \) in \( X \); since \( Y \) is locally closed in \( X \), it is open in \( \overline{Y} \), and the last claim comes from \((9.5.8)\) applied to an open subset \( V \) of \( X \) such that \( Y \) is closed in \( V \).

With the above notation, if the injection \( V \to X \) is quasi-compact, and if \( \mathcal{J} \) is the quasi-coherent sheaf of ideals of \( \mathcal{O}_X|V \) defining the closed subprescheme \( Y \) of \( V \), it follows from Proposition (9.5.1) that the quasi-coherent sheaf of ideals of \( \mathcal{O}_X \) defining \( \overline{Y} \) is the canonical extension \((9.4.1)\) \( \overline{\mathcal{J}} \) of \( \mathcal{J} \), because it is clearly the largest quasi-coherent subsheaf of ideals of \( \mathcal{O}_X \) inducing \( \mathcal{J} \) on \( V \).

Corollary (9.5.11). — Under the hypotheses of Proposition (9.5.10), every section of \( \mathcal{O}_Y \) over an open \( V \) of \( \overline{Y} \) that is zero on \( V \cap Y \) is zero.

Proof. By Proposition (9.5.8), we can reduce to the case where \( V = \overline{Y} \). If we take into account that the sections of \( \mathcal{O}_Y \) over \( \overline{Y} \) canonically correspond to the \( Y \)-sections of \( \overline{Y} \otimes_Z Z[T] \) \((3.3.15)\) and that the latter is separated over \( \overline{Y} \), then the corollary appears as a specific case of \((9.5.6)\).
When there exists a smaller closed subprescheme $Y'$ of $X$ containing a subprescheme $Y$ of $X$, we say that $Y'$ is the closure of $Y$ in $X$, when there is little cause for confusion.

9.6. Quasi-coherent sheaves of algebras; change of structure sheaf

**Proposition (9.6.1).** — Let $X$ be a prescheme, and $B$ a quasi-coherent $\mathcal{O}_X$-algebra (0, 5.1.3). For a $B$-module $\mathcal{F}$ to be quasi-coherent (on the ringed space $(X, \mathcal{O}_X)$) it is necessary and sufficient that $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

**Proof.** Since the question is local, we can assume $X$ to be affine, given by some ring $A$, and thus $B = \tilde{B}$, where $\tilde{B}$ is an $A$-algebra (1.4.3). If $\mathcal{F}$ is quasi-coherent on the ringed space $(X, \mathcal{O}_X)$ then we can also assume that $\mathcal{F}$ is the cokernel of a $B$-homomorphism $B^{(1)} \rightarrow B^{(1)}$; since this homomorphism is also an $\mathcal{O}_X$-homomorphism of $\mathcal{O}_X$-modules, and $B^{(1)}$ and $B^{(1)}$ are quasi-coherent $\mathcal{O}_X$-modules (1.3.9, ii), $\mathcal{F}$ is also a quasi-coherent $\mathcal{O}_X$-module (1.3.9, i).

Conversely, if $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then $\mathcal{F} = M$, where $M$ is a $B$-module (1.4.3); $M$ is isomorphic to the cokernel of a $B$-homomorphism $B^{(1)} \rightarrow B^{(1)}$, so $\mathcal{F}$ is a $B$-module isomorphic to the cokernel of the corresponding homomorphism $B^{(1)} \rightarrow B^{(1)}$ (1.3.13), which finishes the proof. □

In particular, if $\mathcal{F}$ and $\mathcal{G}$ are two quasi-coherent $B$-modules, then $\mathcal{F} \otimes_B \mathcal{G}$ is a quasi-coherent $B$-module; similarly for $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ whenever we further suppose that $\mathcal{F}$ admits a finite presentation (1.3.13).

(9.6.2). Given a prescheme $X$, we say that a quasi-coherent $\mathcal{O}_X$-algebra $B$ is of finite type if, for all $x \in X$, there exists an open affine neighborhood $U$ of $x$ such that $\Gamma(U, \mathcal{O}_X) = A$. We then have that $B|U = \tilde{B}$ and, for all $f \in A$, the induced $(\mathcal{O}_X|D(f))$-algebra $\mathcal{B}|D(f)$ is of finite type, because it is isomorphic to $(B_f)^{\sim}$, and $B_f = B \otimes_A A_f$ is clearly an algebra of finite type over $A_f$. Since the $D(f)$ form a basis for the topology of $U$, we thus conclude that if $B$ is a quasi-coherent $\mathcal{O}_X$-algebra of finite type then, for every open $V$ of $X$, $B|V$ is a quasi-coherent $(\mathcal{O}_X|V)$-algebra of finite type.

**Proposition (9.6.3).** — Let $X$ be a locally Noetherian prescheme. Then every quasi-coherent $\mathcal{O}_X$-algebra $B$ of finite type is a coherent sheaf of rings (0, 5.3.7).

**Proof.** We can once again restrict to the case where $X$ is an affine scheme given by a Noetherian ring $A$, and where $B = \tilde{B}$, with $\tilde{B}$ being an $A$-algebra of finite type; $B$ is then a Noetherian ring. With this, it remains to prove that the kernel $\mathcal{N}$ of a $B$-homomorphism $B^m \rightarrow B$ is a $B$-module of finite type; but it is isomorphic (as a $B$-module) to $\tilde{N}$, where $\tilde{N}$ is the kernel of the corresponding homomorphism of $B$-modules $B^m \rightarrow B$ (1.3.13). Since $B$ is Noetherian, the submodule $N$ of $B^m$ is a $B$-module of finite type, so there exists a homomorphism $B^p \rightarrow B^m$ with image $N$; since the sequence $B^p \rightarrow B^m \rightarrow B$ is exact, so too is the corresponding sequence $B^p \rightarrow B^m \rightarrow B$ (1.3.5), and since $\mathcal{N}$ is the image of $B^p \rightarrow B^m$ (1.3.9, i), this proves the proposition. □

**Corollary (9.6.4).** — Under the hypotheses of (9.6.3), for a $B$-module $\mathcal{F}$ to be coherent, it is necessary and sufficient that it be a quasi-coherent $\mathcal{O}_X$-module and a $B$-module of finite type. If this is the case, and if $\mathcal{G}$ is a $B$-submodule or a quotient module of $\mathcal{F}$, then in order for $\mathcal{G}$ to be a coherent $B$-module, it is necessary and sufficient that it be a quasi-coherent $\mathcal{O}_X$-module.

**Proof.** Taking (9.6.1) into account, the conditions on $\mathcal{F}$ are clearly necessary; we will show that they are sufficient. We can restrict to the case where $X$ is affine, given by some Noetherian ring $A$, where $B = \tilde{B}$, with $\tilde{B}$ an $A$-algebra of finite type, where $\mathcal{F} = M$, with $M$ a $B$-module, and where there exists a surjective $B$-homomorphism $B^m \rightarrow \mathcal{F} \rightarrow 0$. We then have the corresponding exact sequence $B^m \rightarrow M \rightarrow 0$, so $M$ is a $B$-module of finite type; further, the kernel $P$ of the homomorphism $B^m \rightarrow M$ is then a $B$-module of finite type, since $B$ is Noetherian. We thus conclude (1.3.13) that $\mathcal{F}$ is the cokernel of a $B$-homomorphism $B^m \rightarrow \tilde{B}^m$, and is thus coherent, since $\tilde{B}$ is a coherent sheaf of rings (0, 5.3.4). The same reasoning shows that a quasi-coherent $B$-submodule (resp. a quotient $B$-module) of $\mathcal{F}$ is of finite type, from whence the second part of the corollary. □
**Proposition (9.6.5).** — Let $X$ be a quasi-compact scheme, or a prescheme whose underlying space is Noetherian. For all quasi-compact $\mathcal{O}_X$-algebras $\mathcal{B}$ of finite type, there exists a quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{E}$ of $\mathcal{B}$ of finite type such that $\mathcal{E}$ generates (0.4.1.3) the $\mathcal{O}_X$-algebra $\mathcal{B}$.

**Proof.** In fact, by hypothesis, there exists a finite cover $(U_i)$ of $X$ consisting of affine opens such that $\Gamma(U_i, \mathcal{B}) = B_i$ is an algebra of finite type over $\Gamma(U, \mathcal{O}_X) = A_i$; let $E_i$ be a $A_i$-submodule of $B_i$ of finite type that generates the $A_i$-algebra $B_i$; thanks to (9.4.7), there exists a $\mathcal{O}_X$-submodule $\mathcal{E}_i$ of $\mathcal{B}$, quasi-coherent and of finite type, such that $\mathcal{E}_i|U_i = E_i$. It is clear that the sum $\mathcal{E}$ of the $\mathcal{E}_i$ is the desired object. \[\square\]

**Proposition (9.6.6).** — Let $X$ be a prescheme whose underlying space is locally Noetherian, or a quasi-compact scheme. Then every quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{B}$ is the inductive limit of its quasi-coherent $\mathcal{O}_X$-subalgebras of finite type.

**Proof.** In fact, it follows from (9.4.9) that $\mathcal{B}$ is the inductive limit (as an $\mathcal{O}_X$-module) of its quasi-coherent $\mathcal{O}_X$-submodules of finite type; the latter generating quasi-coherent $\mathcal{O}_X$-subalgebras of $\mathcal{B}$ of finite type (1.3.14), and so $\mathcal{B}$ is a fortiori their inductive limit. \[\square\]

## §10. FORMAL SCHEMES

### 10.1. Formal affine schemes

**(10.1.1).** Let $A$ be an admissible topological ring (0.7.1.2); for each ideal of definition $\mathfrak{J}$ of $A$, Spec$(A/\mathfrak{J})$ can be identified with the closed subspace $V(\mathfrak{J})$ of Spec$(A)$ (1.1.11), the set of open prime ideals of $A$; this topological space does not depend on the ideal of definition $\mathfrak{J}$ considered; we denote this topological space by $\mathfrak{X}$. Let $(\mathfrak{J}_\lambda)$ be a fundamental system of neighborhoods of 0 in $A$, consisting of ideals of definition, and for each $\lambda$, let $\mathcal{O}_\lambda$ be the structure sheaf of Spec$(A/\mathfrak{J}_\lambda)$; this sheaf is induced on $\mathfrak{X}$ by $\widetilde{A/\mathfrak{J}_\lambda}$ (which is zero outside of $\mathfrak{X}$). For $\mathfrak{J}_\mu \subset \mathfrak{J}_\lambda$, the canonical homomorphism $A/\mathfrak{J}_\mu \to A/\mathfrak{J}_\lambda$ thus defines a homomorphism $u_{\lambda, \mu} : \mathcal{O}_\mu \to \mathcal{O}_\lambda$ of sheaves of rings (1.6.1), and $(\mathcal{O}_\lambda)$ is a projective system of sheaves of rings for these homomorphisms. Since the topology of $\mathfrak{X}$ admits a basis consisting of quasi-compact open subsets, we can associate to each $\mathcal{O}_\lambda$ a pseudo-discrete sheaf of topological rings (0.3.8.1) which have $\mathcal{O}_\lambda$ as the underlying sheaf of rings (without topologies), and that we denote also by $\mathcal{O}_\lambda$; and the $\mathcal{O}_\lambda$ give again a projective system of sheaves of topological rings (0.3.8.2). We denote by $\mathcal{O}_X$ the sheaf of topological rings on $\mathfrak{X}$, the projective limit of the system $(\mathcal{O}_\lambda)$; for each quasi-compact open subset $U$ of $\mathfrak{X}$, $\Gamma(U, \mathcal{O}_X)$ is a topological ring, the projective limit of the system of discrete rings $\Gamma(U, \mathcal{O}_\lambda)$ (0.3.2.6).

**Definition (10.1.2).** — Given an admissible topological ring $A$, we define the formal spectrum of $A$, denoted by Spec$(\widetilde{A})$, to be the closed subspace $\mathfrak{X}$ of Spec$(A)$ consisting of the open prime ideals of $A$. We say that a topologically ringed space is a formal affine scheme if it is isomorphic to a formal spectrum Spec$(\widetilde{A})$ equipped with a sheaf of topological rings $\mathcal{O}_X$ which is the projective limit of sheaves of pseudo-discrete topological rings $(\widetilde{A/\mathfrak{J}_\lambda})|\mathfrak{X}$, where $\mathfrak{J}_\lambda$ varies over the filtered set of ideals of definition of $A$.

When we speak of a formal spectrum $\mathfrak{X} = \text{Spec}(\widetilde{A})$ as a formal affine scheme, it will always be as the topologically ringed space $(\mathfrak{X}, \mathcal{O}_X)$ where $\mathcal{O}_X$ is defined as above.

We note that every affine scheme $\mathfrak{X} = \text{Spec}(A)$ can be considered as a formal affine scheme in only one way, by considering $A$ as a discrete topological ring: the topological rings $\Gamma(U, \mathcal{O}_\lambda)$ are then discrete whenever $U$ is quasi-compact (but not, in general, when $U$ is an arbitrary open subset of $\mathfrak{X}$).

**Proposition (10.1.3).** — If $\mathfrak{X} = \text{Spec}(A)$, where $A$ is an admissible ring, then $\Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ is topologically isomorphic to $A$.

**Proof.** Indeed, since $\mathfrak{X}$ is closed in Spec$(A)$, it is quasi-compact, and so $\Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ is topologically isomorphic to the projective limit of the discrete rings $\Gamma(\mathfrak{X}, \mathcal{O}_\lambda)$; but $\Gamma(\mathfrak{X}, \mathcal{O}_\lambda)$ is isomorphic to $A/\mathfrak{J}_\lambda$ (1.3.7); since $A$ is separated and complete, it is topologically isomorphic to $\varprojlim A/\mathfrak{J}_\lambda$ (0.7.2.1), whence the proposition. \[\square\]
Proposition (10.1.4). — Let $A$ be an admissible ring, $\mathfrak{X} = \text{Spf}(A)$, and, for every $f \in A$, let $\mathcal{D}(f) = D(f) \cap \mathfrak{X}$; then the topologically ringed space $(\mathcal{D}(f), \mathcal{O}_\mathfrak{X}|\mathcal{D}(f))$ is isomorphic to the formal affine spectrum $\text{Spf}(A_{(f)})$ (0, 7.6.15).

Proof. For every ideal of definition $\mathfrak{I}$ of $A$, the discrete ring $A^{-1}A/\mathfrak{I}A^{-1}$ is canonically identified with $A_{(f)} / \mathfrak{I}(f)$ (0, 7.6.9), so, by (1.2.5) and (1.2.6), the topological space $\text{Spf}(A_{(f)})$ is canonically identified with $\mathfrak{D}(f)$. Further, for every quasi-compact open subset $U$ of $\mathfrak{X}$ contained in $\mathfrak{D}(f)$, $\Gamma(U, \mathcal{O}_\mathfrak{X})$ can be identified with the module of sections of the structure sheaf of $\text{Spec}(A^{-1}A/\mathfrak{I}A^{-1})$ over $U$ (1.3.6), so, setting $\mathfrak{Y} = \text{Spf}(A_{(f)})$, $\Gamma(U, \mathcal{O}_\mathfrak{X})$ can be identified with the module of sections $\Gamma(U, \mathcal{O}_\mathfrak{Y})$, which proves the proposition.

(10.1.5). As a sheaf of rings without topology, the structure sheaf $\mathcal{O}_\mathfrak{X}$ of $\text{Spf}(A)$ admits, for every $x \in \mathfrak{X}$, a fibre which, by (10.1.4), can be identified with the inductive limit $\lim_{\to} A_{(f)}$ for the $f \notin \mathfrak{I}_x$. Then, by (0, 7.6.17) and (0, 7.6.18):

Proposition (10.1.6). — For every $x \in \mathfrak{X} = \text{Spf}(A)$, the fibre $\mathcal{O}_x$ is a local ring whose residue field is isomorphic to $k(x) = A_x / \mathfrak{I}_x A_x$. If, further, $A$ is adic and Noetherian, then $\mathcal{O}_x$ is a Noetherian ring.

Since $k(x)$ is not reduced at $0$, we conclude from this that the support of the ring of sheaves $\mathcal{O}_\mathfrak{X}$ is equal to $\mathfrak{X}$.

10.2. Morphisms of formal affine schemes

(10.2.1). Let $A$, $B$ be two admissible rings, and let $\phi : B \to A$ be a continuous morphism. The continuous map $^a\phi : \text{Spec}(A) \to \text{Spec}(B)$ (1.2.1) then maps $\mathfrak{X} = \text{Spf}(A)$ to $\mathfrak{Y} = \text{Spf}(B)$, since the inverse image under $\phi$ of an open prime ideal of $A$ is an open prime ideal of $B$. Conversely, for all $g \in B$, $\phi$ defines a continuous homomorphism $\Gamma(\mathcal{D}(g), \mathcal{O}_\mathfrak{Y}) \to \Gamma(\mathcal{D}(\phi(g)), \mathcal{O}_\mathfrak{X})$ according to (10.1.4), (10.1.3), and (0, 7.6.7); since these homomorphisms satisfy the compatibility conditions for the restrictions corresponding to the change from $g$ to a multiple of $g$, and since $\mathcal{D}(\phi(g)) = \mathcal{D}(g^{-1})$, they define a continuous homomorphism of sheaves of topological rings $\mathcal{O}_\mathfrak{Y} \to ^a\phi_* (\mathcal{O}_\mathfrak{X})$ (0, 3.2.5) that we denote by $\bar{\phi}$; we have thus defined a morphism $\Phi = (\bar{\phi}, \bar{\phi})$ of topologically ringed spaces $\mathfrak{X} \to \mathfrak{Y}$. We note that, as a homomorphism of sheaves without topology, $\bar{\phi}$ defines a homomorphism $\bar{\phi}_x^\sharp : \mathcal{O}_x|\mathfrak{Y} \to \mathcal{O}_x$ on the stalks, for all $x \in \mathfrak{X}$.

Proposition (10.2.2). — Let $A$ and $B$ be admissible topological rings, and let $\mathfrak{X} = \text{Spf}(A)$ and $\mathfrak{Y} = \text{Spf}(B)$. For a morphism $u = (\psi, \theta) : \mathfrak{X} \to \mathfrak{Y}$ of topologically ringed spaces to be of the form $(\bar{\phi}, \bar{\phi})$, where $\phi$ is a continuous ring homomorphism $B \to A$, it is necessary and sufficient that $\theta_x^\sharp$ be a local homomorphism $\mathcal{O}_x|\mathfrak{Y} \to \mathcal{O}_x$ for all $x \in \mathfrak{X}$.

Proof. The condition is necessary: let $p = j_x \in \text{Spf}(A)$, and let $q = \phi^{-1}(j_x)$; if $g \notin q$, then we have $\phi(g) \notin p$, and it is immediate that the homomorphism $B_{(g)} \to A_{(\phi(g))}$ induced by $\phi$ (0, 7.6.7) sends $q_{(g)}$ to a subset of $\mathcal{P}_{(\phi(g))}$; by passing to the inductive limit, we see (taking (10.1.5) and (0, 7.6.17) into account) that $\theta_x^\sharp$ is a local homomorphism.

Conversely, let $(\psi, \theta)$ be a morphism satisfying the condition of the proposition; by (10.1.3), $\theta$ defines a continuous ring homomorphism $\phi = \Gamma(\theta) : B = \Gamma(\mathfrak{Y}, \mathcal{O}_\mathfrak{Y}) \to \Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X}) = A$.

By virtue of the hypothesis on $\theta$, for the section $\phi(g)$ of $\mathcal{O}_\mathfrak{X}$ over $\mathfrak{X}$ to be an invertible germ at the point $x$, it is necessary and sufficient that $g$ be an invertible germ at the point $\psi(x)$. But, by (0, 7.6.17), the sections of $\mathcal{O}_\mathfrak{X}$ (resp. $\mathcal{O}_\mathfrak{Y}$) over $\mathfrak{X}$ (resp. $\mathfrak{Y}$) that have a non-invertible germ at the point $x$ (resp. $\psi(x)$) are exactly the elements of $j_x$ (resp. $j_{\psi(x)}$); the above remark thus shows that $^a\phi = \psi$. Finally, for all $g \in B$ the diagram

$$
\begin{array}{ccc}
B = \Gamma(\mathfrak{Y}, \mathcal{O}_\mathfrak{Y}) & \xrightarrow{\phi} & \Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X}) = A \\
\downarrow & & \downarrow \\
B_{(g)} = \Gamma(\mathcal{D}(g), \mathcal{O}_\mathfrak{Y}) & \xrightarrow{\Gamma(\theta_{(g)})} & \Gamma(\mathcal{D}(\phi(g)), \mathcal{O}_\mathfrak{X}) = A_{(\phi(g))}
\end{array}
$$
is commutative; by the universal property of completed rings of fractions (0, 7.6.6), \( \theta \) is equal to \( \tilde{\phi} \) for all \( g \in B \), and so (0, 3.2.5) we have \( \theta = \tilde{\phi} \).

We say that a morphism \((\psi, \theta)\) of topologically ringed spaces satisfying the condition of Proposition (10.2.2) is a morphism of formal affine schemes. We can say that the functors \( \text{Spf}(A) \) in \( A \) and \( \Gamma(X, \mathcal{O}_X) \) in \( X \) define an equivalence between the category of admissible rings and the opposite category of formal affine schemes (T, I, 1.2).

(10.2.3). As a particular case of (10.2.2), note that, for \( f \in A \), the canonical injection of the formal affine scheme induced by \( X \) on \( \mathcal{D}(f) \) corresponds to the continuous canonical homomorphism \( A \rightarrow A_f \). Under the hypotheses of Proposition (10.2.2), let \( h \) be an element of \( B \), and \( g \) an element of \( A \) that is a multiple of \( \phi(h) \); we then have \( \psi(D(g)) \subseteq D(h) \); the restriction of \( u \) to \( \mathcal{D}(g) \), considered as a morphism from \( \mathcal{D}(g) \) to \( \mathcal{D}(h) \), is the unique morphism \( v \) making the diagram

\[
\begin{array}{ccc}
D(g) & \xrightarrow{v} & D(h) \\
\downarrow & & \downarrow \\
X & \xrightarrow{u} & Y
\end{array}
\]

commutate.

This morphism corresponds to the unique continuous homomorphism \( \phi': B_{(h)} \rightarrow A_{(g)} \) (0, 7.6.7) making the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\downarrow & & \downarrow \\
A_{(g)} & \xleftarrow{\phi'} & B_{(h)}
\end{array}
\]

commute.

10.3. Ideals of definition for a formal affine scheme

(10.3.1). Let \( A \) be an admissible ring, \( \mathfrak{J} \) an open ideal of \( A \), and \( X \) the formal affine scheme \( \text{Spf}(A) \). Let \( (\mathfrak{J}_f) \) be the set of those ideals of definition for \( A \) that are contained in \( \mathfrak{J} \); then \( \mathfrak{J}/\mathfrak{J}_f \) is a sheaf of ideals of \( \mathfrak{J}/\mathfrak{J}_f \). Denote by \( \mathfrak{J}^{\lambda} \) the projective limit of the sheaves on \( X \) induced by \( \mathfrak{J}/\mathfrak{J}_f \), which is identified with a sheaf of ideals of \( \mathcal{O}_X \) (0, 3.2.6). For every \( f \in A \), \( \Gamma(D(f), \mathfrak{J}^{\lambda}) \) is the projective limit of the \( S_f^{-1}\mathfrak{J}/S_f^{-1}\mathfrak{J}_f \) or, in other words, can be identified with the open ideal \( \mathfrak{J}_{(f)} \) of the ring \( A_{(f)} \) (0, 7.6.9), and, in particular, \( \Gamma(X, \mathfrak{J}^{\lambda}) = \mathfrak{J} \); we conclude (the \( D(f) \) forming a basis for the topology of \( X \)) that

\[
\mathfrak{J}^{\lambda}|D(f) = (\mathfrak{J}_{(f)})^{\lambda}.
\]

(10.3.2). With the notation of (10.3.1), for all \( f \in A \), the canonical map from \( A_{(f)} = \Gamma(D(f), \mathcal{O}_X) \) to \( \Gamma(D(f), (\mathfrak{A}/\mathfrak{J})|X) = S_f^{-1}A/S_f^{-1}\mathfrak{J} \) is surjective and has \( \Gamma(D(f), \mathfrak{J}^{\lambda}) = \mathfrak{J}_{(f)} \) as its kernel (0, 7.6.9); these maps thus define a surjective continuous homomorphism, said to be canonical, from the sheaf of topological rings \( \mathcal{O}_X \) to the sheaf of discrete rings \((\mathfrak{A}/\mathfrak{J})|X\), whose kernel is \( \mathfrak{J}^{\lambda} \); this homomorphism is none other than \( \phi \) (10.2.1), where \( \phi \) is the continuous homomorphism \( A \rightarrow A/\mathfrak{J} \); the morphism \((^a\phi, \phi) : \text{Spec}(A/\mathfrak{J}) \rightarrow X \) of formal affine schemes (where \(^a\phi \) is the identity homeomorphism from \( X \) to itself) is also said to be canonical. We thus have, according to the above, a canonical isomorphism

\[
\mathcal{O}_X/\mathfrak{J}^{\lambda} \simeq (\mathfrak{A}/\mathfrak{J})|X.
\]

It is clear (since \( \Gamma(X, \mathfrak{J}^{\lambda}) = \mathfrak{J} \)) that the map \( \mathfrak{J} \rightarrow \mathfrak{J}^{\lambda} \) is strictly increasing; according to the above, for \( \mathfrak{J} \subset \mathfrak{J}^{\lambda} \), the sheaf \( \mathfrak{J}^{\lambda}/\mathfrak{J}^{\lambda} \) is canonically isomorphic to \( \mathfrak{J}/\mathfrak{J}^{\lambda} = (\mathfrak{J}^{\lambda}/\mathfrak{J})^{-} \).

(10.3.3). The hypotheses and notation being the same as in (10.3.1), we say that a sheaf of ideals \( \mathcal{J} \) of \( \mathcal{O}_X \) is a sheaf of ideals of definition for \( X \) (or an ideal sheaf of definition for \( X \)) if, for all \( x \in X \), there exists an open neighborhood of \( x \) of the form \( D(f) \), where \( f \in A \), such that \( \mathcal{J}|D(f) \) is of the form \( \mathfrak{J}^{\lambda} \), where \( \mathfrak{J}^{\lambda} \) is an ideal of definition for \( A_{(f)} \).
Proposition (10.3.4). — For all \( f \in A \), each sheaf of ideals of definition for \( \mathcal{X} \) induces a sheaf of ideals of definition for \( \mathcal{D}(f) \).

Proof. This follows from (10.3.1.1). \( \square \)

Proposition (10.3.5). — If \( A \) is an admissible ring, then every sheaf of ideals of definition for \( \mathcal{X} = \text{Spf}(A) \) is of the form \( \mathfrak{J}^n \), where \( \mathfrak{J} \) is a uniquely determined ideal of definition for \( A \).

Proof. Let \( \mathcal{J} \) be a sheaf of ideals of definition of \( \mathcal{X} \); by hypothesis, and since \( \mathcal{X} \) is quasi-compact, there are finitely-many elements \( f_i \in A \) such that the \( \mathcal{D}(f_i) \) cover \( \mathcal{X} \) and such that \( \mathcal{J} |_{\mathcal{D}(f_i)} = \mathfrak{J}_{f_i}^n \), where \( \mathfrak{J}_{f_i} \) is an ideal of definition for \( A_{(f_i)} \). For each \( i \), there exists an open ideal \( \mathfrak{R}_i \) of \( A \) such that \( (\mathfrak{R}_i)_f = \mathfrak{R}_i (0, 7.6.9) \); let \( \mathfrak{R} \) be an ideal of definition for \( A \) containing all the \( \mathfrak{R}_i \). The canonical image of \( \mathcal{J} / \mathcal{R}_A \) in the structure sheaf \( (A / \mathcal{R})^{-1} \text{Spec}(A / \mathcal{R}) \) (10.3.2) is thus such that its restriction to \( \mathcal{D}(f_i) \) is equal to its restriction to \( (\mathfrak{R}_i / \mathcal{R})^{-1} \); we conclude that this canonical image is a quasi-coherent sheaf on \( \text{Spec}(A / \mathcal{R}) \), and so is of the form \( (\mathfrak{J} / \mathfrak{R}_A)^n \), where \( \mathfrak{J} \) is an ideal of definition for \( A \) containing \( \mathfrak{R} \), whence \( \mathcal{J} = \mathfrak{J}^n \) (10.3.2); in addition, since for each \( i \) there exists an integer \( n_i \) such that \( \mathfrak{J}_{f_i}^{n_i} \subset \mathfrak{R}(f_i) \), we will have, by setting \( n \) to be the largest of the \( n_i \), that \( (\mathcal{J} / \mathcal{R}_A)^n = 0 \), and, as a result (10.3.2), that \( ((\mathfrak{J} / \mathfrak{R}_A)^n)^n = 0 \), whence finally that \( (\mathfrak{J} / \mathfrak{R})^n = 0 \) (1.3.13), which proves that \( \mathfrak{J} \) is an ideal of definition for \( A \). \( \square \)

Proposition (10.3.6). — Let \( A \) be an adic ring, and \( \mathfrak{J} \) an ideal of definition for \( A \) such that \( \mathfrak{J} / \mathfrak{J}^2 \) is an \( (A / \mathfrak{J}) \)-module of finite type. For any integer \( n > 0 \), we then have \( (\mathfrak{J}^n)^A = (\mathfrak{J}^n)^A \).

Proof. For all \( f \in A \), we have (since \( \mathfrak{J}^n \) is an open ideal)

\[
(\Gamma(\mathcal{D}(f), \mathfrak{J}^A))^n = (\mathfrak{J}(f))^n = (\mathfrak{J}^n)(f) = \Gamma(\mathcal{D}(f^n), (\mathfrak{J}^n)^A)
\]

by (10.3.1.1) and (0, 7.6.12). The result then follows from the fact that \( (\mathfrak{J}^n)^A \) is associated to the presheaf \( U \mapsto (\Gamma(U, \mathfrak{J}^A))^n(0, 4.1.6) \), since the \( \mathcal{D}(f) \) form a basis for the topology of \( \mathcal{X} \). \( \square \)

(10.3.7). We say that a family \( (\mathcal{J}_a) \) of sheaves of ideals of definition for \( \mathcal{X} \) is a fundamental system of sheaves of ideals of definition if each sheaf of ideals of definition for \( \mathcal{X} \) contains one of the \( \mathcal{J}_a \); since \( \mathcal{J}_a = \mathfrak{J}_a^A \), it is equivalent to say that \( \mathfrak{J}_a \) form a fundamental system of neighborhoods of \( 0 \) in \( A \). Let \( (\mathfrak{J}_d) \) be a family of elements of \( A \) such that the \( \mathcal{D}(\mathfrak{J}_d) \) cover \( \mathcal{X} \). If \( (\mathcal{J}_a) \) is a filtered decreasing family of sheaves of ideals of \( \mathcal{O}_X \) such that, for each \( a \), the family \( (\mathcal{J}_a | \mathcal{D}(\mathfrak{J}_d)) \) is a fundamental system of sheaves of ideals of definition for \( \mathcal{D}(\mathfrak{J}_d) \), then \( (\mathcal{J}_a) \) is a fundamental system of sheaves of ideals of definition for \( \mathcal{X} \). Indeed, for each sheaf of ideals of definition \( \mathcal{J} \) for \( \mathcal{X} \), there is a finite cover of \( \mathcal{X} \) by \( \mathcal{D}(f_i) \) such that, for each \( i \), \( \mathcal{J}_a | \mathcal{D}(f_i) \) is a sheaf of ideals of definition for \( \mathcal{D}(f_i) \) that is contained in \( \mathcal{J} | \mathcal{D}(f_i) \). If \( \mu \) is an index such that \( \mathcal{J}_\mu \subset \mathcal{J}_a \) for all \( a \), then it follows from (10.3.3) that \( \mathcal{J}_\mu \) is a sheaf of ideals of definition for \( \mathcal{X} \), evidently contained in \( \mathcal{J} \), whence our claim.

10.4. Formal preschemes and morphisms of formal preschemes

(10.4.1). Given a topologically ringed space \( \mathcal{X} \), we say that an open \( U \subset \mathcal{X} \) is a formal affine open (resp. a formal adic affine open, resp. a formal Noetherian affine open) if the topologically ringed space induced on \( U \) by \( \mathcal{X} \) is a formal affine scheme (resp. a scheme whose ring is adic, resp. adic and Noetherian).

Definition (10.4.2). — A formal prescheme is a topologically ringed space \( \mathcal{X} \) which admits a formal affine open neighborhood for each point. We say that the formal prescheme \( \mathcal{X} \) is adic (resp. locally Noetherian) if each point of \( \mathcal{X} \) admits a formal adic (resp. Noetherian) open neighborhood. We say that \( \mathcal{X} \) is Noetherian if it is locally Noetherian and its underlying space is quasi-compact (and hence Noetherian).

Proposition (10.4.3). — If \( \mathcal{X} \) is a formal prescheme (resp. a locally Noetherian formal prescheme), then the formal affine (resp. Noetherian affine) open sets form a basis for the topology of \( \mathcal{X} \).

Proof. This follows from Definition (10.4.2) and Proposition (10.1.4) by taking into account that, if \( A \) is an adic Noetherian ring, then so too is \( A_{(f)} \) for all \( f \in A \). \( \square \)
**Corollary (10.4.4).** — If $\mathcal{X}$ is a formal prescheme (resp. a locally Noetherian formal prescheme, resp. a Noetherian formal prescheme), then the topologically ringed space induced on each open set of $\mathcal{X}$ is also a formal prescheme (resp. a locally Noetherian formal prescheme, resp. a Noetherian formal prescheme).

**Definition (10.4.5).** — Given two formal preschemes $\mathcal{X}$ and $\mathcal{Y}$, a morphism (of formal preschemes) from $\mathcal{X}$ to $\mathcal{Y}$ is a morphism $(\phi, \theta)$ of topologically ringed spaces such that, for all $x \in \mathcal{X}$, $\theta_x^\phi$ is a local homomorphism $\mathcal{O}_{\phi(x)} \to \mathcal{O}_x$.

It is immediate that the composition of any two morphisms of formal preschemes is again a morphism of formal preschemes; the formal preschemes thus form a category, and we denote by $\text{Hom}(\mathcal{X}, \mathcal{Y})$ the set of morphisms from a formal prescheme $\mathcal{X}$ to a formal prescheme $\mathcal{Y}$.

If $U$ is an open subset of $\mathcal{X}$, then the canonical injection into $\mathcal{X}$ of the formal prescheme induced on $U$ by $\mathcal{X}$ is a morphism of formal preschemes (and in fact a monomorphism of topologically ringed spaces (0, 4.1.1)).

**Proposition (10.4.6).** — Let $\mathcal{X}$ be a formal prescheme, and $\mathcal{S} = \text{Spf}(A)$ a formal affine scheme. There exists a canonical bijective equivalence between the morphisms from the formal prescheme $\mathcal{X}$ to the formal prescheme $\mathcal{S}$ and the continuous homomorphisms from the ring $A$ to the topological ring $\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$.

**Proof.** The proof is similar to that of (2.2.4), by replacing “homomorphism” with “continuous homomorphism”, “affine open” with “formal affine open”, and by using Proposition (10.2.2) instead of Theorem (1.7.3); we leave the details to the reader. □

(10.4.7). Given a formal prescheme $\mathcal{S}$, we say that the data of a formal prescheme $\mathcal{X}$ and a morphism $\phi: \mathcal{X} \to \mathcal{S}$ defines a formal prescheme $\mathcal{X}$ over $\mathcal{S}$ or a formal $\mathcal{S}$-prescheme, $\phi$ being called the structure morphism of the $\mathcal{S}$-prescheme $\mathcal{X}$. If $\mathcal{S} = \text{Spf}(A)$, where $A$ is an admissible ring, then we also say that the formal $\mathcal{S}$-prescheme $\mathcal{X}$ is a formal $A$-prescheme or a formal prescheme over $A$. An arbitrary formal prescheme can be considered as a formal prescheme over $\mathcal{Z}$ (equipped with the discrete topology).

If $\mathcal{X}$ and $\mathcal{Y}$ are formal $\mathcal{S}$-preschemes, then we say that a morphism $u : \mathcal{X} \to \mathcal{Y}$ is a $\mathcal{S}$-morphism if the diagram

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{u} & \mathcal{Y} \\
\downarrow & & \downarrow \\
\mathcal{S} & & \\
\end{array}
$$

(where the downwards arrows are the structure morphisms) is commutative. With this definition, the formal $\mathcal{S}$-preschemes (for some fixed $\mathcal{S}$) form a category. We denote by $\text{Hom}_{\mathcal{S}}(\mathcal{X}, \mathcal{Y})$ the set of $\mathcal{S}$-morphisms from a formal $\mathcal{S}$-prescheme $\mathcal{X}$ to a formal $\mathcal{S}$-prescheme $\mathcal{Y}$. When $\mathcal{S} = \text{Spf}(A)$, we sometimes say $A$-morphism instead of $\mathcal{S}$-morphism.

(10.4.8). Since every affine scheme can be considered as a formal affine scheme (10.1.2), every (usual) prescheme can be considered as a formal prescheme. In addition, it follows from Definition (10.4.5) that, for the usual preschemes, the morphisms (resp. $S$-morphisms) of formal preschemes coincide with the morphisms (resp. $S$-morphisms) defined in §2.

### 10.5. Sheaves of ideals of definition for formal preschemes

(10.5.1). Let $\mathcal{X}$ be a formal prescheme; we say that an $\mathcal{O}_\mathcal{X}$-ideal $\mathcal{I}$ is a sheaf of ideals of definition (or an ideal sheaf of definition) for $\mathcal{X}$ if every $x \in \mathcal{X}$ has a formal affine open neighborhood $U$ such that $\mathcal{I}|U$ is a sheaf of ideals of definition for the formal affine scheme induced on $U$ by $\mathcal{X}$ (10.3.3); by (10.3.1.1) and Proposition (10.4.3), for each open $V \subset \mathcal{X}$, $\mathcal{I}|V$ is then a sheaf of ideals of definition for the formal prescheme induced on $V$ by $\mathcal{X}$.

We say that a family $(\mathcal{I}_a)$ of sheaves of ideals of definition for $\mathcal{X}$ is a fundamental system of sheaves of ideals of definition if there exists a cover $(U_a)$ of $\mathcal{X}$ by formal affine open sets such that, for each $a$, the family of $\mathcal{I}_a|U_a$ is a fundamental system of sheaves of ideals of definition (10.3.6) for the formal affine scheme induced on $U_a$ by $\mathcal{X}$. It follows from the last remark of (10.3.7) that, when $\mathcal{X}$ is a formal affine scheme, this definition coincides with the definition given in (10.3.7). For an open subset $V$ of $\mathcal{X}$, the restrictions $\mathcal{I}_a|V$ then form a fundamental system of sheaves of ideals of definition for the formal prescheme induced on $V$, according to (10.3.1.1). If $\mathcal{X}$ is a locally Noetherian...
formal prescheme, and $\mathcal{J}$ is a sheaf of ideals of definition for $\mathcal{X}$, then it follows from Proposition (10.3.6) that the powers $\mathcal{J}^n$ form a fundamental system of sheaves of ideals of definition for $\mathcal{X}$.

(10.5.2). Let $\mathcal{X}$ be a formal prescheme, and $\mathcal{J}$ a sheaf of ideals of definition for $\mathcal{X}$. Then the ringed space $(\mathcal{X}, \mathcal{O}_X / \mathcal{J})$ is a (usual) prescheme, which is affine (resp. locally Noetherian, resp. Noetherian) when $\mathcal{X}$ is a formal affine scheme (resp. a locally Noetherian formal scheme, resp. a Noetherian formal scheme); we can reduce to the affine case, and then the proposition has already been proved in (10.3.2). In addition, if $\theta : \mathcal{O}_X \to \mathcal{O}_X / \mathcal{J}$ is the canonical homomorphism, then $u = (1, \theta)$ is a morphism (said to be canonical) of formal preschemes $(\mathcal{X}, \mathcal{O}_X / \mathcal{J}) \to (\mathcal{X}, \mathcal{O}_X)$, because, again, this was proved in the affine case (10.3.2), to which it is immediately reduced.

Proposition (10.5.3). — Let $\mathcal{X}$ be a formal prescheme, and $(\mathcal{J}_\lambda)$ a fundamental system of sheaves of ideals of definition for $\mathcal{X}$. Then the sheaf of topological rings $\mathcal{O}_{\mathcal{X}}$ is the projective limit of the pseudo-discrete sheaves of rings $(0, 3.8.1) \mathcal{O}_X / \mathcal{J}_\lambda$.

Proof. Since the topology of $\mathcal{X}$ admits a basis of formal quasi-compact affine open sets (10.4.3), we reduce to the affine case, where the proposition is a consequence of Proposition (10.3.5), (10.3.2), and Definition (10.1.1).

It is not true that any formal prescheme admits a sheaf of ideals of definition. However:

Proposition (10.5.4). — Let $\mathcal{X}$ be a locally Noetherian formal prescheme. There exists a largest sheaf of ideals of definition $\mathcal{I}$ for $\mathcal{X}$; this is the unique sheaf of ideals of definition $\mathcal{J}$ such that the prescheme $(\mathcal{X}, \mathcal{O}_X / \mathcal{J})$ is reduced. If $\mathcal{J}$ is a sheaf of ideals of definition for $\mathcal{X}$, then $\mathcal{I}$ is the inverse image under $\mathcal{O}_X / \mathcal{J}$ of the nilradical of $\mathcal{O}_X / \mathcal{J}$.

Proof. Suppose first that $\mathcal{X} = \text{Spf}(A)$, where $A$ is an adic Noetherian ring. The existence and the properties of $\mathcal{I}$ follow immediately from Propositions (10.3.5) and (5.1.1), taking into account the existence and the properties of the largest ideal of definition for $A$ ((0, 7.1.6) and (0, 7.1.7)).

To prove the existence and the properties of $\mathcal{I}$ in the general case, it suffices to show that, if $U \supset V$ are two Noetherian formal affine open subsets of $\mathcal{X}$, then the largest sheaf of ideals of definition $\mathcal{I}_U$ for $U$ induces the largest sheaf of ideals of definition $\mathcal{I}_V$ for $V$; but as $(\mathcal{V}, (\mathcal{O}_X | V)/(\mathcal{I}_U | V))$ is reduced, this follows from the above.

We denote by $\mathcal{X}_{\text{red}}$ the (usual) reduced prescheme $(\mathcal{X}, \mathcal{O}_X / \mathcal{I})$.

Corollary (10.5.5). — Let $\mathcal{X}$ be a locally Noetherian formal prescheme, and $\mathcal{I}$ the largest sheaf of ideals of definition for $\mathcal{X}$; for each open subset $V$ of $\mathcal{X}$, $\mathcal{I}|_V$ is the largest sheaf of ideals of definition for the formal prescheme induced on $V$ by $\mathcal{X}$.

Proposition (10.5.6). — Let $\mathcal{X}$ and $\mathcal{Y}$ be formal preschemes, $\mathcal{J}$ (resp. $\mathcal{K}$) be a sheaf of ideals of definition for $\mathcal{X}$ (resp. $\mathcal{Y}$), and $f : \mathcal{X} \to \mathcal{Y}$ a morphism of formal preschemes.

(i) If $f^*(\mathcal{K})\mathcal{O}_X \subset \mathcal{J}$, then there exists a unique morphism $f' : (\mathcal{X}, \mathcal{O}_X / \mathcal{J}) \to (\mathcal{Y}, \mathcal{O}_Y / \mathcal{K})$ of usual preschemes making the diagram

\[
\begin{array}{ccc}
(\mathcal{X}, \mathcal{O}_X) & \xrightarrow{f} & (\mathcal{Y}, \mathcal{O}_Y) \\
\uparrow & & \uparrow \\
(\mathcal{X}, \mathcal{O}_X / \mathcal{J}) & \xrightarrow{f'} & (\mathcal{Y}, \mathcal{O}_Y / \mathcal{K})
\end{array}
\]

commutate, where the vertical arrows are the canonical morphisms.

(ii) Suppose that $\mathcal{X} = \text{Spf}(A)$ and $\mathcal{Y} = \text{Spf}(B)$ are formal affine schemes, $\mathcal{J} = \mathfrak{J}^\Lambda$ and $\mathcal{K} = \mathfrak{K}^\Lambda$, where $\mathfrak{J}$ (resp. $\mathfrak{K}$) is an ideal of definition for $A$ (resp. $B$), and $f = (\phi, \tilde{\phi})$, where $\phi : B \to A$ is a continuous homomorphism; for $f^*(\mathcal{K})\mathcal{O}_X \subset \mathcal{J}$ to hold, it is necessary and sufficient that $\phi(\mathfrak{K}) \subset \mathfrak{J}$, and, in this case, $f'$ is then the morphism $(\phi', \tilde{\phi}')$, where $\phi' : B / \mathfrak{K} \to A / \mathfrak{J}$ is the homomorphism induced from $\phi$ by passing to quotients.

Proof.
(i) If \( f = (\psi, \theta) \), then the hypotheses imply that the image under \( \theta^2 : \psi^*(\mathcal{O}_Y) \to \mathcal{O}_X \) of the sheaf of ideals \( \psi^*(\mathcal{I}_X) \) of \( \psi^*(\mathcal{O}_Y) \) is contained in \( \mathcal{I} \) (0, 4.3.5). By passing to quotients, we thus obtain from \( \theta^1 \) a homomorphism of sheaves of rings

\[
\omega : \psi^*(\mathcal{O}_Y) / \psi^*(\mathcal{I}_X) \to \mathcal{O}_X / \mathcal{I} ;
\]

furthermore, since, for all \( x \in X \), \( \theta^1_x \) is a local homomorphism, so too is \( \omega_x \). The morphism of ringed spaces \( (\psi, \omega^2) \) is thus (2.2.1) the unique morphism \( f' \) of ringed spaces whose existence was claimed.

(ii) The canonical functorial correspondence between morphisms of formal affine schemes and continuous homomorphisms of rings (10.2.2) shows that, in the case considered, the relation \( f^*(\mathcal{O}_X) \mathcal{O}_Y \subset f' \) implies first of all that the \( \lambda \) that \( \psi \) lies in the \( \mathfrak{A} \) homomorphism \( \mathcal{O}_Y \mathcal{O}_X \to \mathcal{O}_X \mathcal{O}_Y \) on ringed spaces is the \( \omega \) homomorphism of homomorphisms of sheaves of rings.

By passing to the projective limit, there is an induced \( \theta \) on \( A \mathcal{O}_X \mathcal{O}_Y \), an inductive limit (r, I, 1.8) of the system

\[ (10.6.2) \]

With the notation of Proposition (10.6.2). — The canonical formal preschemes as inductive limits of preschemes

\[ (10.5.2) \]


It is clear that the correspondence \( f \mapsto f' \) defined above is functorial.

10.6. Formal preschemes as inductive limits of preschemes

(10.6.1). Let \( X \) be a formal prescheme, and \( (\mathcal{I}_\lambda) \) a fundamental system of sheaves of ideals of definition for \( X \); for each \( \lambda \), let \( f_\lambda \) be the canonical morphism \( (X, \mathcal{O}_X / \mathcal{I}_\lambda) \to X \) (10.5.2); for \( \mathcal{I}_\mu \subset \mathcal{I}_\lambda \), the canonical morphism \( \mathcal{O}_X / \mathcal{I}_\mu \to \mathcal{O}_X / \mathcal{I}_\lambda \) defines a canonical morphism \( f_\mu \) : \( (X, \mathcal{O}_X / \mathcal{I}_\lambda) \to (X, \mathcal{O}_X / \mathcal{I}_\mu) \) of (usual) preschemes such that \( f_\lambda = f_\mu \circ f_\mu \). The preschemes \( X_\lambda = (X, \mathcal{O}_X / \mathcal{I}_\lambda) \) and the morphisms \( f_\mu \) thus form (by (10.4.8)) an inductive system in the category of formal preschemes.

**Proposition (10.6.2).** — With the notation of (10.6.1), the formal prescheme \( X \) and the morphisms \( f_\lambda \) form an inductive limit \( (T, I, 1.8) \) of the system \( (X_\lambda, f_\mu) \) in the category of formal preschemes.

**Proof.** Let \( Y \) be a formal prescheme, and, for each index \( \lambda \), let

\[
g_\lambda = (\psi_\lambda, \theta_\lambda) : X_\lambda \to Y
\]

be a morphism such that \( g_\mu = g_\mu \circ f_\mu \) for \( \mathcal{I}_\mu \subset \mathcal{I}_\lambda \). This latter condition and the definition of the \( X_\lambda \) imply first of all that the \( \psi_\lambda \) are identical to a single continuous map \( \psi : X \to Y \) of the underlying spaces; in addition, the homomorphisms \( \theta^2_\lambda : \psi^*(\mathcal{O}_Y) \to \mathcal{O}_X / \mathcal{I}_\lambda \) form a projective system of homomorphisms of sheaves of rings. By passing to the projective limit, there is an induced homomorphism \( \omega : \psi^*(\mathcal{O}_Y) \to \lim \mathcal{O}_X / \mathcal{I}_\lambda = \mathcal{O}_X \), and it is clear that the morphism \( g = (\psi, \omega) \) of ringed spaces is the unique morphism making the diagrams

\[
\begin{array}{ccc}
X_\lambda & \xrightarrow{g_\lambda} & Y \\
\downarrow f_\lambda & & \downarrow g \\
X & \xrightarrow{g} & Y
\end{array}
\]

commutative. It remains to prove that \( g \) is a morphism of formal preschemes; the question is local on \( X \) and \( Y \), so we can assume that \( X = \text{Spf}(A) \) and \( Y = \text{Spf}(B) \), with \( A \) and \( B \) admissible rings, and with \( \mathcal{I}_\lambda = \mathfrak{A}_\lambda \), where \( (\mathfrak{A}_\lambda) \) is a fundamental system of ideals of definition for \( A \) (10.3.5);
we can cover \( (10.3.2) \), with \( (10.6.2.1) \) commutate then follows from the bijective correspondence \( (10.2.2) \) between morphisms of formal affine schemes and continuous ring homomorphisms, and from the definition of the projective limit. But the uniqueness of \( g \) as a morphism of ringed spaces shows that it coincides with the morphism in the beginning of the proof. \( \square \)

The following proposition establishes, under certain additional conditions, the existence of the inductive limit of a given inductive system of (usual) preschemes in the category of formal preschemes:

**Proposition (10.6.3).** — Let \( \mathcal{X} \) be a topological space, and \( (\mathcal{O}_i, u_{ji}) \) a projective system of sheaves of rings on \( \mathcal{X} \), with \( \mathbb{N} \) for its set of indices. Let \( \mathcal{J}_i \) be the kernel of \( u_0 : \mathcal{O}_1 \to \mathcal{O}_0 \). Suppose that:

(a) the ringed space \( (\mathcal{X}, \mathcal{O}_i) \) is a prescheme \( X_i \);
(b) for all \( x \in \mathcal{X} \) and all \( i \), there exists an open neighborhood \( U_i \) of \( x \) in \( \mathcal{X} \) such that the restriction \( \mathcal{J}_i|U_i \) is nilpotent; and
(c) the homomorphisms \( u_{ji} \) are surjective.

Let \( \mathcal{O}_X \) be the sheaf of topological rings given by the projective limit of the pseudo-discrete sheaves of rings \( \mathcal{O}_i \), and let \( u_i : \mathcal{O}_X \to \mathcal{O}_i \) be the canonical homomorphism. Then the topologically ringed space \( (\mathcal{X}, \mathcal{O}_X) \) is a formal prescheme; the homomorphisms \( u_i \) are surjective; their kernels \( \mathcal{J}^{(i)} \) form a fundamental system of sheaves of ideals of definition for \( \mathcal{X} \), and \( \mathcal{J}^{(0)} \) is the projective limit of the sheaves of ideals \( \mathcal{J}_i \).

**Proof.** We first note that, on each stalk, \( u_{ji} \) is a surjective homomorphism and a fortiori a local homomorphism; thus \( v_{ij} = (1_X, u_{ji}) \) is a morphism of preschemes \( X_j \to X_i (i \geq j) \). Suppose first that each \( X_i \) is an affine scheme of ring \( A_i \). There exists a ring homomorphism \( \phi_{ji} : A_i \to A_j \) such that \( u_{ij} = \phi_{ji} (1.7.3) \); as a result (1.6.3), the sheaf \( \mathcal{O}_i \) is a quasi-coherent \( \mathcal{O}_j \)-module over \( X_i \) (for the external law defined by \( u_{ij} \)), associated to \( A_j \)-module by means of \( \phi_{ji} \). For all \( f \in A_i \), let \( f' = \phi_{ji}(f) \); by hypothesis, the open sets \( D(f) \) and \( D(f') \) are identical in \( \mathcal{X} \), and the homomorphism \( \Gamma(D(f), \mathcal{O}_i) = (A_i)_f \to \Gamma(D(f'), \mathcal{O}_j) = (A_j)_{f'} \) corresponding to \( u_{ij} \) is exactly \( (\phi_{ji})_f (1.6.1) \). But when we consider \( A_j \) as an \( \mathcal{A}_j \)-module, \( (A_j)_{f'} \) is the \( (A_j)_{f'} \)-module \( (A_j)_{f} \), so we also have \( u_{ji} = \phi_{ji} \), where \( \phi_{ji} \) is now considered as a homomorphism of \( \mathcal{A}_j \)-modules. Then, since \( u_{ji} \) is surjective, we conclude that \( \phi_{ji} \) is also surjective (1.3.9), and if \( \tilde{J}_{ji} \) is the kernel of \( \phi_{ji} \), then the kernel of \( u_{ji} \) is a quasi-coherent \( \mathcal{O}_j \)-module equal to \( \tilde{J}_{ji} \). In particular, we have \( \mathcal{J}_i = \tilde{J}_{ji} \), where \( \tilde{J}_{ji} \) is the kernel of \( \phi_{ij} \). Hypothesis (b) implies that \( \mathcal{J}_i \) is nilpotent: indeed, since \( \mathcal{X} \) is quasi-compact, we can cover \( \mathcal{X} \) by a finite number of open sets \( U_k \) such that \( (\mathcal{J}_i|U_k)^{n_k} = 0 \), and, by setting \( n = \text{the largest of the } n_k \), we have \( \mathcal{J}_i^n = 0 \). We thus conclude that \( \tilde{J}_{ji} \) is nilpotent (1.3.13). Then the ring \( A = \lim A_i \) is admissible (0, 7.2.2), the canonical homomorphism \( \phi : A \to A_i \) is surjective, and its kernel \( \tilde{J}^{(i)} \) is equal to the projective limit of the \( \tilde{J}_{ji} \) for \( k \geq i \); the \( \tilde{J}^{(i)} \) form a fundamental system of sheaves of ideals of \( 0 \) in \( A \). The claims of Proposition (10.6.3) follow in this case from (10.1.1) and (10.3.2), with \( (\mathcal{X}, \mathcal{O}_X) \) being \( \text{Spf}(A) \).

Again, in this particular case, we note that, if \( f = (f_i) \) is an element of the projective limit \( A = \lim A_i \), then all the open sets \( D(f_i) \) (which are affine open sets in \( X_i \)) can be identified with the open subset \( D(f) \) of \( \mathcal{X} \), and the prescheme induced on \( D(f) \) by \( X_i \) is thus identified with the affine scheme \( \text{Spec}(A_{f_i}) \).

In the general case, we remark first that, for every quasi-compact open subset \( U \) of \( \mathcal{X} \), each of the \( \mathcal{J}_i|U \) is nilpotent, as shown by the above reasoning. We will show that, for every \( x \in \mathcal{X} \), there exists an open neighborhood \( U \) of \( x \) in \( \mathcal{X} \) which is an affine open set for all the \( X_i \). Indeed, we take \( U \) to be an affine open set for \( X_0 \), and observe that \( \mathcal{O}_{X_0} = \mathcal{O}_X / \mathcal{J}_i \). Since, by the above, \( \mathcal{J}_i|U \) is nilpotent, \( U \) is also an affine open set for each \( X_i \) by Proposition (5.1.9). This being so, for each \( U \) satisfying the preceding conditions, the study of the affine case as above shows that \( (U, \mathcal{O}_X|U) \) is a formal prescheme whose \( \mathcal{J}^{(i)}|U \) form a fundamental system of sheaves of ideals of definition, and \( \mathcal{J}^{(0)}|U \) is the projective limit of the \( \mathcal{J}_i|U \); whence the conclusion. \( \square \)

**Corollary (10.6.4).** — Suppose that, for \( i \geq j \), the kernel of \( u_{ji} \) is \( \mathcal{J}_i^{j+1} \) and that \( \mathcal{J}_i / \mathcal{J}_i^2 \) is of finite type over \( \mathcal{O}_0 = \mathcal{O}_1 / \mathcal{J}_1 \). Then \( \mathcal{X} \) is an adic formal prescheme, and if \( \mathcal{J}^{(n)} \) is the kernel of \( \mathcal{O}_X \to \mathcal{O}_n \),
then $\mathcal{F}^{(n)} = \mathcal{F}^{n+1}$ and $\mathcal{F} / \mathcal{F}^2$ is isomorphic to $\mathcal{F}_1$. If, in addition, $X_0$ is locally Noetherian (resp. Noetherian), then $X$ is locally Noetherian (resp. Noetherian).

PROOF. Since the underlying spaces of $X$ and $X_0$ are the same, the question is local, and we can suppose that all the $X_i$ are affine; taking into account the fact that $\mathcal{F}_{ij} = \mathcal{F}_{ji}$ (with the notation of Proposition (10.3.6)), we can immediately reduce the problem to the corresponding claims of Proposition (0, 7.2.7) and Corollary (0, 7.2.8), by noting that $\mathcal{F}_{ij} / \mathcal{F}_{ji}^2$ then is an $A_0$-module of finite type (1.3.9).

In particular, every locally Noetherian formal prescheme $X$ is the inductive limit of a sequence $(X_n)$ of locally Noetherian (usual) preschemes satisfying the conditions (b) and (c) of Proposition (10.6.3), and let $X_n = (X, \mathcal{O}_X / \mathcal{F}^{n+1})$ (10.5.1) and Proposition (10.6.2).

Corollary (10.6.5). — Let $A$ be an admissible ring. For the formal affine scheme $X = \text{Spf}(A)$ to be Noetherian, it is necessary and sufficient for $A$ to be adic and Noetherian.

PROOF. The condition is evidently sufficient. Conversely, suppose that $X$ is Noetherian, and let $\mathcal{F}$ be an ideal of definition for $A$, and $\mathcal{F} = \mathcal{F}^\Delta$ the corresponding sheaf of ideals of definition for $X$. The (usual) preschemes $X_n = (X, \mathcal{O}_X / \mathcal{F}^{n+1})$ are then affine and Noetherian, so the rings $A_n = A / \mathcal{F}^{n+1}$ are Noetherian (6.1.3), whence we conclude that $\mathcal{F} / \mathcal{F}^2$ is an $A_0$-module of finite type.

Since the $\mathcal{F}^n$ form a fundamental system of sheaves of ideals of definition for $X$ (10.5.1), we have $\mathcal{O}_X = \lim \mathcal{O}_X / \mathcal{F}^n$ (10.5.3); we thus conclude (10.1.3) that $A$ is topologically isomorphic to $\lim A / \mathcal{F}^n$, which is adic and Noetherian (0, 7.2.8). □

Remark (10.6.6). — With the notation of Proposition (10.6.3), let $\mathcal{F}_i$ be an $\mathcal{O}_X$-module, and suppose we are given, for $i \geq i$, a $v_{ij}$-morphism $\theta_{ij} : \mathcal{F}_i \to \mathcal{F}_j$ such that $\theta_{kj} \circ \theta_{ij} = \theta_{ki}$ for $k \leq j \leq i$. Since the underlying continuous map of $v_{ij}$ is the identity, $\theta_{ij}$ is a homomorphism of sheaves of abelian groups to the space $X$; in addition, if $\mathcal{F}$ is the projective limit of the projective system $(\mathcal{F}_i)$ of sheaves of abelian groups, then the fact that the $\theta_{ij}$ are $v_{ij}$-morphisms lets us define an $\mathcal{O}_X$-module structure on $\mathcal{F}$ by passing to the projective limit; when equipped with this structure, we say that $\mathcal{F}$ is the projective limit (with respect to the $\theta_{ij}$) of the system of $\mathcal{O}_X$-modules $(\mathcal{F}_i)$. In the particular case where $v_{ij}^* (\mathcal{F}_i) = \mathcal{F}_j$ and $\theta_{ij}$ is the identity, we say that $\mathcal{F}$ is the projective limit of a system $(\mathcal{F}_i)$ such that $v_{ij}^* (\mathcal{F}_i) = \mathcal{F}_j$ for $j \leq i$ (without mentioning the $\theta_{ij}$).

(10.6.7). Let $\mathcal{X}$ and $\mathcal{Y}$ be formal preschemes, $\mathcal{F}$ (resp. $\mathcal{H}$) a sheaf of ideals of definition for $\mathcal{X}$ (resp. $\mathcal{Y}$), and $f : \mathcal{X} \to \mathcal{Y}$ a morphism such that $f^* (\mathcal{H}) \mathcal{O}_X \subset \mathcal{F}$. We then have, for every integer $n > 0$, that $f^* (\mathcal{H}^n) \mathcal{O}_X = (f^* (\mathcal{H}) \mathcal{O}_X)^n \subset \mathcal{F}^n$; $f$ thus induces (10.5.6) a morphism of (usual) preschemes $f_n : X_n \to Y_n$ by setting $X_n = (\mathcal{X}, \mathcal{O}_X / \mathcal{F}^{n+1})$ and $Y_n = (\mathcal{Y}, \mathcal{O}_Y / \mathcal{H}^{n+1})$, and it immediately follows from the definitions that the diagrams

\[
\begin{array}{ccc}
X_m & \xrightarrow{f_m} & Y_m \\
\downarrow & & \downarrow \\
X_n & \xrightarrow{f_n} & Y_n
\end{array}
\]

commute for $m \leq n$; in other words, $(f_n)$ is an inductive system of morphisms.

(10.6.8). Conversely, let $(X_n)$ (resp. $(Y_n)$) be an inductive system of (usual) preschemes satisfying conditions (b) and (c) of Proposition (10.6.3), and let $\mathcal{X}$ (resp. $\mathcal{Y}$) be its inductive limit. By definition of the inductive limit, each sequence $(f_n)$ of morphisms $X_n \to Y_n$ forms an inductive system that admits an inductive limit $f : \mathcal{X} \to \mathcal{Y}$, which is the unique morphism of formal preschemes that makes the diagrams

\[
\begin{array}{ccc}
X_n & \xrightarrow{f_n} & Y_n \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{f} & \mathcal{Y}
\end{array}
\]

commutate.
Proposition (10.6.9). — Let $\mathcal{X}$ and $\mathcal{Y}$ be locally Noetherian formal preschemes, and $\mathcal{J}$ (resp. $\mathcal{K}$) be a sheaf of ideals of definition for $\mathcal{X}$ (resp. $\mathcal{Y}$); the map $f \mapsto (f_n)$ defined in (10.6.7) is a bijection from the set of morphisms $f : \mathcal{X} \to \mathcal{Y}$ such that $f^* (\mathcal{K}) \mathcal{O}_X \subset \mathcal{J}$ to the set of sequences $(f_n)$ of morphisms that make the diagrams (10.6.7.1) commute.

Proof. If $f$ is the inductive limit of this sequence, then it remains to show that $f^* (\mathcal{K}) \mathcal{O}_X \subset \mathcal{J}$. The statement, being local on $\mathcal{X}$ and $\mathcal{Y}$, can be reduced to the case where $\mathcal{X} = \text{Spf}(A)$ and $\mathcal{Y} = \text{Spf}(B)$ are affine, with $A$ and $B$ adic Noetherian rings, and with $\mathcal{J} = \bar{\mathfrak{J}}^A$ and $\mathcal{K} = \bar{\mathfrak{K}}^A$, where $\bar{\mathfrak{J}}$ (resp. $\bar{\mathfrak{K}}$) is an ideal of definition for $A$ (resp. $B$). We then have that $X_n = \text{Spec}(A_n)$ and $Y_n = \text{Spec}(B_n)$, with $A_n = A/\mathfrak{J}^{n+1}$ and $B_n = B/\mathfrak{K}^{n+1}$, by Proposition (10.3.6) and (10.3.2); $f_n = (\phi_n, \phi_n)$, where the homomorphisms $\phi_n : B_n \to A_n$ forms a projective system, thus $f = (\phi, \phi)$, and so $f = (\phi, \phi)$, where $\phi = \varprojlim \phi_n$. The commutativity of the diagram (10.6.7.1) for $m = 0$ then gives the condition $\phi_n(\bar{\mathfrak{J}}/\mathfrak{K}^{n+1}) \subset \bar{\mathfrak{J}}/\mathfrak{K}^{n+1}$ for all $n$, so, by passing to the projective limit, we have $\phi(\bar{\mathfrak{J}}) \subset \mathfrak{J}$, which implies that $f^* (\mathcal{K}) \mathcal{O}_X \subset \mathcal{J}$.

□

Corollary (10.6.10). — Let $\mathcal{X}$ and $\mathcal{Y}$ be locally Noetherian formal preschemes, and $\mathcal{J}$ the largest sheaf of ideals of definition for $\mathcal{X}$ (10.5.4).

(i) For every sheaf of ideals of definition $\mathcal{K}$ for $\mathcal{Y}$ and every morphism $f : \mathcal{X} \to \mathcal{Y}$, we have $f^* (\mathcal{K}) \mathcal{O}_X \subset \mathcal{J}$.

(ii) There is a canonical bijective correspondence between $\text{Hom}(\mathcal{X}, \mathcal{Y})$ and the set of sequences $(f_n)$ of morphisms making the diagrams (10.6.7.1) commute, where $X_n = (\mathcal{X}, \mathcal{O}_X / \mathcal{J}^{n+1})$ and $Y_n = (\mathcal{Y}, \mathcal{O}_Y / \mathcal{J}^{n+1})$.

Proof. (ii) follows immediately from (i) and Proposition (10.6.9). To prove (i), we can reduce to the case where $\mathcal{X} = \text{Spf}(A)$ and $\mathcal{Y} = \text{Spf}(B)$, with $A$ and $B$ Noetherian, and with $\mathcal{J} = \mathfrak{I}^A$ and $\mathcal{K} = \mathfrak{K}^A$, where $\mathfrak{I}$ is the largest ideal of definition for $A$ and $\mathfrak{K}$ is an ideal of definition for $B$. Let $f = (\phi, \phi)$, where $\phi : B \to A$ is a continuous homomorphism; since the elements of $\mathfrak{K}$ are topologically nilpotent (0,7.1.4, ii), so too are those of $\phi(\mathfrak{K})$, and so $\phi(\mathfrak{K}) \subset \mathfrak{I}$, since $\mathfrak{I}$ is the set of topologically nilpotent elements of $A$ (0,7.1.6); hence, by Proposition (10.5.6, ii), we are done.

□

Corollary (10.6.11). — Let $\mathcal{S}$, $\mathcal{X}$, $\mathcal{Y}$ be locally Noetherian formal preschemes, and $f : \mathcal{X} \to \mathcal{S}$ and $g : \mathcal{Y} \to \mathcal{S}$ the morphisms that make $\mathcal{X}$ and $\mathcal{Y}$ formal $\mathcal{S}$-preschemes. Let $\mathcal{J}$ (resp. $\mathcal{K}$) be a sheaf of ideals of definition for $\mathcal{X}$ (resp. $\mathcal{Y}$), and suppose that $f^* (\mathcal{J}) \mathcal{O}_X \subset \mathcal{K}$ and $g^* (\mathcal{J}) \mathcal{O}_Y = \mathcal{L}$; set $S_n = (\mathcal{S}, \mathcal{O}_S / \mathcal{J}^{n+1})$, $X_n = (\mathcal{X}, \mathcal{O}_X / \mathcal{K}^{n+1})$, and $Y_n = (\mathcal{Y}, \mathcal{O}_Y / \mathcal{L}^{n+1})$. Then there exists a canonical bijective correspondence between $\text{Hom}_{\mathcal{S}}(\mathcal{X}, \mathcal{Y})$ and the set of sequences $(u_n)$ of $S_n$-morphisms $u_n : X_n \to Y_n$ making the diagrams (10.6.7.1) commute.

Proof. For each $\mathcal{S}$-morphism $u : \mathcal{X} \to \mathcal{Y}$, we have by definition that $f = g \circ u$, whence $u^* (\mathcal{L}) \mathcal{O}_X = u^* (g^* (\mathcal{J}) \mathcal{O}_Y) \mathcal{O}_X = f^* (\mathcal{J}) \mathcal{O}_X \subset \mathcal{K}$, and the corollary then follows from Proposition (10.6.9).
10.7. Products of formal preschemes

(10.7.1). Let \( \mathcal{S} \) be a formal prescheme; formal \( \mathcal{S} \)-preschemes form a category, and we can define a notion of a product of formal \( \mathcal{S} \)-preschemes.

**Proposition (10.7.2).** — Let \( \mathcal{X} = \text{Spf}(B) \) and \( \mathcal{Y} = \text{Spf}(C) \) be formal affine schemes over a formal affine scheme \( \mathcal{S} = \text{Spf}(A) \). Let \( \mathcal{Z} = \text{Spf}(B \hat{\otimes}_A C) \), and let \( p_1 \) and \( p_2 \) be the \( \mathcal{S} \)-morphisms corresponding (10.2.2) to the canonical (continuous) \( A \)-homomorphisms \( \rho : B \to B \hat{\otimes}_A C \) and \( \sigma : C \to B \hat{\otimes}_A C \); then \((\mathcal{Z}, p_1, p_2)\) is a product of the formal affine \( \mathcal{S} \)-schemes \( \mathcal{X} \) and \( \mathcal{Y} \).

**Proof.** By Proposition (10.4.6), it suffices to check that, if we associate to each continuous \( A \)-homomorphism \( \phi : B \to B \hat{\otimes}_A C \) (where \( B \) is an admissible ring which is a topological \( A \)-algebra), the pair \((\phi \circ \rho, \phi \circ \sigma)\), then this defines a bijection

\[
\text{Hom}_A(B \hat{\otimes}_A C, D) \simeq \text{Hom}_A(B, D) \times \text{Hom}_A(C, D),
\]

which is exactly the universal property of the completed tensor product \((0, 7.7.6)\). \( \square \)

**Proposition (10.7.3).** — Given formal \( \mathcal{S} \)-preschemes \( \mathcal{X} \) and \( \mathcal{Y} \), their product \( \mathcal{X} \times_\mathcal{S} \mathcal{Y} \) exists.

**Proof.** The proof is similar to that of Theorem (3.2.6), replacing affine schemes (resp. affine open sets) by formal affine schemes (resp. formal affine open sets), and replacing Proposition (3.2.2) by Proposition (10.7.2). \( \square \)

All the formal properties of the product of preschemes ((3.2.7) and (3.2.8), (3.3.1) and (3.3.12)) hold true without modification for the product of formal preschemes.

(10.7.4). Let \( \mathcal{S} \), \( \mathcal{X} \), and \( \mathcal{Y} \) be formal preschemes, and let \( f : \mathcal{X} \to \mathcal{S} \) and \( g : \mathcal{Y} \to \mathcal{S} \) be morphisms. Suppose that there exist, in \( \mathcal{S} \), \( \mathcal{X} \), and \( \mathcal{Y} \) respectively, three fundamental systems of sheaves of ideals of definitions \( (\mathscr{I}_\lambda), (\mathscr{X}_\lambda), \) and \( (\mathscr{L}_\lambda) \), all having the same set \( I \) of indices, and such that \( f^*(\mathscr{I}_\lambda) \mathscr{O}_\mathcal{X} \subseteq \mathscr{X}_\lambda \) and \( g^*(\mathscr{I}_\lambda) \mathscr{O}_\mathcal{Y} \subseteq \mathscr{L}_\lambda \) for all \( \lambda \). Set \( S_\lambda = (\mathcal{S}, \mathscr{O}_\mathcal{S} / \mathscr{I}_\lambda), \) \( X_\lambda = (\mathcal{X}, \mathscr{O}_\mathcal{X} / \mathscr{X}_\lambda) \), and \( Y_\lambda = (\mathcal{Y}, \mathscr{O}_\mathcal{Y} / \mathscr{L}_\lambda) \); for \( J \subseteq \mathfrak{I}_\lambda, K \subseteq \mathfrak{X}_\lambda \), and \( \mathfrak{L}_\lambda \subseteq \mathfrak{L}_\lambda \), note that \( S_\lambda \) (resp. \( X_\lambda, Y_\lambda \)) is a closed subscheme of \( S_\mu \) (resp. \( X_\mu, Y_\mu \)) that has the same underlying space (10.6.1). Since \( S_\lambda \to S_\mu \) is a monomorphism of preschemes, we see that the products \( X_\lambda \times_{S_\lambda} Y_\lambda \) and \( X_\lambda \times_{S_\mu} Y_\lambda \) are identical (3.2.4), since \( X_\lambda \times_{S_\mu} Y_\lambda \) can be identified with a closed subscheme of \( X_\mu \times_{S_\mu} Y_\mu \) that has the same underlying space (4.3.1). With this in mind, the product \( \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \) is the inductive limit of the usual preschemes \( X_\lambda \times_{S_\lambda} Y_\lambda \); indeed, as we see in Proposition (10.6.2), we can reduce to the case where \( \mathcal{S}, \mathcal{X}, \) and \( \mathcal{Y} \) are formal affine schemes. Taking into account both Proposition (10.5.6, ii) and the hypotheses on the fundamental systems of sheaves of ideals of definition for \( \mathcal{S}, \mathcal{X}, \) and \( \mathcal{Y} \), we immediately see that our claim follows from the definition of the completed tensor product of two algebras (0, 7.7.1).

Furthermore, let \( \mathfrak{I} \) be a formal \( \mathcal{S} \)-prescheme, \( (\mathfrak{I}_\lambda) \) a fundamental system of ideals of definition for \( \mathfrak{I} \) having \( I \) for its set of indices, and let \( u : \mathfrak{I} \to \mathcal{X} \) and \( v : \mathfrak{I} \to \mathcal{Y} \) be \( \mathcal{S} \)-morphisms such that \( u^*(\mathfrak{I}_\lambda) \mathscr{O}_\mathcal{X} \subseteq \mathfrak{I}_\lambda \) and \( v^*(\mathfrak{I}_\lambda) \mathscr{O}_\mathcal{Y} \subseteq \mathfrak{I}_\lambda \) for all \( \lambda \). If we set \( Z_\lambda = (\mathfrak{I}_\lambda, \mathcal{O}_Z / \mathfrak{I}_\lambda), \) and if \( u_\lambda : Z_\lambda \to X_\lambda \) and \( v_\lambda : Z_\lambda \to Y_\lambda \) are the \( S_\lambda \)-morphisms corresponding to \( u \) and \( v \) (10.5.6), then we immediately have that \((u, v)_{\mathfrak{I}}\) is the inductive limit of the \( S_\lambda \)-morphisms \((u_\lambda, v_\lambda)_{\mathfrak{I}_\lambda}\).

The ideas of this section apply, in particular, to the case where \( \mathcal{S}, \mathcal{X}, \) and \( \mathcal{Y} \) are locally Noetherian, taking the systems consisting of the powers of a sheaf of ideals of definition (10.5.1) as the fundamental systems of sheaves of ideals of definition. However, we note that \( \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \) is not necessarily locally Noetherian (see however (10.13.5)).
10.8. Formal completion of a prescheme along a closed subset

(10.8.1). Let $X$ be a locally Noetherian (usual) prescheme, and $X'$ a closed subset of the underlying space of $X$; we denote by $\Phi$ the set of coherent sheaves of ideals $\mathcal{J}$ of $\mathcal{O}_X$ such that the support of $\mathcal{O}_X/\mathcal{J}$ is $X'$. The set $\Phi$ is nonempty ((5.2.1), (4.1.4), (6.1.1)); we order it by the relation $\supset$.

Lemma (10.8.2). — The ordered set $\Phi$ is filtered; if $X$ is Noetherian, then, for all $\mathcal{J}_0 \in \Phi$, the set of powers $\mathcal{J}_0^n$ ($n > 0$) is cofinal in $\Phi$.

Proof. If $\mathcal{J}_1$ and $\mathcal{J}_2$ are in $\Phi$, and if we set $\mathcal{J} = \mathcal{J}_1 \cap \mathcal{J}_2$, then $\mathcal{J}$ is coherent since $\mathcal{O}_X$ is coherent (6.1.1) and (0.5.3.4), and we have $\mathcal{J}_x = (\mathcal{J}_1)_x \cap (\mathcal{J}_2)_x$ for all $x \in X$, whence $\mathcal{J}_x = \mathcal{O}_x$ for $x \notin X'$, and $\mathcal{J}_x \neq \mathcal{O}_x$ for $x \in X'$, which proves that $\mathcal{J} \in \Phi$. On the other hand, if $X$ is Noetherian and if $\mathcal{J}_0$ and $\mathcal{J}$ are in $\Phi$, then there exists an integer $n > 0$ such that $\mathcal{J}_0^n(\mathcal{O}_X/\mathcal{J}) = 0$ (9.3.4), which implies that $\mathcal{J}_0^n \subset \mathcal{J}$. $\square$

(10.8.3). Now let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module; for all $\mathcal{J} \in \Phi$, we have that $\mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{J})$ is a coherent $\mathcal{O}_X$-module (9.1.1) with support contained in $X'$, and we will usually identify it with its restriction to $X'$. When $\mathcal{J}$ varies over $\Phi$, these sheaves form a projective system of sheaves of abelian groups.

Definition (10.8.4). — Given a closed subset $X'$ of a locally Noetherian prescheme $X$ and a coherent $\mathcal{O}_X$-module $\mathcal{F}$, we define the completion of $\mathcal{F}$ along $X'$, denoted by $\mathcal{F}/X'$ (or $\hat{\mathcal{F}}$, if there is little chance of confusion), to be the restriction to $X'$ of the sheaf $\lim_{\mathcal{J} \in \Phi} (\mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{J}))$; we say that its sections over $X'$ are the formal sections of $\mathcal{F}$ along $X'$.

It is immediate that, for every open $U \subset X$, we have $(\mathcal{F}|U)/(U \cap X') = (\mathcal{F}/X')(U \cap X')$.

By passing to the projective limit, it is clear that $(\mathcal{O}_X/X')$ is a sheaf of rings, and that $\mathcal{F}/X'$ can be considered as an $(\mathcal{O}_X/X')$-module. In addition, since there exists a basis for the topology of $X'$ consisting of quasi-compact open sets, we can consider $(\mathcal{O}_X/X')$ (resp. $\mathcal{F}/X'$) as a sheaf of topological rings (resp. of topological groups), the projective limit of the pseudo-discrete sheaves of rings (resp. groups) $\mathcal{O}_X/\mathcal{J}$ (resp. $\mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{J}) = \mathcal{F}/\mathcal{J}$), and, by passing to the projective limit, $\mathcal{F}/X'$ then becomes a topological $(\mathcal{O}_X/X')$-module ((0.3.8.1) and (0.8.2)); recall that, for every quasi-compact open $U \subset X$, $\Gamma(U \cap X', (\mathcal{O}_X/X'))$ (resp. $\Gamma(U \cap X', (\mathcal{F}/X'))$) is then the projective limit of the discrete rings (resp. groups) $\Gamma(U, \mathcal{O}_X/\mathcal{J})$ (resp. $\Gamma(U, \mathcal{F}/\mathcal{J})$).

Now, if $u : \mathcal{F} \to \mathcal{G}$ is a homomorphism of $\mathcal{O}_X$-modules, then there are canonically induced homomorphisms $u_\mathcal{J} : \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{J}) \to \mathcal{G} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{J})$ for all $\mathcal{J} \in \Phi$, and these homomorphisms form a projective system. By passing to the projective limit and restricting to $X'$, these give a continuous $(\mathcal{O}_X/X')$-homomorphism $\mathcal{F}/X' \to \mathcal{G}/X'$, denoted $u_{X'}$, and called the completion of the homomorphism $u$ along $X'$. It is clear that, if $v : \mathcal{F} \to \mathcal{H}$ is a second homomorphism of $\mathcal{O}_X$-modules, then we have $(v \circ u)_{X'} = (v_{X'}) \circ (u_{X'})$, hence $\mathcal{F}/X'$ is a covariant additive functor in $\mathcal{F}$ from the category of coherent $\mathcal{O}_X$-modules to the category of topological $(\mathcal{O}_X/X')$-modules.

Proposition (10.8.5). — The support of $(\mathcal{O}_X/X')$ is $X'$; the topologically ringed space $(X', (\mathcal{O}_X/X'))$ is a locally Noetherian formal prescheme, and, if $\mathcal{J} \in \Phi$, then $\mathcal{J}/X'$ is a sheaf of ideals of definition for this formal prescheme. If $X = \text{Spec}(A)$ is an affine scheme with Noetherian ring $A$, $\mathcal{J} = \mathfrak{a}$ for some ideal $\mathfrak{a}$ of $A$, and $X' = V(\mathfrak{a})$, then $X'/(\mathcal{O}_X/X')$ is canonically identified with $\text{Spf}(\mathfrak{A})$, where $\mathfrak{A}$ is the separated completion of $A$ with respect to the $\mathfrak{a}$-adic topology.

Proof. We can evidently reduce to proving the latter claim. We know (0, 7.3.3) that the separated completion $\mathfrak{a}$ of $\mathfrak{a}$ with respect to the $\mathfrak{a}$-adic topology can be identified with the ideal $\mathfrak{a}^n$ of $\hat{A}$, where $\hat{A}$ is the Noetherian $\mathfrak{a}$-adic ring such that $\hat{A}/\mathfrak{a}^n = A/\mathfrak{a}^n$ (0, 7.2.6). This latter equation shows that the open prime ideals of $\hat{A}$ are the ideals $\mathfrak{p} \cap \hat{A} = \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of $A$ containing $\mathfrak{a}$, and that we have $\hat{A}/\mathfrak{a}^n = (A/\mathfrak{a}^n)^\wedge$. The proposition follows immediately from the definitions. $\square$

We say that the formal prescheme defined above is the completion of $X$ along $X'$, and we denote it by $X_{/X'}$ or $\hat{X}$ when there is little chance of confusion. When we take $X' = X$, we can set $\mathcal{J} = 0$, and we thus have $X_{/X} = X$. 3
It is clear that, if \( U \) is a prescheme induced on an open subset of \( X \), then \( U/(U\cap X') \) is canonically identified with the formal prescheme induced on \( X/X' \) by the open subset \( U \cap X' \) of \( X' \).

**Corollary (10.8.6).** — The (usual) prescheme \( \tilde{X}_{\text{red}} \) is the unique reduced prescheme of \( X \) having \( X' \) as its underlying space (5.2.1). For \( X \) to be Noetherian, it is necessary and sufficient for \( \tilde{X}_{\text{red}} \) to be Noetherian, and it suffices that \( X \) be Noetherian.

**Proof.** Since \( \tilde{X}_{\text{red}} \) is determined locally (10.5.4), we can assume that \( X \) is an affine scheme of some Noetherian ring \( A \); with the notation of Proposition (10.8.5), the ideal \( \mathfrak{A} \) of topologically nilpotent elements of \( \hat{A} \) is the inverse image under the canonical map \( \hat{A} \to \hat{A}/\hat{\mathfrak{A}} = A/\mathfrak{A} \) of the nilradical of \( A/\mathfrak{A} \) (0, 7.1.3), so \( \hat{A}/\mathfrak{A} \) is isomorphic to the quotient of \( A/\mathfrak{A} \) by its nilradical. The first claim then follows from Propositions (10.5.4) and (5.1.1). If \( \tilde{X}_{\text{red}} \) is Noetherian, then so too is its underlying space \( X' \), and so the \( X' = \text{Spec}(\mathcal{O}_X/\mathcal{J}) \) are Noetherian (6.1.2), and thus so too is \( \hat{X} \) (10.6.4); the converse is immediate, by Proposition (6.1.2).

**Proposition (10.8.7).** The canonical homomorphisms \( \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J} \) (for \( \mathcal{J} \in \Phi \)) form a projective system, and give, by passing to the projective limit, a homomorphism of sheaves of rings \( \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J} \), where \( \Phi \) denotes the canonical injection \( X' \to X \) of the underlying spaces. We denote by \( i \) (or \( i_X \)) the morphism (said to be canonical)

\[
(p, \theta) : X/X' \to X
\]

of ringed spaces.

By taking tensor products, for every coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \), the canonical homomorphisms \( \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J} \) give homomorphisms \( \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{J}) \) of \( \mathcal{O}_X \)-modules which form a projective system, and thus give, by passing to the projective limit, a canonical functorial homomorphism \( \gamma : \mathcal{F} \to \mathcal{F}/\mathcal{J} \) of \( \mathcal{O}_X \)-modules.

**Proposition (10.8.8).** —

(i) The functor \( \mathcal{F}/\mathcal{J} \) (in \( \Phi \)) is exact.

(ii) The functorial homomorphism \( \gamma^2 : i^*(\mathcal{F}) \to \mathcal{F}/\mathcal{J} \) of \( \mathcal{O}_X/\mathcal{J} \)-modules is an isomorphism.

**Proof.**

(i) It suffices to prove that, if \( 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \) is an exact sequence of coherent \( \mathcal{O}_X \)-modules, and if \( U \) is an affine open subset of \( X \) with Noetherian ring \( A \), then the sequence

\[
0 \to \Gamma(U \cap X', \mathcal{F}/\mathcal{J}) \to \Gamma(U \cap X', \mathcal{F}) \to \Gamma(U \cap X', \mathcal{F}'') \to 0
\]

is exact. We have that \( \mathcal{F}|U = \hat{M}, \mathcal{F}'|U = \hat{M}', \) and \( \mathcal{F}''|U = \hat{M}'' \), where \( M, M', \) and \( M'' \) are three \( A \)-modules of finite type such that the sequence \( 0 \to M' \to M \to M'' \to 0 \) is exact ((1.5.1) and (1.3.11)); let \( \mathcal{J} \in \Phi \), and let \( \hat{\mathfrak{A}} \) be an ideal of \( A \) such that \( \mathcal{J} \cap U = \hat{\mathfrak{A}} \). We then have

\[
\Gamma(U \cap X', \mathcal{F}/\mathcal{J}) = \lim_{\rightarrow} (M \otimes_A (A/\hat{\mathfrak{A}})) = \hat{M},
\]

(3.12); so, by definition of the projective limit, we have

\[
\Gamma(U \cap X', \mathcal{F}/\mathcal{J}) = \hat{M}, \quad \Gamma(U \cap X', \mathcal{F}'', \mathcal{F}/\mathcal{J}) = \hat{M}^0;
\]

our claim then follows, since \( A \) is Noetherian, and since the functor \( \hat{M} \) in \( M \) is exact on the category of \( A \)-modules of finite type (0, 7.3.3).
(ii) The question is local, so we can assume that we have an exact sequence \( \mathcal{O}_X^m \to \mathcal{O}_X^n \to \mathcal{F} \to 0 \) (0, 5.3.2); since \( \gamma^1 \) is functorial, and the functors \( i^*(\mathcal{F}) \) and \( \mathcal{F}/X' \) are right exact by (i) and (0, 4.3.1), we have the commutative diagram

\[
\begin{array}{ccc}
i^*(\mathcal{O}_X^m) & \to & i^*(\mathcal{O}_X^n) \to i^*(\mathcal{F}) \to 0 \\
\gamma^1 & \downarrow & \gamma^2 \\
(\mathcal{O}_X^m)/X' & \to & (\mathcal{O}_X^n)/X' \to \mathcal{F}/X' \to 0
\end{array}
\]

whose rows are exact. Furthermore, the functors \( i^*(\mathcal{F}) \) and \( \mathcal{F}/X' \) commute with finite direct sums ((0, 3.2.6) and (0, 4.3.2)), and we thus reduce to proving our claim for \( \mathcal{F} = \mathcal{O}_X \). We have \( i^*(\mathcal{O}_X) = (\mathcal{O}_X)/X' = \mathcal{O}_X(0, 4.3.4) \), and that \( \gamma^2 \) is a homomorphism of \( \mathcal{O}_X \)-modules; so it suffices to check that \( \gamma^2 \) sends the unit section of \( \mathcal{O}_X \) over an open subset of \( X' \) to itself, which is immediate, and shows, in this case, that \( \gamma^2 \) is the identity.

\[\square\]

**Corollary (10.8.9).** — The morphism \( i : X_{/X'} \to X \) of ringed spaces is flat.

**Proof.** This follows from (0, 6.7.3) and Proposition (10.8.8, i).

\[\square\]

**Corollary (10.8.10).** — If \( \mathcal{F} \) and \( \mathcal{G} \) are coherent \( \mathcal{O}_X \)-modules, then there exist canonical functorial (in \( \mathcal{F} \) and \( \mathcal{G} \)) isomorphisms

\[
(\mathcal{F}/X') \otimes (\mathcal{O}_X/X') (\mathcal{G}/X') \simeq (\mathcal{F} \otimes \mathcal{O}_X \mathcal{G})/X',
\]

(10.8.10.2)

\[
(\mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))/X' \simeq \mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}/X', \mathcal{G}/X').
\]

**Proof.** This follows from the canonical identification of \( i^*(\mathcal{F}) \) with \( \mathcal{F}/X' \); the existence of the first isomorphism is then a result which holds for all morphisms of ringed spaces (0, 4.3.1), and the second is a result which holds for all flat morphisms (0, 6.7.6), by Corollary (10.8.9).

\[\square\]

**Proposition (10.8.11).** — For every coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \), the canonical homomorphism \( \Gamma(X, \mathcal{F}) \to \Gamma(X', \mathcal{F}/X') \) induced by \( \mathcal{F} \to \mathcal{F}/X' \) has kernel consisting of the zero sections in some neighborhood of \( X' \).

**Proof.** It follows from the definition of \( \mathcal{F}/X' \) that the canonical image of such a section is zero. Conversely, if \( s \in \Gamma(X, \mathcal{F}) \) has a zero image in \( \Gamma(X', \mathcal{F}/X') \), then it suffices to see that every \( x \in X' \) admits a neighborhood in \( X \) in which \( s \) is zero, and we can thus reduce to the case where \( X = \text{Spec}(A) \) is affine, \( A \) Noetherian, \( X' = V(J) \) for some ideal \( J \) of \( A \), and \( \mathcal{F} = \hat{M} \) for some \( A \)-module \( M \) of finite type. Then \( \Gamma(X', \mathcal{F}/X') \) is the separated completion \( \hat{M} \) of \( M \) for the \( J \)-adic topology, and the homomorphism \( \Gamma(X, \mathcal{F}) \to \Gamma(X', \mathcal{F}/X') \) is the canonical homomorphism \( M \to \hat{M} \). We know (0, 7.3.7) that the kernel of this homomorphism is the set of the \( z \in M \) killed by an element of \( 1 + J \). So we have \( (1 + f)s = 0 \) for some \( f \in J \); for every \( x \in X' \) we have \( (1 + f_x)s_x = 0 \), and, since \( 1_x + f_x \) is invertible in \( \mathcal{O}_x(J_x \mathcal{O}_x) \) being contained in the maximal ideal of \( \mathcal{O}_x \), we have \( s_x = 0 \), which proves the proposition.

\[\square\]

**Corollary (10.8.12).** — The support of \( \mathcal{F}/X' \) is equal to \( \text{Supp}(\mathcal{F}) \cap X' \).

**Proof.** It is clear that \( \mathcal{F}/X' \) is an \( (\mathcal{O}_X/X') \)-module of finite type ((10.8.8, ii) and (0, 5.2.4)), so its support is closed (0, 5.2.2) and evidently contained in \( \text{Supp}(\mathcal{F}) \cap X' \). To show that it is equal to the latter set, we immediately reduce to proving that the equation \( \Gamma(X', \mathcal{F}/X') = 0 \) implies that \( \text{Supp}(\mathcal{F}) \cap X' = \emptyset \); this follows from Proposition (10.8.11) and Theorem (1.4.1).

\[\square\]

**Corollary (10.8.13).** — Let \( u : \mathcal{F} \to \mathcal{G} \) be a homomorphism of coherent \( \mathcal{O}_X \)-modules. For \( u_{/X'} : \mathcal{F}/X' \to \mathcal{G}/X' \) to be zero, it is necessary and sufficient for \( u \) to be zero on a neighborhood of \( X' \).

**Proof.** By Proposition (10.8.8, ii), \( u_{/X'} \) can be identified with \( i^*(u) \), so, if we consider \( u \) as a section over \( X \) of the sheaf \( \mathcal{H} = \mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \), then \( u_{/X'} \) is the section over \( X' \) of \( i^*(\mathcal{H}) = \mathcal{H}/X' \), to which it canonically corresponds (10.8.10.2 and (0, 4.4.6)). It thus suffices to apply Proposition (10.8.11) to the coherent \( \mathcal{O}_X \)-module \( \mathcal{H} \).

\[\square\]
Corollary (10.8.14). — Let \( u : \mathcal{F} \to \mathcal{G} \) be a homomorphism of coherent \( \mathcal{O}_X \)-modules. For \( u_{/X'} \) to be a monomorphism (resp. an epimorphism), it is necessary and sufficient for \( u \) to be a monomorphism (resp. an epimorphism) on a neighborhood of \( X' \).

Proof. Let \( \mathcal{P} \) and \( \mathcal{N} \) be the cokernel and kernel (respectively) of \( u \), so that we have the exact sequence \( 0 \to \mathcal{N} \mathord{\xrightarrow{\delta}} \mathcal{F} \mathord{\xrightarrow{u}} \mathcal{G} \mathord{\xrightarrow{w}} \mathcal{P} \to 0 \), hence (10.8.8, i) the exact sequence

\[
0 \longrightarrow \mathcal{N}_{/X'} \mathord{\xrightarrow{v_{/X'}}} \mathcal{F}_{/X'} \mathord{\xrightarrow{u_{/X'}}} \mathcal{G}_{/X'} \mathord{\xrightarrow{w_{/X'}}} \mathcal{P}_{/X'} \longrightarrow 0.
\]

If \( u_{/X'} \) is a monomorphism (resp. an epimorphism), then we have \( v_{/X'} = 0 \) (resp. \( w_{/X'} = 0 \)), so there exists a neighborhood of \( X' \) on which \( v = 0 \) (resp. \( w = 0 \)) by Corollary (10.8.13).

10.9. Extension of morphisms to completions

(10.9.1). Let \( X \) and \( Y \) be locally Noetherian (usual) preschemes, \( f : X \to Y \) a morphism, and \( X' \) (resp. \( Y' \)) a closed subset of the underlying space \( X \) (resp. \( Y \)) such that \( f(X') \subset Y' \). Let \( \mathcal{J} \) (resp. \( \mathcal{K} \)) be a sheaf of ideals of \( \mathcal{O}_X \) (resp. \( \mathcal{O}_Y \)) such that the support of \( \mathcal{O}_X / \mathcal{J} \) (resp. \( \mathcal{O}_Y / \mathcal{K} \)) is \( X' \) (resp. \( Y' \)) and \( \mathcal{J}(X') \subset \mathcal{J} \); we note that there always exist such sheaves of ideals, since, for example, we can take \( \mathcal{J} \) to be the largest sheaf of ideals of \( \mathcal{O}_X \) defining a subscheme of \( X \) with underlying space \( X' \) (5.2.1), and the hypothesis \( f(X') \subset Y' \) implies that \( f^*(\mathcal{K}) \mathcal{O}_X \subset \mathcal{J} \) (5.2.4). For every integer \( n > 0 \) we have \( f^*(\mathcal{K}^n) \mathcal{O}_X \subset \mathcal{J}^n \) (0.4.3.5); as a result (4.4.6), if we set \( X'_n = (X', \mathcal{O}_X / \mathcal{J}^n) \) and \( Y'_n = (Y', \mathcal{O}_Y / \mathcal{K}^n) \), then \( f \) induces a morphism \( f_n : X'_n \to Y'_n \), and it is immediate that the \( f_n \) form an inductive system. We denote its inductive limit (10.8.6) by \( \hat{f} : X'/X \to Y'/Y \), and we say (by abuse of language) that \( \hat{f} \) is the extension of \( f \) to the completions of \( X \) and \( Y \) along \( X' \) and \( Y' \). It can be checked immediately that this morphism does not depend on the choice of sheaves of ideals \( \mathcal{J} \) and \( \mathcal{K} \) satisfying the above conditions. It suffices to consider the case where \( X \) and \( Y \) are Noetherian affine schemes with rings \( A \) and \( B \) (respectively); then \( \mathcal{J} = \mathfrak{J} \) and \( \mathcal{K} = \mathfrak{K} \), where \( \mathfrak{J} \) (resp. \( \mathfrak{K} \)) is an ideal of \( A \) (resp. \( B \)), \( f \) corresponds to a ring homomorphism \( \phi : B \to A \) such that \( \phi(\mathfrak{K}) \subset \mathfrak{J} \) (4.4.6 and (1.7.4)); \( \hat{f} \) is then the morphism corresponding (10.2.2) to the continuous homomorphism \( \hat{f} : \hat{B} \to \hat{A} \), where \( \hat{A} \) (resp. \( \hat{B} \)) is the separated completion of \( A \) (resp. \( B \)) with respect to the \( \mathfrak{J} \)-adic (resp. \( \mathfrak{K} \)-adic) topology (10.6.8); we know that, if we replace \( \mathcal{J} \) by another sheaf of ideals \( \mathcal{J}' = \mathfrak{J}' \) such that the support of \( \mathcal{O}_X / \mathcal{J}' \) is \( X' \), then the \( \mathfrak{J} \)-adic and \( \mathfrak{J}' \)-adic topologies on \( A \) are the same (10.82).

We note that, by definition, the continuous map \( X' \to Y' \) of the underlying spaces of \( X'/X \) and \( Y'/Y \) corresponding to \( \hat{f} \) is exactly the restriction to \( X' \) of \( f \).

(10.9.2). It follows immediately from the above definition that the diagram of morphisms of ringed spaces

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{\hat{f}} & \hat{Y} \\
ix \downarrow & & \downarrow iy \\
X & \xrightarrow{f} & Y
\end{array}
\]

commutes, with the vertical arrows being the canonical morphisms (10.8.7).

(10.9.3). Let \( Z \) be a third prescheme, \( g : Y \to Z \) a morphism, and \( Z' \) a closed subset of \( Z \) such that \( g(Y') \subset Z' \). If \( \hat{g} \) denotes the completion of the morphism \( g \) along \( Y' \) and \( Z' \), then it immediately follows from (10.9.1) that we have \( (g \circ f)^{\wedge} = \hat{g} \circ \hat{f} \).

Proposition (10.9.4). — Let \( X \) and \( Y \) be locally Noetherian \( S \)-preschemes, with \( Y \) of finite type over \( S \). Let \( f \) and \( g \) be \( S \)-morphisms from \( X \) to \( Y \) such that \( f(X') \subset Y' \) and \( g(X') \subset Y' \). For \( \hat{f} = \hat{g} \) to hold, it is necessary and sufficient for \( f \) and \( g \) to coincide on a neighborhood of \( X' \).

Proof. The condition is evidently sufficient (even without the finiteness hypothesis on \( Y \)). To see that it is necessary, we remark first that the hypothesis \( \hat{f} = \hat{g} \) implies that \( f(x) = g(x) \) for all \( x \in X' \). Also, since the questions is local, we can assume that \( X \) and \( Y \) are affine open neighborhoods of \( x \) and \( y = f(x) = g(x) \) respectively (with Noetherian rings), that \( S \) is affine, and that \( \Gamma(Y, \mathcal{O}_Y) \) is a \( \Gamma(S, \mathcal{O}_S) \)-algebra of finite type (6.3.3). Then \( f \) and \( g \) correspond to \( \Gamma(S, \mathcal{O}_S) \)-homomorphisms \( \rho \) and
\( \sigma \) (respectively) from \( \Gamma(\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \) to \( \Gamma(X, \mathcal{O}_X) \) (1.7.3), and, by hypothesis, the extensions by continuity of these homomorphisms to the separated completion of \( \Gamma(\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \) are the same. We conclude from Proposition (10.8.11) that, for every section \( s \in \Gamma(\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \), the sections \( \rho(s) \) and \( \sigma(s) \) coincide on a neighborhood (depending on \( s \)) of \( X' \); since \( \Gamma(\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \) is an algebra of finite type over \( \Gamma(S, \mathcal{O}_S) \), we have that there exists a neighborhood \( V \) of \( X' \) such that \( \rho(s) \) and \( \sigma(s) \) coincide on \( V \) for every section \( s \in \Gamma(\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \). If \( h \in \Gamma(X, \mathcal{O}_X) \) is such that \( D(h) \) is a neighborhood of \( x \) contained in \( V \), then we conclude from the above and from Theorem (1.4.1, d) that \( f \) and \( g \) coincide on \( D(h) \). \( \square \)

**Proposition (10.9.5).** — Under the hypotheses of (10.9.1), for every coherent \( \mathcal{O}_\mathcal{Y} \)-module \( \mathcal{G} \), there exists a canonical functorial isomorphism of \( (\mathcal{O}_X)_{\mathcal{X}} \)-modules

\[
(f^*(\mathcal{G}))_{X'} \simeq \hat{\mathcal{F}}^*(\mathcal{G}_{\mathcal{Y}'}).
\]

**Proof.** If we canonically identify \( (f^*(\mathcal{G}))_{X'} \) with \( i_X^*(f^*(\mathcal{G})) \), and \( \hat{\mathcal{F}}^*(\mathcal{G}_{\mathcal{Y}'}) \) with \( \hat{\mathcal{F}}^*(i_Y^*(\mathcal{G})) \) (10.8.8), then the proposition follows immediately from the commutativity of the diagram in (10.9.2). \( \square \)

\( I \mid 200 \)

**Proposition (10.9.7).** — Let \( S, X, \) and \( Y \) be locally Noetherian preschemes, \( g : X \to S \) and \( h : Y \to S \) morphisms, \( S' \) a closed subset of \( S \), and \( X' \) (resp. \( Y' \)) a closed subset of \( X \) (resp. \( Y \)) such that \( g(X') \subset S' \) (resp. \( h(Y') \subset S' \)); let \( Z = X \times_S Y \); suppose that \( Z \) is locally Noetherian, and let \( Z' = p^{-1}(X') \cap q^{-1}(Y') \), where \( p \) and \( q \) are the projections of \( X \times_S Y \). With these conditions, the completion \( Z_{/Z'} \) can be identified with the product \( (X_{/X'}) \times_{S_{/S'}} (Y_{/Y'}) \) of formal \( S_{/S'} \)-preschemes, where the structure morphisms are identified with \( \hat{g} \) and \( \hat{h} \), and the projections with \( \tilde{p} \) and \( \tilde{q} \).

**Proof.** The fact that the completion is local for \( S, X, \) and \( Y \), and we thus reduce to the case where \( S = \text{Spec}(A) \), \( X = \text{Spec}(B) \), \( Y = \text{Spec}(C) \), \( S' = V(\mathfrak{I}) \), \( X' = V(\mathfrak{R}) \), and \( Y' = V(\mathfrak{L}) \), with \( \mathfrak{J}, \mathfrak{R}, \) and \( \mathfrak{L} \) ideals such that \( \mathfrak{J} \subset \mathfrak{R} \) and \( \mathfrak{J} \subset \mathfrak{L} \), where we denote by \( \bar{\mathfrak{I}} \) and \( \bar{\mathfrak{R}} \) the homomorphisms \( A \to B \) and \( A \to C \) which correspond to \( g \) and \( h \) (respectively). We know that \( Z = \text{Spec}(B \otimes_A C) \) and that \( Z' = V(\mathfrak{M}) \), where \( \mathfrak{M} \) is the ideal \( \text{Im}(\bar{\mathfrak{I}} \otimes_A \mathfrak{R}) + \text{Im}(\mathfrak{R} \otimes_A \mathfrak{L}) \). The conclusion follows (10.7.2) from the fact that the completed tensor product \( \hat{B} \otimes_{\bar{A}} \hat{C} \) (where \( \hat{A}, \hat{B}, \) and \( \hat{C} \) are, respectively, the separated completions of \( A, B, \) and \( C \) with respect to the \( \mathfrak{J}, \mathfrak{R}, \) and \( \mathfrak{L} \)-preadic topologies) is the separated completion of the tensor product \( B \otimes_A C \) with respect to the \( \mathfrak{M} \)-preadic topology (0, 7.7.2). \( \square \)

In addition, we note that, if \( T \) is a locally Noetherian \( S \)-prescheme, \( u : T \to X \) and \( v : T \to Y \) both \( S \)-morphisms, and \( T' \) a closed subset of \( T \) such that \( u(T') \subset X' \) and \( v(T') \subset Y' \), then the extension to the completion \( (u, v)(T')_{S/\mathcal{S}} \) can be identified with \( (\bar{u}, \bar{v})(T')_{S/\mathcal{S}} \).

**Corollary (10.9.8).** — Let \( X \) and \( Y \) be locally Noetherian \( S \)-preschemes such that \( X \times_S Y \) is locally Noetherian; let \( S' \) be a closed subset of \( S \), and \( X' \) (resp. \( Y' \)) a closed subset of \( X \) (resp. \( Y \)) whose image in \( S \) is contained in \( S' \). For every \( S \)-morphism \( f : X \to Y \) such that \( f(X') \subset Y' \), the graph morphism \( \Gamma_f \) can be identified with the extension \( (\Gamma_f)^{\hat{\lambda}} \) of the graph morphism of \( f \).
Corollary (10.9.9). — Let $X$ and $Y$ be locally Noetherian preschemes, $f : X \to Y$ a morphism, $Y'$ a closed subset of $Y$, and $X' = f^{-1}(Y')$. Then the prescheme $X_{/X'}$ can be identified, by the commutative diagram

$$
\begin{array}{ccc}
X & \xleftarrow{f} & X_{/X'} \\
\downarrow & & \downarrow \tilde{f} \\
Y & \xleftarrow{g} & Y_{/Y'}
\end{array}
$$

with the product $X \times_Y (Y_{/Y'})$ of formal preschemes.

PROOF. It suffices to apply Proposition (10.9.7), replacing $S$ and $S'$ by $Y$, and $X$ and $X'$ by $X$. □

Remark (10.9.10). — If $S$ is the sum $X_1 \sqcup X_2$ (3.1), $X'$ the union $X_1' \sqcup X_2'$, where $X_i'$ is a closed subset of $X_i$ ($i = 1, 2$), then we have $X_{/X'} = X_1_{/X_1'} \sqcup X_2_{/X_2'}$.

10.10. Application to coherent sheaves on formal affine schemes

(10.10.1). In this paragraph, $A$ denotes an adic Noetherian ring, and $\mathfrak{J}$ an ideal of definition for $A$. Let $X = \text{Spec}(A)$, and $\mathfrak{X} = \text{Spf}(A)$, which can be identified with the closed subset $V(\mathfrak{J})$ of $X$ (10.1.2). In addition, Definitions (10.1.2) and (10.8.4) show that the formal affine scheme $\mathfrak{X}$ is identical to the completion $X_{/\mathfrak{X}}$ of the affine scheme $X$ along the closed subset $\mathfrak{X}$ of its underlying space. To every coherent $\mathcal{O}_X$-module $\mathcal{F}$, there corresponds an $\mathcal{O}_{X_{/\mathfrak{X}}}$-module of finite type $\mathcal{F}_{/\mathfrak{X}}$, which is a sheaf of topological modules over the sheaf of topological rings $\mathcal{O}_{X_{/\mathfrak{X}}}$. Every coherent $\mathcal{O}_X$-module $\mathcal{F}$ is of the form $M$, where $M$ is an $A$-module of finite type (1.5.1); we set $(\tilde{M})_{/X} = M_{/\mathfrak{X}}$. In addition, if $u : M \to N$ is an $A$-homomorphism of $A$-modules of finite type, then it corresponds to a homomorphism $\tilde{u} : \tilde{M} \to \tilde{N}$, and, as a result, to a continuous homomorphism $\tilde{u}_{/X} : (\tilde{M})_{/X} \to (\tilde{N})_{/X}$, which we denote by $u^A$. It is immediate that $(v \circ u)^A = v^A \circ u^A$; we have thus defined a covariant additive functor $M^A$ from the category of $A$-modules of finite type to the category of $\mathcal{O}_{X_{/\mathfrak{X}}}$-modules of finite type. When $A$ is a discrete ring, we have $M^A = \tilde{M}$.

Proposition (10.10.2). —

(i) $M^A$ is an exact functor in $M$, and there exists a canonical functorial isomorphism of $A$-modules

$$
\Gamma(\mathfrak{X}, M^A) \simeq M.
$$

(ii) If $M$ and $N$ are $A$-modules of finite type, then there exist canonical functorial isomorphisms

$$
(M \otimes_A N)^A \simeq M^A \otimes_{\mathcal{O}_{X_{/\mathfrak{X}}}} N^A,
$$

$$
(\text{Hom}_A(M, N))^A \simeq \text{Hom}_{\mathcal{O}_{X_{/\mathfrak{X}}}}(M^A, N^A).
$$

(iii) The map $u \mapsto u^A$ is a functorial isomorphism

$$
\text{Hom}_A(M, N) \simeq \text{Hom}_{\mathcal{O}_{X_{/\mathfrak{X}}}}(M^A, N^A).
$$

PROOF. The exactness of $M^A$ follows from the exactness of the functors $\tilde{M}$ (1.3.5) and $\mathcal{F}_{/X}$ (10.8.8). By definition, $\Gamma(X, M^A)$ is the separated completion of the $A$-module $\Gamma(X, \tilde{M}) = M$ with respect to the $\mathfrak{J}$-preadic topology; but, since $A$ is complete and $M$ is of finite type, we know (0.7.3.6) that $M$ is separated and complete, which proves (i). The isomorphism (10.10.2.1) (resp. (10.10.2.2)) comes from the composition of the isomorphisms (1.3.12, i) and (10.8.10.1) (resp. (1.3.12, ii) and (10.8.10.2)). Finally, since $\text{Hom}_A(M, N)$ is an $A$-module of finite type, we can apply (i), which identifies $\Gamma(\mathfrak{X}, (\text{Hom}_A(M, N))^A)$ with $\text{Hom}_A(M, N)$, and we can use (10.10.2.2), which proves that the homomorphism (10.10.2.3) is an isomorphism. □

We deduce from Proposition (10.10.2) a series of results analogous to those of Theorem (1.3.7) and Corollary (1.3.12), whose formulation we leave to the reader.

We note that the exactness property of $M^A$, applied to the exact sequence $0 \to \mathfrak{J} \to A \to A/\mathfrak{J} \to 0$, shows that the sheaf of ideals of $\mathcal{O}_X$ denoted here by $\mathfrak{J}^A$ coincides with the one denoted also by $\mathfrak{J}^A$ in (10.3.1), by (10.3.2).

Proposition (10.10.3). — Under the hypotheses of (10.10.1), $\mathcal{O}_X$ is a coherent sheaf of rings.
Proof. If \( f \in A \), then we know that \( A_f \) is an adic Noetherian ring (0.7.6.11), and since the question is local, we reduce (10.1.4) to proving that the kernel of the homomorphism \( v : O^n_X \to O_X \) is an \( \mathcal{O}_X \)-module of finite type. We then have \( v = u^A \), where \( u \) is an \( A \)-homomorphism \( A^n \to A \) (10.10.2); since \( A \) is Noetherian, the kernel of \( u \) is of finite type, or, equivalently, we have a homomorphism \( A^m \to A^n \) such that the sequence \( A^m \to A^n \xrightarrow{\phi} A \) is exact. We conclude (10.10.2) that the sequence \( O^m \xrightarrow{u^A} O^n \xrightarrow{v} O_X \) is exact, which proves that the kernel of \( v \) is of finite type. \( \square \)

(10.10.4). With the above notation, set \( A_n = A / \mathfrak{J}^{n+1} \), and let \( S_n \) be the affine scheme \( \text{Spec}(A_n) = (X, \mathcal{O}_X / \mathfrak{J}^{n+1}) \), with \( \mathfrak{J} = \mathfrak{J}^n \) the sheaf of ideals of definition for \( \mathcal{O}_X \) corresponding to the ideal \( \mathfrak{J} \). Let \( u_{mn} \) be the morphism of preschemes \( X_m \to X_n \) corresponding to the canonical homomorphism \( A_n \to A_m \) for \( m \leq n \); the formal scheme \( X \) is the inductive limit of the \( X_n \) with respect to the \( u_{mn} \) (10.6.3).

Proposition (10.10.5). Under the hypothesis of (10.10.1), let \( \mathcal{F} \) be an \( \mathcal{O}_X \)-module. The following conditions are equivalent:

(a) \( \mathcal{F} \) is a coherent \( \mathcal{O}_X \)-module;
(b) \( \mathcal{F} \) is isomorphic to the projective limit (10.6.6) of a sequence \((\mathcal{F}_n)\) of coherent \( \mathcal{O}_{X_n} \)-modules such that \( u_{n+m}^*(\mathcal{F}_n) = \mathcal{F}_m \); and
(c) there exists an \( A \)-module \( M \) of finite type (determined up to canonical isomorphism by Proposition (10.10.2), i) such that \( \mathcal{F} \) is isomorphic to \( M^A \).

Proof. We first show that (b) implies (c). We have \( \mathcal{F}_n = \overline{M}_n \), where \( M_n \) is an \( A_n \)-module of finite type, and the hypotheses imply that \( M_m = M_n \otimes_{A_n} A_m \) for \( m \leq n \) (1.6.5); the \( M_n \) thus form a projective system for the canonical di-homomorphisms \( M_n \to M_m \) (\( m \leq n \)), and it follows immediately from the definition of the \( A_n \) that this projective system satisfies the conditions of (0.7.9); as a result, its projective limit \( M \) is a module of finite type such that \( M_n = M \otimes_{A_n} A_m \) for all \( n \). We deduce that \( \mathcal{F}_n \) is induced over \( X_n \) by \( \overline{M} \otimes_{\mathcal{O}_X} (\mathcal{O}_X / \mathfrak{J}^{n+1}) \), and so \( \mathcal{F} = M^A \) by Definition (10.8.4).

Conversely, (c) implies (b); indeed, if \( u_n \) is the immersion morphism \( X_n \to X \), then \( u_{n+m}^*(\overline{M}) = (M \otimes_{A_n} A_m)^A \) is induced over \( X_n \) by \( \overline{M} \otimes_{\mathcal{O}_X} (\mathcal{O}_X / \mathfrak{J}^{n+1}) \), and \( M^A = \lim_{\leftarrow} u_{n+m}^*(\overline{M}) \) by Definition (10.8.4); since \( u_m = u_n \circ u_{mn} \) for \( m \leq n \), the \( \mathcal{F}_n = u_{n+m}^*(\overline{M}) \) satisfy the conditions of (b), whence our claim.

We now show that (c) implies (a): indeed, we have, by definition, that \( \mathcal{O}_X = A \); since \( M \) is the cokernel of a homomorphism \( A^m \to A^n \), it follows from Proposition (10.10.2) that \( M^n \) is the cokernel of a homomorphism \( O^m_X \to O^n_X \) and, since the sheaf of rings \( \mathcal{O}_X \) is coherent (10.10.3), so too is \( M^n \) (0.5.3.4).

Finally, (a) implies (b). Considered as an \( \mathcal{O}_X \)-module, we have that \( \mathcal{O}_{X_n} = \mathcal{O}_X / \mathfrak{J}^{n+1} = A^n_n \), but \( \mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n} \) is a coherent \( \mathcal{O}_X \)-module (0.5.3.5), and, since it is also an \( \mathcal{O}_{X_n} \)-module, and \( \mathfrak{J}^{n+1} \) is coherent, we conclude that \( \mathcal{F}_n \) is a coherent \( \mathcal{O}_{X_n} \)-module (0.5.3.10), and it is immediate that \( u_{n+m}^*(\mathcal{F}_n) = \mathcal{F}_m \) for \( m \leq n \) (recalling that the continuous map \( X_m \to X_n \) of the underlying spaces is the identity on \( X \)). The sheaf \( \mathcal{F} = \varprojlim \mathcal{F}_n \) is thus a coherent \( \mathcal{O}_X \)-module, since we have seen that (b) implies (a). The canonical homomorphisms \( \mathcal{F} \to \mathcal{F}_n \) form a projective system, which, by passing to the limit, gives a canonical homomorphism \( w : \mathcal{F} \to \mathcal{G} \), and it remains only to prove that \( w \) is bijective. The question is now local, so we can reduce to the case where \( \mathcal{F} \) is the cokernel of a homomorphism \( O^m_X \to O^n_X \); since this homomorphism is of the form \( v^A \), where \( v \) is a homomorphism \( A^m \to A^n \) (10.10.2), \( \mathcal{F} \) is isomorphic to \( M^A \), where \( M = \text{Coker} v \) (10.10.2). We then have, by Proposition (10.10.2), that \( \mathcal{F}_n = M^A \otimes_{\mathcal{O}_X} A^A_n = (M \otimes_{A_n} A_n)^A \), and, since the \( \mathfrak{J} \)-adic topology on \( M \otimes_{A_n} A_n \) is discrete, we have \((M \otimes_{A_n} A_n)^A = (M \otimes_{A_n} A_n)^A \) (as an \( \mathcal{O}_{X_n} \)-module); we have seen above that \( M^A = \varprojlim \mathcal{F}_n \), and \( w \) is thus the identity in this case. \( \square \)

Corollary (10.10.6). If \( \mathcal{F} \) satisfies condition (b) of Proposition (10.10.5), then the projective system \((\mathcal{F}_n)\) is isomorphic to the system of the \( \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n} \).

(10.10.7). Now let \( A \) and \( B \) be adic Noetherian rings, and \( \phi : B \to A \) a continuous homomorphism; we denote by \( \mathfrak{J} \) (resp. \( \mathfrak{R} \)) an ideal of definition for \( A \) (resp. \( B \)) such that \( \phi(\mathfrak{R}) \subset \mathfrak{J} \), and we set \( X = \text{Spec}(A), Y = \text{Spec}(B), X = \text{Spf}(A), \) and \( \mathfrak{J} = \text{Spf}(B) \). Let \( f : X \to Y \) be the morphism of
preschemes corresponding to $\phi$ (1.6.1), and $\hat{f} : \mathcal{X} \to \mathcal{Y}$ its extension to the completions (10.9.1), which is also a morphism of formal preschemes that corresponds to $\phi$ (10.2.2).

**Proposition (10.10.8).** — For every $B$-module $N$ of finite type, there exists a canonical functorial isomorphism of $\mathcal{O}_X$-modules

$$\hat{f}^*(N^A) \simeq (N \otimes_B A)^A.$$

**Proof.** Denoting by $i_X : \mathcal{X} \to X$ and $i_Y : \mathcal{Y} \to Y$ the canonical morphisms, we have (10.8.8), up to canonical functorial isomorphisms, $N^A = i_Y^*(\tilde{N})$ and

$$(N \otimes_B A)^A = i_X^*((N \otimes_B A)^\gamma) = i_X^*(f^*(\tilde{N}))$$

(1.6.5); the proposition then follows from the commutativity of the diagram in (10.9.2). \[ \square \]

**Corollary.** — For every ideal $b$ of $B$, we have $\hat{f}^*(b^A)\mathcal{O}_X = (bA)^A$.

**Proof.** Let $j$ be the canonical injection $b \to B$, to which corresponds the canonical injection $j^A : b^A \to \mathcal{O}_Y$ of sheaves of $\mathcal{O}_Y$-modules; by definition, $\hat{f}^*(b^A)\mathcal{O}_X$ is the image of the homomorphism $\hat{f}^*(j^A) : \hat{f}^*(b^A) \to \hat{f}^*(\mathcal{O}_Y)$; but this homomorphism can be identified with $(j \otimes 1)^A : (b \otimes_B A)^A \to \mathcal{O}_X = (B \otimes_B A)^A$ by Proposition (10.10.8). Since the image of $j \otimes 1$ is the ideal $bA$ of $A$, the image of $(j \otimes 1)^A$ is thus $(bA)^A$, by Proposition (10.10.2), whence the conclusion. \[ \square \]

### 10.11. Coherent sheaves on formal preschemes

**Proposition (10.11.1).** — If $\mathcal{X}$ is a locally Noetherian formal prescheme, then the sheaf of rings $\mathcal{O}_X$ is coherent, and every sheaf of ideals of definition for $\mathcal{X}$ is coherent.

**Proof.** The question is local, so we can reduce to the case of a Noetherian affine formal scheme, and the proposition then follows from Propositions (10.10.3) and (10.10.5). \[ \square \]

**Theorem (10.11.3).** — For an $\mathcal{O}_X$-module $\mathcal{F}$ to be coherent, it is necessary and sufficient for it to be isomorphic to a projective limit of a sequence $(\mathcal{F}_n)$, where the $\mathcal{F}_n$ are coherent $\mathcal{O}_{X_n}$-modules such that $u^*_{nm}(\mathcal{F}_n) = \mathcal{F}_m$ for $m \leq n$ (10.6.6). The projective system $(\mathcal{F}_n)$ is then isomorphic to the system of the $u^*_n(\mathcal{F}) = \mathcal{F} \otimes \mathcal{O}_{X_n}$, where $u_n$ is the canonical morphism $X_n \to X$.

**Proof.** The question is local, so we can reduce to the case where $\mathcal{X}$ is a Noetherian affine formal scheme, and the theorem then is a consequence of Proposition (10.10.5) and Corollary (10.10.6). \[ \square \]

We can thus say that the data of a coherent $\mathcal{O}_X$-module is equivalent to the data of a projective system $(\mathcal{F}_n)$ of coherent $\mathcal{O}_{X_n}$-modules such that $u^*_{nm}(\mathcal{F}_n) = \mathcal{F}_m$ for $m \leq n$.

**Corollary (10.11.4).** — If $\mathcal{F}$ and $\mathcal{G}$ are coherent $\mathcal{O}_X$-modules, then we can (with the notation of Theorem (10.11.3)) define a canonical functorial isomorphism

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \simeq \lim_{\leftarrow n} \text{Hom}_{\mathcal{O}_{X_n}}(\mathcal{F}_n, \mathcal{G}_n).$$

**Proof.** The projective limit on the right-hand side is understood to be taken with respect to the maps $\theta_n \mapsto u^*_{nm}(\theta_n)$ ($m \leq n$) from $\text{Hom}_{\mathcal{O}_{X_n}}(\mathcal{F}_n, \mathcal{G}_n)$ to $\text{Hom}_{\mathcal{O}_{X_m}}(\mathcal{F}_m, \mathcal{G}_m)$. The homomorphism (10.11.4.1) sends an element $\theta \in \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ to the sequence $(u^*_n(\theta))$; we see that we can define an inverse homomorphism of the above by sending a projective system $(\theta_n) \in \lim_{\to n} \text{Hom}_{\mathcal{O}_{X_n}}(\mathcal{F}_n, \mathcal{G}_n)$ to its projective limit in $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, taking into account Theorem (10.11.3). \[ \square \]

**Corollary (10.11.5).** — For a homomorphism $\theta : \mathcal{F} \to \mathcal{G}$ to be surjective, it is necessary and sufficient for the corresponding homomorphism $\theta_0 = u^*_0(\theta) : \mathcal{F}_0 \to \mathcal{G}_0$ to be surjective.
Theorem (10.11.3) shows that we can consider every coherent $\mathcal{O}_X$-module $\mathcal{F}$ as a topological $\mathcal{O}_X$-module, considering it as a projective limit of pseudo-discrete sheaves of groups $\mathcal{F}_n$ (0, 3.8.1). It then follows from Corollary (10.11.4) that every homomorphism $u : \mathcal{F} \to \mathcal{G}$ of coherent $\mathcal{O}_X$-modules is automatically continuous (0, 3.8.2). Furthermore, if $\mathcal{H}$ is a coherent $\mathcal{O}_X$-submodule of a coherent $\mathcal{O}_X$-module $\mathcal{F}$, then, for every open $U \subset X$, $\Gamma(U, \mathcal{H})$ is a closed subgroup of the topological group $\Gamma(U, \mathcal{F})$, since the functor $\Gamma(U, -)$ is left exact, and $\Gamma(U, \mathcal{H})$ is the kernel of the homomorphism $\Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{F}/\mathcal{H})$, which is continuous by the above, since $\mathcal{F}/\mathcal{H}$ is coherent (0, 5.3.4); our claim follows from the fact that $\Gamma(U, \mathcal{F}/\mathcal{H})$ is a separated topological group.

**Proposition (10.11.7).** — Let $\mathcal{F}$ and $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. We can define (with the notation of Theorem (10.11.3)) canonical functorial isomorphisms of topological $\mathcal{O}_X$-modules (10.11.6)

\[
\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \simeq \lim_{\mathcal{n}} (\mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{G}_n),
\]

\[
\mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \simeq \lim_{\mathcal{n}} \mathcal{H}\text{om}_{\mathcal{O}_{X_n}}(\mathcal{F}_n, \mathcal{G}_n).
\]

**Proof.** The existence of the isomorphism (10.11.7.1) follows from the formula

\[
\mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{G}_0 = (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}_0) \otimes_{\mathcal{O}_{X_n}} (\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{G}_0) = (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{G}_0
\]

and from Theorem (10.11.3). The isomorphism (10.11.7.2), where both sides are considered as sheaves of modules without topology, follows from the definition of the sections of $\mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ and $\mathcal{H}\text{om}_{\mathcal{O}_{X_n}}(\mathcal{F}_n, \mathcal{G}_n)$, and from the existence of the isomorphism (10.11.4.1), mapping a presheaf induced on an arbitrary Noetherian formal affine open set to $X$. It remains to prove that the isomorphism (10.11.7.2) is bicontinuous over a quasi-compact set, and we can thus reduce to the case where $X = \text{Sp}(A)$ with $A$ an adic Noetherian ring, and hence (10.10.5) to the case where $\mathcal{F} = M^\Lambda$ and $\mathcal{G} = N^\Lambda$, with $M$ and $N$ both $A$-modules of finite type; taking (10.10.2.1), (10.10.2.3), and Corollary (1.3.12, ii) into account, we reduce to showing that the canonical isomorphism $\text{Hom}_A(M, N) \simeq \lim_{\mathcal{n}} \text{Hom}_{A_n}(M_n, N_n)$ (with $M_n = M \otimes_A A_n$ and $N_n = N \otimes_A A_n$) is continuous, which has already been proved in (0, 7.8.2).

**Proposition (10.11.8).** Since $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the group of sections of the sheaf of topological groups $\mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, it is equipped with a group topology. If $X$ is Noetherian, then it follows from (10.11.7.2) that the subgroups $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}^n)$ (for arbitrary $n$) form a fundamental system of neighborhoods of $0$ in this group.

**Proposition (10.11.9).** — Let $X$ be a Noetherian formal prescheme, and $\mathcal{F}$ and $\mathcal{G}$ coherent $\mathcal{O}_X$-modules. In the topological group $\mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, the surjective (resp. injective, bijective) homomorphisms form an open set.

**Proof.** By Corollary (10.11.5), the set of surjective homomorphisms in $\mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the inverse image under the continuous map $\mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}_0, \mathcal{G}_0)$ of a subset of the discrete group $\mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}_0, \mathcal{G}_0)$, whence the first claim. To show the second, we cover $X$ by a finite number of Noetherian formal affine subsets $U_i$. For $\theta \in \mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ to be injective, it is necessary and sufficient for all of the images under the (continuous) restriction maps $\mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}\text{om}_{\mathcal{O}_X(U_i)}(\mathcal{F}|U_i, \mathcal{G}|U_i)$ to be injective; we can thus reduce to the affine case, and then this has already been proved in (0, 7.8.3).
10.12. Adic morphisms of formal preschemes

(10.12.1). Let $\mathfrak{X}$ and $\mathfrak{S}$ be locally Noetherian formal preschemes; we say that a morphism $f : \mathfrak{X} \to \mathfrak{S}$ is adic if there exists an ideal of definition $\mathfrak{J}$ of $\mathfrak{S}$ such that $\mathfrak{K} = f^*(\mathfrak{J})\mathcal{O}_X$ is an ideal of definition of $\mathfrak{X}$; we then also say that $\mathfrak{X}$ is an adic $\mathfrak{S}$-prescheme (for $f$). Whenever this is the case, for every ideal of definition $\mathfrak{J}_1$ of $\mathfrak{S}$, $\mathfrak{J}_1 = f^*(\mathfrak{J}_1)\mathcal{O}_X$ is an ideal of definition of $\mathfrak{X}$. Indeed, since the questions is local on $\mathcal{O}_{X, x}$, we can assume that $\mathfrak{X}$ and $\mathfrak{S}$ are Noetherian and affine; then there exists a whole number $n$ such that $\mathfrak{J}^n \subseteq \mathfrak{J}$ and $\mathfrak{J}_1^n \subseteq \mathfrak{J}$ (see (10.3.6) and (0, 7.14)), whence $\mathfrak{K}^n \subseteq \mathfrak{K}_1$ and $\mathfrak{K}_1^n \subseteq \mathfrak{K}$. The first of these relations shows that $\mathfrak{J}_1 = \mathfrak{J}^n_1$, where $\mathfrak{J}^1_1$ is an open ideal of $A = \Gamma(\mathfrak{X}, \mathcal{O}_X)$, and the second shows that $\mathfrak{J}_1$ is an ideal of definition of $A$ (0.7.14), whence our claim.

It follows immediately from the above that, if $\mathfrak{X}$ and $\mathfrak{Y}$ are adic $\mathfrak{S}$-preschemes, then every $\mathfrak{S}$-morphism $u : \mathfrak{X} \to \mathfrak{Y}$ is adic: indeed, if $f : \mathfrak{X} \to \mathfrak{S}$ and $g : \mathfrak{Y} \to \mathfrak{S}$ are the structure morphisms, and $\mathfrak{J}$ is an ideal of definition of $\mathfrak{S}$, then we have $f = g \circ u$, and so $u^*(g^*(\mathfrak{J})\mathcal{O}_Y)\mathcal{O}_X = f^*(\mathfrak{J})\mathcal{O}_X$ is an ideal of definition of $\mathfrak{X}$, and, by hypothesis, $g^*(\mathfrak{J})\mathcal{O}_Y$ is an ideal of definition of $\mathfrak{Y}$.

(10.12.2). In what follows, we suppose that we have some fixed locally Noetherian formal prescheme $\mathfrak{S}$, and some ideal of definition $\mathfrak{J}$ of $\mathfrak{S}$; we set $S_n = (\mathfrak{S}, \mathcal{O}_\mathfrak{S}/\mathfrak{J}^{n+1})$. The (locally Noetherian) adic $\mathfrak{S}$-preschemes clearly form a category. We say that an inductive system $(X_n)$ of locally Noetherian (usual) $S_n$-preschemes is an adic inductive $(S_n)$-system if the structure morphisms $f_n : X_n \to S_n$ are such that, for $m \leq n$, the diagrams

\[
\begin{array}{ccc}
X_n & \xrightarrow{f_n} & X_m \\
\downarrow & & \downarrow f_m \\
S_n & \xleftarrow{u} & S_m
\end{array}
\]

commute and identify $X_m$ with the product $X_n \times_{S_n} S_m = (X_n)(S_m)$. The adic inductive systems form a category: it suffices in fact to define a morphism $(X_n) \to (Y_n)$ of such systems to be an inductive system of $S_n$-morphisms $u_n : X_n \to Y_n$ such that $u_m$ is identified with $(u_n)(S_m)$ for $m \leq n$. With this in mind:

**Theorem (10.12.3).** — There is a canonical equivalence between the category of adic $\mathfrak{S}$-preschemes and the category of adic inductive $(S_n)$-systems.

The equivalence in question is obtained in the following way: if $\mathfrak{X}$ is an adic $\mathfrak{S}$-prescheme, and $f : \mathfrak{X} \to \mathfrak{S}$ is the structure morphism, then $\mathfrak{K} = f^*(\mathfrak{J})\mathcal{O}_X$ is an ideal of definition of $\mathfrak{X}$, and we associate to $\mathfrak{X}$ the inductive system of the $X_n = (\mathfrak{X}, \mathcal{O}_\mathfrak{X}/\mathfrak{J}^{n+1})$, with the structure morphism $f_n : X_n \to S_n$ corresponding to $f$ (10.5.6). We first show that $(X_n)$ is an adic inductive system: if $f = (\psi, \theta)$, then $\psi^*(\mathfrak{J})\mathcal{O}_X = \mathfrak{K}$, so $\psi^*(\mathfrak{J}^n)\mathcal{O}_X = \mathfrak{K}^n$ for all $n$, and (by exactness of the functor $\psi^*$) $\mathfrak{K}^{n+1}/\mathfrak{K}^n = \psi^*(\mathfrak{J}^n/\mathfrak{J}^{n+1})\mathcal{O}_X$ for $m \leq n$; our conclusion thus follows from (4.4.5). Furthermore, it can be immediately verified that a $\mathfrak{S}$-morphism $u : \mathfrak{X} \to \mathfrak{Y}$ of adic $\mathfrak{S}$-preschemes corresponds (with the obvious notation) to an inductive system of $S_n$-morphisms $u_n : X_n \to Y_n$ such that $u_m$ is identified with $(u_n)(S_m)$ for $m \leq n$.

The fact that this equivalence is well defined will follow from the more-precise following proposition.

**Proposition (10.12.3.1).** — Let $(X_n)$ be an inductive system of $S_n$-preschemes; suppose that the structure morphisms $f_n : X_n \to S_n$ are such that the diagrams in (10.12.2.1) commute and identify $X_m$ with $X_n \times_{S_n} S_m$ for $m < n$. Then the inductive system $(X_n)$ satisfies conditions (b) and (c) of (10.6.3); let $\mathfrak{X}$ be the inductive limit, and $f : \mathfrak{X} \to \mathfrak{S}$ the morphism given by the inductive limit of the inductive system $(f_n)$. Then, if $\mathfrak{X}_0$ is locally Noetherian, $\mathfrak{X}$ is locally Noetherian, and $f$ is an adic morphism.

**Proof.** Since the sheaf of ideals of $\mathcal{O}_{S_n}$ that defines the subscheme $S_n$ of $S_n$ is nilpotent, by (4.4.5), so too is the sheaf of ideals of $\mathcal{O}_X$, that defines the subscheme $X_m$ of $X_n$, and so the conditions of (10.6.3) are satisfied. Since the questions is local on $\mathfrak{X}$ and $\mathfrak{S}$, we can assume that $\mathfrak{S} = \text{Spf}(A)$, $\mathfrak{J} = \mathfrak{J}^n$ (with $A$ a Noetherian $\mathfrak{J}$-adic ring), and $X_n = \text{Spec}(B_n)$; if $A_n = A/\mathfrak{J}^{n+1}$, then the hypothesis implies that $B_0$ is Noetherian, and if we set $\mathfrak{J}_0 = \mathfrak{J}/\mathfrak{J}^{n+1}$, then $B_n = B_\mathfrak{J}/B_\mathfrak{J}^{m+1}$. The kernel of $B_n \to B_0$ is thus $\mathfrak{J}_n = \mathfrak{J}_nB_n$, and the kernel of $B_n \to B_m$ is $\mathfrak{J}_n^{m+1}$ for $m < n$; further, since $A_1$ is Noetherian, $\mathfrak{J}_1$ is of finite type over $A_1$, and so $\mathfrak{J}_1 = \mathfrak{J}_1/\mathfrak{J}_1^2$ is of finite type over $B_1$, and
where the projective limit is relative to the maps $u\colon X \to Y$.

The above equivalence gives, for adic $\mathcal{O}$-preschemes $X$ and $Y$, a canonical bijection

$$\text{Hom}_{\mathcal{O}}(X, Y) \simeq \varprojlim_n \text{Hom}_{\mathcal{O}}(X_n, Y_n)$$

where the projective limit is relative to the maps $u_n \to (u_n)_{(S_n)}$ for $m \leq n$.

### 10.13. Morphisms of finite type

**Proposition (10.13.1).** Let $Y$ be a locally Noetherian formal prescheme, $\mathcal{X}$ an ideal of definition of $Y$, and $f : X \to Y$ a morphism of formal preschemes. Then the following conditions are equivalent.

(a) $X$ is locally Noetherian, $f$ is an adic morphism (10.12.1), and, if we set $\mathcal{J} = f^*(\mathcal{X})\mathcal{O}_X$, then the morphism $f_0 : (X, \mathcal{O}_X / \mathcal{J}) \to (Y, \mathcal{O}_Y / \mathcal{K})$ induced by $f$ is of finite type.

(b) $X$ is locally Noetherian, and is the inductive limit of an adic inductive $(Y_n)$-system $(X_n)$ such that the morphism $X_0 \to Y_0$ is of finite type.

(1) Every point of $Y$ has a Noetherian formal affine open neighbourhood $V$ which satisfies the following property:

(Q) $f^{-1}(V)$ is a finite union of Noetherian formal affine open subsets $U_i$ such that the Noetherian adic ring $\Gamma(U_i, \mathcal{O}_X)$ is topologically isomorphic to the quotient of a formal series algebra, restricted (0, 7.5.1) to $\Gamma(V, \mathcal{O}_Y)$, by an ideal (which is necessarily closed).

**Proof.** It is immediate that (a) implies (b), by (10.12.3). To show that (b) implies (c), we can, since the question is local on $Y$, assume that $Y = \text{Spf}(B)$, where $B$ is Noetherian and adic; let $\mathcal{X} = \mathfrak{A}$, with $\mathfrak{A}$ an ideal of definition of $B$. Since, by hypothesis, $X_0$ is of finite type over $Y_0$, $X_0$ is a finite union of affine open subsets $U_i$ such that the ring $A_{U_i}$ of the affine scheme induced by $X_0$ on $U_i$ is an algebra of finite type over the ring $B / \mathfrak{A}$ of $Y_0$ (6.3.2). By (5.1.9), $U_i$ is also an affine open subset in each of the Noetherian preschemes $X_n$, and, if $A_{U_i}$ is the ring of the affine scheme induced by $X_n$ on $U_i$, then hypothesis (b) implies, for $m \leq n$, that $A_{U_i}$ is isomorphic to $A_{U_i} / B / \mathfrak{A}$ (10.6.4); consequently, the formal prescheme induced by $X$ on $U_i$ is isomorphic to $\text{Spf}(A_{U_i})$, where $A_{U_i} = \lim_n A_{U_i}$ (10.6.4); $A_{U_i}$ is a $\mathfrak{A}$-adic ring, and $A_{U_i} / \mathfrak{A}$ is an algebra of finite type over $B / \mathfrak{A}$. We thus conclude (0, 7.5.5) that $A_{U_i}$ is topologically isomorphic to a quotient of a formal series algebra restricted to $B$ (by a necessarily closed ideal, because such an algebra is Noetherian (0, 7.5.4)).

To show that (c) implies (a), we can restrict to the case where $X = \text{Spf}(A)$ is also affine, with $A$ a Noetherian adic ring isomorphic to a quotient of a formal series algebra, restricted to $B$, by a closed ideal. Then (0, 7.5.5) $A / \mathfrak{A}$ is an algebra of finite type over $B / \mathfrak{A}$, and $\mathfrak{A} / \mathfrak{A}$ is an ideal of definition of $A$, and so, by (10.10.9), the conditions of (a) are satisfied.

We note that, if the conditions of Proposition (10.13.1) are satisfied, then property (a) holds true for any ideal of definition $\mathcal{X}$ of $Y$ (by (c)), and so, in property (b), all the $f_n$ are morphisms of finite type.

**Corollary (10.13.2).** If the conditions of (10.13.1) are satisfied, then every Noetherian formal affine open subset $V$ of $Y$ has property (Q), and, if $Y$ is Noetherian, then so too is $X$.

**Proof.** This follows immediately from (10.13.1) and (6.3.2).

**Definition (10.13.3).** When the equivalent properties (a), (b), and (c) of (10.13.1) are satisfied, we say that the morphism $f$ is of finite type, or that $X$ is a formal $\mathcal{O}$-prescheme of finite type, or a formal prescheme of finite type over $Y$. 

Corollary (10.13.4). — Let \( \mathfrak{X} = \text{Spf}(A) \) and \( \mathfrak{Y} = \text{Spf}(B) \) be Noetherian formal affine schemes; for \( \mathfrak{X} \) to be of finite type over \( \mathfrak{Y} \), it is necessary and sufficient for the Noetherian adic ring \( A \) to be isomorphic to the quotient of a formal series algebra, restricted to \( B \), by some closed ideal.

PROOF. With the notation of (10.13.1), if \( \mathfrak{X} \) is of finite type over \( \mathfrak{Y} \), then \( A/\mathfrak{R}A \) is a \((B/\mathfrak{R})\) algebra of finite type by (6.3.3), and \( \mathfrak{R}A \) is an ideal of definition of \( A \) (10.10.9). We are then done, by (0, 7.5.5).

Proposition (10.13.5). — 

(i) The composition of any two morphisms (of formal preschemes) of finite type is again of finite type. 

(ii) Let \( \mathfrak{X}, \mathfrak{S}, \) and \( \mathfrak{S}' \) be locally Noetherian (resp. Noetherian) formal preschemes, and \( f : \mathfrak{X} \rightarrow \mathfrak{S} \) and \( \mathfrak{X} \rightarrow \mathfrak{S}' \) morphisms. If \( f \) is of finite type, then \( \mathfrak{X} \times \mathfrak{S} \) is locally Noetherian (resp. Noetherian) and of finite type over \( \mathfrak{S}' \).

(iii) Let \( \mathfrak{S} \) be a locally Noetherian formal prescheme, and \( \mathfrak{X}' \) and \( \mathfrak{Y}' \) formal \( \mathfrak{S} \)-preschemes such that \( \mathfrak{X}' \times \mathfrak{S} \) is locally Noetherian. If \( \mathfrak{X} \) and \( \mathfrak{Y} \) are locally Noetherian formal \( \mathfrak{S} \)-preschemes, and \( f : \mathfrak{X} \rightarrow \mathfrak{X}' \) and \( g : \mathfrak{Y} \rightarrow \mathfrak{Y}' \) are \( \mathfrak{S} \)-morphisms of finite type, then \( \mathfrak{X} \times \mathfrak{S} \mathfrak{Y} \) is locally Noetherian, and \( f \times \mathfrak{S} \mathfrak{g} \) is an \( \mathfrak{S} \)-morphism of finite type.

PROOF. By the formal argument of (3.5.1), (iii) follows from (i) and (ii), so it suffices to prove (i) and (ii).

Let \( \mathfrak{X}, \mathfrak{Y}, \) and \( \mathfrak{Z} \) be locally Noetherian formal preschemes, and \( f : \mathfrak{X} \rightarrow \mathfrak{Y} \) and \( g : \mathfrak{Y} \rightarrow \mathfrak{Z} \) morphisms of finite type. If \( \mathfrak{J} \) is an ideal of definition of \( \mathfrak{Z} \), then \( \mathfrak{X}' = \mathfrak{g}^*(\mathcal{O}\mathfrak{Z})/\mathfrak{J} \mathfrak{Z} \) is an ideal of definition of \( \mathfrak{Y} \), and \( \mathfrak{J} = \mathfrak{g}^*(\mathcal{O}\mathfrak{Z})/\mathfrak{O}\mathfrak{X} \) is an ideal of definition for \( \mathfrak{X} \). Let \( X_0 = (\mathfrak{X}, \mathcal{O}\mathfrak{X}/\mathfrak{J}) \), \( Y_0 = (\mathfrak{Y}, \mathcal{O}\mathfrak{Y}/\mathfrak{J}) \), and \( Z_0 = (\mathfrak{Z}, \mathcal{O}\mathfrak{Z}/\mathfrak{J}) \), and let \( f_0 : X_0 \rightarrow Y_0 \) and \( g_0 : Y_0 \rightarrow Z_0 \) be the morphisms corresponding to \( f \) and \( g \) (respectively). Since, by hypothesis, \( f_0 \) and \( g_0 \) are of finite type, so too is \( g_0 \circ f_0 \) (6.3.4), which corresponds to \( g \circ f \); thus \( g \circ f \) is of finite type, by (10.13.1).

Under the conditions of (ii), \( \mathfrak{S} \) (resp. \( \mathfrak{X}, \mathfrak{S}' \)) is the inductive limit of a sequence \( (S_n) \) (resp. \( (X_n), (S'_n) \)) of locally Noetherian preschemes, and we can assume (10.13.1) that \( X_m = X_n \times_{S_n} S_m \) for \( m \leq n \). The formal prescheme \( \mathfrak{X} \times \mathfrak{S} \mathfrak{S}' \) is then the inductive limit of the preschemes \( X_n \times_{S_n} S'_n \) (10.7.4), and we have that 

\[
X_m \times_{S_m} S'_m = (X_n \times_{S_n} S_m) \times_{S_n} S'_m = (X_n \times_{S_n} S'_n) \times_{S'_n} S'_m.
\]

Furthermore, \( X_0 \times_{S_0} S'_0 \) is locally Noetherian, since \( X_0 \) is of finite type over \( S_0 \) (6.3.8). We thus conclude (10.12.3.1), first of all, that \( \mathfrak{X} \times \mathfrak{S} \mathfrak{S}' \) is locally Noetherian; then, since \( X_0 \times_{S_0} S'_0 \) is of finite type over \( S'_0 \) (6.3.8), it follows from (10.12.3.1) and (10.13.1) that \( \mathfrak{X} \times \mathfrak{S} \mathfrak{S}' \) is of finite type over \( \mathfrak{S}' \), which proves (ii) (the claim about Noetherian preschemes being an immediate consequence of (6.3.8)).

Corollary (10.13.6). — Under the hypotheses of (10.9.9), if \( f \) is a morphism of finite type, then so too is its extension \( \bar{f} \) to the completions.

10.14. Closed subpreschemes of formal schemes

Proposition (10.14.1). — Let \( \mathfrak{X} \) be a locally Noetherian formal prescheme, and \( \mathfrak{A} \) a coherent sheaf of ideals of \( \mathcal{O}_{\mathfrak{X}} \). If \( \mathfrak{Y} \) is the (closed) support of \( \mathcal{O}_\mathfrak{X}/\mathfrak{A} \), then the topologically ringed space \( (\mathfrak{Y}, (\mathcal{O}_\mathfrak{X}/\mathfrak{A})|_\mathfrak{Y} \) is a locally Noetherian formal prescheme that is Noetherian if \( \mathfrak{X} \) is.

PROOF. Note that \( \mathcal{O}_\mathfrak{X}/\mathfrak{A} \) is coherent by (10.10.3) and (0, 5.3.4), so its support \( \mathfrak{Y} \) is closed (0, 5.2.2).

Let \( \mathfrak{J} \) be an ideal of definition of \( \mathfrak{X} \), and let \( X_0 = (\mathfrak{X}/\mathcal{O}_\mathfrak{X}/\mathfrak{J}^{n+1}) \); the sheaf of rings \( \mathcal{O}_\mathfrak{X}/\mathfrak{A} \) is the projective limit of the sheaves \( \mathcal{O}_\mathfrak{X}/(\mathfrak{A} + \mathfrak{J}^{n+1}) = (\mathcal{O}_\mathfrak{X}/\mathfrak{A}) \otimes_{\mathcal{O}_\mathfrak{X}} (\mathcal{O}_\mathfrak{X}/\mathfrak{J}^{n+1}) \) (10.11.3), all of which have support \( \mathfrak{Y} \). The sheaf \( (\mathfrak{A} + \mathfrak{J}^{n+1})/\mathfrak{J}^{n+1} \) is a coherent \( \mathcal{O}_\mathfrak{X} \)-module, since \( \mathfrak{J}^{n+1} \) is coherent, and so \( (\mathfrak{A} + \mathfrak{J}^{n+1})/\mathfrak{J}^{n+1} \) is also a coherent \( (\mathcal{O}_\mathfrak{X}/\mathfrak{J}^{n+1}) \)-module (0, 5.3.10); if \( Y_0 \) is the closed subprescheme of \( X_0 \) defined by this sheaf of ideals, it is immediate that \( (\mathfrak{Y}, (\mathcal{O}_\mathfrak{X}/\mathfrak{A})|_\mathfrak{Y} \) is the formal prescheme given by the inductive limit of the \( Y_n \), and, since the conditions of (10.6.4) are satisfied, this proves that this formal prescheme is locally Noetherian, and further Noetherian if \( \mathfrak{X} \) is (since then \( Y_0 \) is, by (6.1.4)).

□
Definition (10.14.2). — We define a closed subscheme of a formal prescheme $X$ to be any formal prescheme of the form $(\mathfrak{Y}, (\mathcal{O}_X / \mathfrak{A})|\mathfrak{Y})$ with $\mathfrak{A}$ a coherent ideal of $\mathcal{O}_X$; we say that this prescheme is the subscheme defined by $\mathfrak{A}$.

It is clear that the correspondence thus defined between coherent ideals of $\mathcal{O}_X$ and closed subpreschemes of $X$ is bijective.

The morphism of topologically ringed spaces $j = (\psi, \theta) : \mathfrak{Y} \to X$, where $\psi$ is the injection $\mathfrak{Y} \to X$ and $\theta$ the canonical homomorphism $\mathcal{O}_X \to \mathcal{O}_X / \mathfrak{A}$, is evidently (10.4.5) a morphism of formal preschemes, and we call it the canonical injection from $\mathfrak{Y}$ to $X$. Note that, if $X = \text{Spf}(A)$, or if $A$ is Noetherian and adic, then $\mathfrak{A} = a^A$, where $a$ is an ideal of $A$ (10.10.5), and it then follows immediately from the above that $\mathfrak{Y} = \text{Spf}(A / a)$, up to isomorphism, and that $j$ corresponds (10.2.2) to the canonical homomorphism $A \to A / a$.

We say that a morphism $f : \mathfrak{Z} \to X$ of locally Noetherian formal preschemes is a closed immersion if it factors as $\mathfrak{Z} \xrightarrow{\alpha} \mathfrak{Y} \xrightarrow{j} X$, where $g$ is an isomorphism from $\mathfrak{Z}$ to a closed subscheme $\mathfrak{Y}$ of $X$, and $j$ is the canonical injection. Since $j$ is a monomorphism of ringed spaces, $g$ and $\mathfrak{Y}$ are necessarily unique.

Proposition (10.14.3). — A closed immersion is a morphism of finite type.

Proof. We can immediately restrict to the case where $X$ is a formal affine scheme $\text{Spf}(A)$, and $\mathfrak{Y} = \text{Spf}(A / a)$; the proposition then follows from Proposition (10.13.1, c). □

Lemma (10.14.4). — Let $f : \mathfrak{Y} \to X$ be a morphism of locally Noetherian formal preschemes, and let $(U_a)$ be a cover of $f(\mathfrak{Y})$ by Noetherian formal affine open subsets of $X$ such that the $f^{-1}(U_a)$ are Noetherian formal affine open subsets of $\mathfrak{Y}$. For $f$ to be a closed immersion, it is necessary and sufficient for $f(\mathfrak{Y})$ to be a closed subset of $X$ and, for all $a$, for the restriction of $f$ to $f^{-1}(U_a)$ to correspond (10.4.6) to a surjective homomorphism $\Gamma(U_a, \mathcal{O}_X) \to \Gamma(f^{-1}(U_a), \mathcal{O}_\mathfrak{Y})$.

Proof. The conditions are clearly necessary. Conversely, if the conditions are satisfied, and if we denote by $a_a$ the kernel of $\Gamma(U_a, \mathcal{O}_X) \to \Gamma(f^{-1}(U_a), \mathcal{O}_\mathfrak{Y})$, then we can define a coherent sheaf of ideals $\mathfrak{A}$ of $\mathcal{O}_X$ by setting $\mathfrak{A}|U_a = a^A_a$ and taking $\mathfrak{A}$ to be zero on the complement of the union of the $U_a$. Since $f(\mathfrak{Y})$ is closed, and since the support of $a^A_a$ is $U_a \cap f(\mathfrak{Y})$, everything reduces to proving that $a^A_a$ and $a^A_b$ induce the same sheaf on any Noetherian formal affine open subset $V \subset U_a \cap U_b$. But the restriction to $f^{-1}(U_a)$ of $f$ is a closed immersion of this formal prescheme into $U_a$, and $f^{-1}(V)$ is a Noetherian formal affine open subset of $f^{-1}(U_a)$, and the restriction of $f$ to $f^{-1}(V)$ is a closed immersion; if $b$ is the kernel of the surjective homomorphism $\Gamma(V, \mathcal{O}_X) \to \Gamma(f^{-1}(V), \mathcal{O}_\mathfrak{Y})$ corresponding to this restriction, then it is immediate (10.10.2) that $a^A_a$ induces $b^b$ on $V$. The sheaf of ideals $\mathfrak{A}$ being thus defined, it is then clear that $f = g \circ j$, where $j : \mathfrak{Z} \to X$ is the canonical injection of the closed subscheme $\mathfrak{Z}$ of $X$ defined by $\mathfrak{A}$, and that $g$ is an isomorphism from $\mathfrak{Y}$ to $\mathfrak{Z}$. □

Proposition (10.14.5). —

(i) If $f : \mathfrak{Z} \to \mathfrak{Y}$ and $g : \mathfrak{Y} \to X$ are closed immersions of locally Noetherian formal preschemes, then $g \circ f$ is a closed immersion.

(ii) Let $X$, $\mathfrak{Y}$, and $\mathfrak{S}$ be locally Noetherian formal preschemes, $f : X \to \mathfrak{S}$ a closed immersion, and $g : \mathfrak{Y} \to \mathfrak{S}$ a morphism. Then the morphism $X \times_\mathfrak{S} \mathfrak{Y} \to \mathfrak{Y}$ is a closed immersion.

(iii) Let $\mathfrak{S}$ be a locally Noetherian formal prescheme, and $X'$ and $\mathfrak{Y}'$ locally Noetherian formal $\mathfrak{S}$-preschemes such that $X' \times_\mathfrak{S} \mathfrak{Y}'$ is locally Noetherian. If $X$ and $\mathfrak{Y}$ are locally Noetherian $\mathfrak{S}$-preschemes, and $f : X \to X'$ and $g : \mathfrak{Y} \to \mathfrak{Y}'$ are $\mathfrak{S}$-morphisms that are closed immersions, then $f \times_\mathfrak{S} g$ is a closed immersion.

Proof. By (3.5.1), it again suffices to prove (i) and (ii).

To prove (i), we can assume that $\mathfrak{Y}$ (resp. $\mathfrak{Z}$) is a closed subscheme of $X$ (resp. $\mathfrak{Y}$) defined by a coherent sheaf $\mathcal{F}$ (resp. $\mathcal{X}$) of ideals of $\mathcal{O}_X$ (resp. $\mathcal{O}_\mathfrak{Y}$); if $\psi$ is the injection $\mathfrak{Y} \to X$ of underlying spaces, then $\psi_*\mathcal{X}$ is a coherent sheaf of ideals of $\psi_*\mathcal{O}_\mathfrak{Y} = \mathcal{O}_X / \mathcal{F}$ (0.5.12), and thus also a coherent $\mathcal{O}_X$-module (0.5.10); the kernel $\mathcal{X}_1$ of $\mathcal{O}_X \to (\mathcal{O}_X / \mathcal{F})/\psi_*\mathcal{X}$ is thus a coherent sheaf of ideals of $\mathcal{O}_X$ (0.5.4), and $\mathcal{O}_X / \mathcal{X}_1$ is isomorphic to $\psi_*\mathcal{O}_\mathfrak{Y}$, which proves that $\mathfrak{Z}$ is an isomorphism to a closed subscheme of $X$. 

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To prove (ii), we can immediately restrict to the case where \( \mathcal{E} = \text{Spf}(A) \), \( \mathfrak{X} = \text{Spf}(B) \), and \( \mathfrak{Y} = \text{Spf}(C) \), with \( A \) a Noetherian \( \mathfrak{p} \)-adic ring, \( B = A/a \) (where \( a \) is an ideal of \( A \)), and \( C \) a Noetherian topological adic \( A \)-algebra. Everything then reduces to proving that the homomorphism \( C \rightarrow C \otimes_A (A/a) \) is surjective: but \( A/a \) is an \( A \)-module of finite type, and its topology is the \( \mathfrak{p} \)-adic topology; it then follows from (0, 7.7.8) that \( C \otimes_A (A/a) \) can be identified with \( C \otimes_A (A/a) = C/aC \), whence our claim.

\[ \square \]

**Corollary (10.14.6).** — Under the hypotheses of (10.14.5, ii), let \( p : \mathfrak{X} \times_{\mathcal{E}} \mathfrak{Y} \rightarrow \mathfrak{X} \) and \( q : \mathfrak{X} \times_{\mathcal{E}} \mathfrak{Y} \rightarrow \mathfrak{Y} \) be the projections, so that the diagram

\[
\begin{array}{ccc}
\mathfrak{X} & \xrightarrow{p} & \mathfrak{X} \times_{\mathcal{E}} \mathfrak{Y} \\
\downarrow{f} & & \downarrow{q} \\
\mathcal{E} & \xrightarrow{g} & \mathfrak{Y}
\end{array}
\]

commutes. For every coherent \( \mathcal{O}_{\mathfrak{X}} \) module \( \mathcal{F} \), we then have a canonical isomorphism of \( \mathcal{O}_{\mathfrak{Y}} \)-modules

\[
(10.14.6.1) \quad u : g^* f_*(\mathcal{F}) \simeq q_* p^*(\mathcal{F}).
\]

**Proof.** We know that defining a homomorphism \( g^* f_*(\mathcal{F}) \rightarrow q_* p^*(\mathcal{F}) \) is equivalent to defining a homomorphism \( f_*(\mathcal{F}) \rightarrow g_* q_* p^*(\mathcal{F}) = f_* p_* p^*(\mathcal{F}) \) (0, 4.4.3): we take \( u = f_*(\rho) \), where \( \rho \) is the canonical homomorphism \( \mathcal{F} \rightarrow p_* p^*(\mathcal{F}) \) (0, 4.4.3). To see that \( u \) is an isomorphism, we can immediately restrict to the case where \( \mathcal{E} \), \( \mathfrak{X} \), and \( \mathfrak{Y} \) are formal spectra of Noetherian adic rings \( A \), \( B \), and \( C \) (respectively), satisfying the conditions in (10.14.5, ii) above; we then have \( \mathcal{F} = M^\Delta \), where \( M \) is an \( (A/a) \)-module of finite type (10.10.5), and the two sides of (10.14.6.1) can then be identified, respectively, by (10.10.8), with \( (C \otimes_A M)^\Delta \) and \( ((C/aC) \otimes_{A/a} M)^\Delta \), whence the corollary, since \( (C/aC) \otimes_{A/a} M = (C \otimes_A (A/a)) \otimes_{A/a} M \) is canonically identified with \( C \otimes_A M \).

\[ \square \]

**Corollary (10.14.7).** — Let \( \mathfrak{X} \) be a locally Noetherian usual prescheme, \( \mathfrak{Y} \) a closed subscheme of \( \mathfrak{X} \), \( j \) the canonical injection \( \mathfrak{Y} \rightarrow \mathfrak{X}, X' \) a closed subset of \( X \), and \( Y' = Y \cap X' \); then \( \tilde{j} : Y/Y' \rightarrow X/X' \) is a closed immersion, and, for every coherent \( \mathcal{O}_Y \)-module \( \mathcal{F} \), we have

\[
\tilde{j}_*(\mathcal{F}/Y') = (j_*(\mathcal{F}))/X'.
\]

**Proof.** Since \( Y' = \tilde{j}^{-1}(X') \), it suffices to use (10.9.9) and to apply (10.14.5) and (10.14.6).

\[ \square \]

## 10.15. Separated formal presheves

**Definition (10.15.1).** — Let \( \mathcal{E} \) be a formal presheve, \( \mathfrak{X} \) a formal \( \mathcal{E} \)-presheve, and \( f : \mathfrak{X} \rightarrow \mathcal{E} \) the structure morphism. We define the diagonal morphism \( \Delta_{\mathfrak{X}|\mathcal{E}} : \mathfrak{X} \rightarrow \mathfrak{X} \times_{\mathcal{E}} \mathfrak{X} \) (also denoted by \( \Delta_{\mathfrak{X}} \)) to be the morphism \( (1_{\mathfrak{X}}, 1_{\mathfrak{X}})_{\mathcal{E}} \). We say that \( \mathfrak{X} \) is separated over \( \mathcal{E} \), or is a formal \( \mathcal{E} \)-scheme, or that \( f \) is a separated morphism, if the image of the underlying space of \( \mathfrak{X} \) under \( \Delta_{\mathfrak{X}} \) is a closed subset of the underlying space of \( \mathfrak{X} \times_{\mathcal{E}} \mathfrak{X} \). We say that a formal presheve \( \mathfrak{X} \) is separated, or is a formal scheme, if it is separated over \( \mathcal{Z} \).

**Proposition (10.15.2).** — Suppose that the formal presheves \( \mathcal{E} \) and \( \mathfrak{X} \) are inductive limits of sequences \( (S_n) \) and \( (X_n) \) (respectively) of usual presheves, and that the morphism \( f : \mathfrak{X} \rightarrow \mathcal{E} \) is the inductive limit of a sequence of morphisms \( f_n : X_n \rightarrow S_n \). For \( f \) to be separated, it is necessary and sufficient for the morphism \( f_0 : X_0 \rightarrow S_0 \) to be separated.

**Proof.** Indeed, \( \Delta_{\mathfrak{X}|\mathcal{E}} \) is then the inductive limit of the sequence of morphisms \( \Delta_{X_n|S_n} \) (10.7.4), and the image of the underlying space of \( \mathfrak{X} \) (resp. \( \mathfrak{X} \times_{\mathcal{E}} \mathfrak{X} \) under \( \Delta_{\mathfrak{X}|\mathcal{E}} \)) is identical to the image of the underlying space of \( X_0 \) (resp. \( X_0 \times_{S_0} X_0 \)) under \( \Delta_{X_0|S_0} \), whence the conclusion.

\[ \square \]

**Proposition (10.15.3).** — Suppose that all the formal presheves \( \mathfrak{E} \) of formal presheves in what follows are inductive limits of sequences of usual presheves (resp. of morphisms of usual presheves).

(i) The composition of any two separated morphisms is separated.

(ii) If \( f : \mathfrak{X} \rightarrow \mathfrak{X}' \) and \( g : \mathfrak{Y} \rightarrow \mathfrak{Y}' \) are separated \( \mathfrak{E} \)-morphisms, then \( f \times_{\mathfrak{E}} g \) is separated.

(iii) If \( f : \mathfrak{X} \rightarrow \mathfrak{Y} \) is a separated \( \mathfrak{E} \)-morphism, then the \( \mathfrak{E}' \)-morphism \( f_{(\mathfrak{E}')} \) is separated for every extension \( \mathfrak{E}' \rightarrow \mathfrak{E} \) of the base formal presheve.
(iv) If the composition \( g \circ f \) of two morphisms is separated, then \( f \) is mentioned.

(In the above, it is implicit that if the same formal prescheme \( \mathfrak{Z} \) is mentioned more than once in the same proposition, we consider it as the inductive limit of the same sequence \( (Z_n) \) of usual preschemes wherever it is mentioned, and the morphisms from \( \mathfrak{Z} \) to another formal prescheme (resp. from a formal prescheme to \( \mathfrak{Z} \)) as inductive limits of morphisms from \( Z_n \) to usual preschemes (resp. from usual preschemes to \( Z_n \)).)

**Proof.** With the notation of (10.15.2), we have, in fact, that \( (g \circ f)_0 = g_0 \circ f_0 \) and \( (f \times g)_0 = f_0 \times g_0 \); the claims of (10.15.3) are then immediate consequences of (10.15.2) and the corresponding claims in (5.5.1) for usual preschemes. \( \square \)

We leave it to the reader to state, for the same type of formal preschemes and morphisms as in (10.15.3), the propositions corresponding to (5.5.5), (5.5.9), and (5.5.10) (by replacing “affine open subset” by “formal affine open subset satisfying condition (b) of (10.6.3)”).

A similar argument also shows that every Noetherian formal affine scheme is separated, which justifies the terminology.

**Proposition (10.15.4).** — Let \( \mathfrak{S} \) be a locally Noetherian formal prescheme, and \( \mathfrak{X} \) and \( \mathfrak{Y} \) locally Noetherian formal \( \mathfrak{S} \)-preschemes such that \( \mathfrak{X} \) or \( \mathfrak{Y} \) is of finite type over \( \mathfrak{S} \) (so that \( \mathfrak{X} \times \mathfrak{S} \mathfrak{Y} \) is locally Noetherian) and such that \( \mathfrak{Y} \) is separated over \( \mathfrak{S} \). Let \( f : \mathfrak{X} \to \mathfrak{Y} \) be an \( \mathfrak{S} \)-morphism; then the graph morphism

\[ \Gamma_f(1_{\mathfrak{X}}, f)_0 : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{S} \mathfrak{Y} \]

is a closed immersion.

**Proof.** We can assume that \( \mathfrak{S} \) is the inductive limit of a sequence \( (S_n) \) of locally Noetherian preschemes, \( \mathfrak{X} \) (resp. \( \mathfrak{Y} \)) the inductive limit of a sequence \( (X_n) \) (resp. \( (Y_n) \)) of \( S_n \)-preschemes, and \( f \) the inductive limit of a sequence \( (f_n : X_n \to Y_n) \) of \( S_n \)-morphisms; then \( \mathfrak{X} \times \mathfrak{S} \mathfrak{Y} \) is the inductive limit of the sequence \( (X_n \times_{S_n} Y_n) \), and \( \Gamma_f \) the inductive limit of the sequence \( (\Gamma_{f_n}) \) (10.7.4); by hypothesis, \( Y_0 \) is separated over \( S_0 \) (10.15.2), so the space \( \Gamma_{f_0}(X_0) \) is a closed subspace of \( X_0 \times_{S_0} Y_0 \); since the underlying spaces of \( \mathfrak{X} \times \mathfrak{S} \mathfrak{Y} \) (resp. \( \Gamma_f(\mathfrak{X}) \)) and \( X_0 \times_{S_0} Y_0 \) (resp. \( \Gamma_{f_0}(X_0) \)) are identical, we already see that \( \Gamma_f(\mathfrak{X}) \) is a closed subspace of \( \mathfrak{X} \times \mathfrak{S} \mathfrak{Y} \). Now note that, when \( \langle U, V \rangle \) runs over the set of pairs consisting of a Noetherian formal affine open subset \( U \) (resp. \( V \)) of \( \mathfrak{X} \) (resp \( \mathfrak{Y} \)), such that \( f(U) \subset V \), the open subsets \( U \times V \) form a cover of \( \Gamma_f(\mathfrak{X}) \) in \( \mathfrak{X} \times \mathfrak{S} \mathfrak{Y} \), and, if \( f_U : U \to V \) is the restriction of \( f \) to \( U \), then \( \Gamma_{f_U} : U \to U \times \mathfrak{S} \mathfrak{Y} \) is the restriction of \( \Gamma_f \) to \( U \). If we show that \( \Gamma_{f_U} \) is a closed immersion, then \( \Gamma_f \) will be a closed immersion (10.14.4), or, in other words, we are led to consider the case where \( \mathfrak{S} = \text{Spf}(A) \), \( \mathfrak{X} = \text{Spf}(B) \), and \( \mathfrak{Y} = \text{Spf}(C) \) are affine (with \( A, B, \) and \( C \) Noetherian adics), with \( f \) corresponding to a continuous \( A \)-homomorphism \( \phi : C \to B \); then \( \Gamma_f \) corresponds to the unique continuous homomorphism \( \omega : B \otimes_A C \to B \) which, when composed with the canonical homomorphisms \( B \to B \otimes_A C \) and \( C \to B \otimes_A C \), gives the identity and \( \phi \) (respectively). But it is clear that \( \omega \) is surjective, whence our claim. \( \square \)

**Corollary (10.15.5).** — Let \( \mathfrak{S} \) be a locally Noetherian formal prescheme, and \( \mathfrak{X} \) an \( \mathfrak{S} \)-prescheme of finite type; for \( \mathfrak{X} \) to be separated over \( \mathfrak{S} \), it is necessary and sufficient for the diagonal morphism \( \mathfrak{X} \to \mathfrak{X} \times \mathfrak{S} \mathfrak{X} \) to be a closed immersion.

**Proposition (10.15.6).** — A closed immersion \( j : \mathfrak{Y} \to \mathfrak{X} \) of locally Noetherian formal preschemes is a separated morphism.

**Proof.** With the notation of (10.14.2), \( j_0 : Y_0 \to X_0 \) is a closed immersion, thus a separated morphism, and so it suffices to apply (10.15.2). \( \square \)

**Proposition (10.15.7).** — Let \( X \) be a locally Noetherian (usual) prescheme, \( X' \) a closed subset of \( X \), and \( \hat{X} = X/\!\!/X' \). For \( \hat{X} \) to be separated, it is necessary and sufficient that \( \hat{X}_\text{red} \) be separated, and it is sufficient that \( X \) be separated.

**Proof.** With the notation of (10.8.5), for \( \hat{X} \) to be separated, it is necessary and sufficient for \( X' \) to be separated (10.15.2), and since \( \hat{X}_\text{red} = (X'_0)_\text{red} \), it is equivalent to ask for \( \hat{X}_\text{red} \) to be separated (5.5.1, vi). \( \square \)
CHAPTER II

Elementary global study of some classes of morphisms (EGA II)

SUMMARY

§1. Affine morphisms.
§2. Homogeneous prime spectra.
§3. Homogeneous prime spectrum of a sheaf of graded algebras.
§4. Projective bundles; ample sheaves.
§5. Quasi-affine morphisms; quasi-projective morphisms; proper morphisms; projective morphisms.
§6. Integral morphisms and finite morphisms.
§7. Valuative criteria.
§8. Blowup schemes; projective cones; projective closure.

The various classes of morphisms studied in this chapter are used extensively in cohomological methods; further study using these methods will be done in Chapter III, where we make particular use of §§2, 4, and 5 of Chapter II. On a first reading, §8 can be omitted: it supplements the formalism developed in §§1 and 3, reducing to easy applications of this formalism, and we will use it less consistently than the other results of this chapter.

§1. AFFINE MORPHISMS

1.1. S-preschemes and \( \mathcal{O}_S \)-algebras

(1.1.1). Let \( S \) be a prescheme, \( X \) an \( S \)-prescheme, and \( f : X \to S \) its structure morphism. We know (0, 4.2.4) that the direct image \( f_* (\mathcal{O}_X) \) is an \( \mathcal{O}_S \)-algebra, which we denote \( \mathcal{A}(X) \) when there is little chance of confusion; if \( U \) is an open subset of \( S \), then we have

\[
\mathcal{A}(f^{-1}(U)) = \mathcal{A}(X)|U.
\]

Similarly, for every \( \mathcal{O}_X \)-module \( \mathcal{F} \) (resp. every \( \mathcal{O}_S \)-algebra \( \mathcal{B} \)), we write \( \mathcal{A}(\mathcal{F}) \) (resp. \( \mathcal{A}(\mathcal{B}) \)) for the direct image \( f_* (\mathcal{F}) \) (resp. \( f_* (\mathcal{B}) \)) which is an \( \mathcal{A}(X) \)-module (resp. an \( \mathcal{A}(X) \)-algebra) and not only an \( \mathcal{O}_S \)-module (resp. an \( \mathcal{O}_S \)-algebra).

(1.1.2). Let \( Y \) be a second \( S \)-prescheme, \( g : Y \to S \) its structure morphism, and \( h : X \to Y \) an \( S \)-morphism; we then have the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\downarrow{f} & & \downarrow{g} \\
S & \xrightarrow{} & \,
\end{array}
\]

We have by definition \( h = (\psi, \theta) \), where \( \theta : \mathcal{O}_Y \to h_* (\mathcal{O}_X) = \psi_* (\mathcal{O}_X) \) is a homomorphism of sheaves of rings; we induce (0, 4.2.2) a homomorphism of \( \mathcal{O}_S \)-algebras \( g_*(\theta) : g_*(\mathcal{O}_Y) \to g_*(h_* (\mathcal{O}_X)) = f_* (\mathcal{O}_X) \), in other words, a homomorphism of \( \mathcal{O}_S \)-algebras \( \mathcal{A}(Y) \to \mathcal{A}(X) \), which we denote by \( \mathcal{A}(h) \). If \( h' : Y \to Z \) is a second \( S \)-morphism, then it is immediate that \( \mathcal{A}(h' \circ h) = \mathcal{A}(h) \circ \mathcal{A}(h') \). We havve thus define a contravariant functor \( \mathcal{A}(X) \) from the category of \( S \)-preschemes to the category of \( \mathcal{O}_S \)-algebras.

Now let \( \mathcal{F} \) be an \( \mathcal{O}_X \)-module, \( \mathcal{G} \) an \( \mathcal{O}_Y \)-module, and \( u : \mathcal{G} \to \mathcal{F} \) an \( h \)-morphism, that is (0, 4.4.1) a homomorphism of \( \mathcal{O}_Y \)-modules \( \mathcal{G} \to h_* (\mathcal{F}) \). Then \( g_*(u) : g_*(\mathcal{G}) \to g_*(h_* (\mathcal{F})) = f_* (\mathcal{F}) \) is a...
homomorphism $\mathcal{A}(Y) \to \mathcal{A}(\mathcal{F})$ of $\mathcal{O}_S$-modules, which we denote by $\mathcal{A}(u)$; in addition, the pair $(\mathcal{A}(h), \mathcal{A}(u))$ form a di-homomorphism from the $\mathcal{A}(Y)$-module $\mathcal{A}(Y)$ to the $\mathcal{A}(X)$-module $\mathcal{A}(\mathcal{F})$.

(1.1.3). If we fix the prescheme $S$, then we can consider the pairs $(X, \mathcal{F})$, where $X$ is an $S$-prescheme and $\mathcal{F}$ is an $\mathcal{O}_X$-module, as forming a category, by defining a morphism $(X, \mathcal{F}) \to (Y, \mathcal{G})$ as a pair $(h, u)$, where $h : X \to Y$ is an $S$-morphism and $u : \mathcal{F} \to \mathcal{G}$ is an $h$-morphism. We can then say that $(\mathcal{A}(X), \mathcal{A}(\mathcal{F}))$ is a contravariant functor with values in the category whose objects are pairs consisting of an $\mathcal{O}_S$-algebra and a module over that algebra, and the morphisms are the di-homomorphisms.

1.2. Affine preschemes over a prescheme

Definition (1.2.1). — Let $X$ be an $S$-prescheme, $f : X \to S$ its structure morphism. We say that $X$ is affine over $S$ if there exists a cover $(S_a)$ of $S$ by affine open sets such that for all $\alpha$, the prescheme induced by $X$ on the open set $f^{-1}(S_a)$ is affine.

Example (1.2.2). — Every closed subscheme of $S$ is an affine $S$-prescheme over $S$ ((I, 4.2.3) and (I, 4.2.4)).

Remark (1.2.3). — An affine prescheme $X$ over $S$ is not necessarily an affine scheme, as the example $X = S$ shows (1.2.2). On the other hand, if an affine scheme $X$ is an $S$-prescheme, then $X$ is not necessarily affine over $S$ (see Example (1.3.3)). However, remember that if $S$ is a scheme, then every $S$-prescheme which is an affine scheme is affine over $S$ (I, 5.5.10).

Proposition (1.2.4). — Every $S$-prescheme which is affine over $S$ is separated over $S$ (in other words, it is an $S$-scheme).

Proof. This follows immediately from (I, 5.5.5) and (I, 5.5.8). □

Proposition (1.2.5). — Let $X$ be an $S$-scheme affine over $S$, $f : X \to S$ its structure morphism. For every open $U \subset S$, $f^{-1}(U)$ is affine over $U$.

Proof. By Definition (1.2.1), we can reduce to the case where $S = \text{Spec}(A)$ and $X = \text{Spec}(B)$ are affine; then $f = (\phi, \phi')$, where $\phi : A \to B$ is a homomorphism. As the $D(g)$ for $g \in A$ form a basis for $S$, we reduce to the case where $U = D(g)$; but we then know that $f^{-1}(U) = D(\phi(g))$ (I, 1.2.2.2), hence the proposition. □

Proposition (1.2.6). — Let $X$ be an $S$-scheme affine over $S$, $f : X \to S$ its structure morphism. For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, $f_* (\mathcal{F})$ is a quasi-coherent $\mathcal{O}_S$-module.

Proof. Taking into account Proposition (1.2.4), this follows from (I, 9.2.2, a). □

In particular, the $\mathcal{O}_S$-algebra $\mathcal{A}(X) = f_*(\mathcal{O}_X)$ is quasi-coherent.

Proposition (1.2.7). — Let $X$ be an $S$-scheme affine over $S$. For every $S$-prescheme $Y$, the map $h \mapsto \mathcal{A}(h)$ from the set $\text{Hom}_S(Y, X)$ to the set $\text{Hom}(\mathcal{A}(X), \mathcal{A}(Y))$ (1.1.2) is bijective.

Proof. Let $f : X \to S$ and $g : Y \to S$ be the structure morphisms. First, suppose that $S = \text{Spec}(A)$ and $X = \text{Spec}(B)$ are affine; we must prove that for every homomorphism $\omega : f_* (\mathcal{O}_X) \to g_* (\mathcal{O}_Y)$ of $\mathcal{O}_S$-algebras, there exists a unique $S$-morphism $h : Y \to X$ such that $\mathcal{A}(h) = \omega$. By definition, for every open $U \subset S$, $\omega$ defines a homomorphism $\omega_U = \Gamma(U, \omega) : \Gamma(f^{-1}(U), \mathcal{O}_X) \to \Gamma(g^{-1}(U), \mathcal{O}_Y)$ of $\Gamma(U, \mathcal{O}_S)$-algebras. In particular, for $U = S$, this gives a homomorphism $\phi : \Gamma(X, \mathcal{O}_X) \to \Gamma(Y, \mathcal{O}_Y)$ of $\Gamma(S, \mathcal{O}_S)$-algebras, to which corresponds a well-defined $S$-morphism $h : Y \to X$, since $X$ is affine (I, 2.2.4). It remains to prove that $\mathcal{A}(h) = \omega$, in other words, for every open set $U$ of a basis for $S$, $\omega_U$ coincides with the homomorphism of algebras $\phi_U$ corresponding to the $S$-morphism $g^{-1}(U) \to f^{-1}(U)$, a restriction of $h$. We can reduce to the case where $U = D(\lambda)$, with $\lambda \in S$; then, if $f = (\rho, \phi)$, where $\rho : A \to B$ is a ring homomorphism, we have $f^{-1}(U) = D(\mu)$, where $\mu = \rho(\lambda)$, and $\Gamma(f^{-1}(U), \mathcal{O}_X)$ is the ring of fractions $B_{\mu}$; the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\phi} & \Gamma(Y, \mathcal{O}_Y) \\
\downarrow & & \downarrow \\
B_{\mu} & \xrightarrow{\phi_U} & \Gamma(g^{-1}(U), \mathcal{O}_Y)
\end{array}
\]
is commutative, and so is the analogous diagram where \( \phi_U \) is replaced by \( \omega_U \); the equality \( \phi_U = \omega_U \) then follows from the universal property of rings of fractions (0, 1.2.4).

We now pass to the general case; let \( (S_a) \) be a cover of \( S \) by affine open sets such that the \( f^{-1}(S_a) \) are affine. Then every homomorphism \( \omega : \mathcal{A}(X) \to \mathcal{A}(Y) \) of \( \mathcal{O}_S \)-algebras gives by restriction a family of homomorphisms

\[
\omega_a : \mathcal{A}(f^{-1}(S_a)) \to \mathcal{A}(g^{-1}(S_a))
\]

of \( \mathcal{O}_{S_a} \)-algebras, hence a family of \( S_a \)-morphisms \( h_a : g^{-1}(S_a) \to f^{-1}(S_a) \) by the above. It remains to see that for every affine open set of a basis for \( S_a \cap S_B \), the restriction of \( h_a \) and \( h_B \) to \( g^{-1}(U) \) coincide, which is evident since by the above, these restrictions both correspond to the homomorphism \( \mathcal{A}(X)|_U \to \mathcal{A}(Y)|_U \), a restriction of \( \omega \).

\[ \square \]

**Corollary (1.2.8).** — Let \( X \) and \( Y \) be two \( S \)-schemes which are affine over \( S \). For an \( S \)-morphism \( h : Y \to X \) to be an isomorphism, it is necessary and sufficient for \( \mathcal{A}(h) : \mathcal{A}(X) \to \mathcal{A}(Y) \) to be an isomorphism.

**Proof.** This follows immediately from Proposition (1.2.7) and from the functorial nature of \( \mathcal{A}(X) \).

\[ \square \]

### 1.3. Affine preschemes over \( S \) associated to an \( \mathcal{O}_S \)-algebra

**Proposition (1.3.1).** — Let \( S \) be a prescheme. For every quasi-coherent \( \mathcal{O}_S \)-algebra \( B \), there exists a prescheme \( X \) affine over \( S \), defined up to unique \( S \)-isomorphism, such that \( \mathcal{A}(X) = B \).

**Proof.** The uniqueness follows from Corollary (1.2.8); we prove the existence of \( X \). For every affine open \( U \subset S \), let \( X_U \) be the prescheme Spec(\( \Gamma(U, B) \)); as \( \Gamma(U, B) \) is a \( \Gamma(U, \mathcal{O}_S) \)-algebra, \( X_U \) is an \( S \)-prescheme (I, 1.6.1). In addition, as \( B \) is quasi-coherent, the \( \mathcal{O}_S \)-algebra \( \mathcal{A}(X_U) \) canonically identifies with \( B|U \) ((I, 1.3.7), (I, 1.3.13), (I, 1.6.3)). Let \( V \) be a second affine open subset of \( S \), and let \( X_{U,V} \) be the prescheme induced by \( X_U \) on \( f^{-1}_U(U \cap V) \), where \( f_U \) denotes the structure morphism \( X_U \to S \); \( X_{U,V} \) and \( X_{V,U} \) are affine over \( U \cap V \) (1.2.5), and by definition \( \mathcal{A}(X_{U,V}) \) and \( \mathcal{A}(X_{V,U}) \) canonically identify with \( B|(U \cap V) \). Hence there is (1.2.8) a canonical \( S \)-isomorphism \( \theta_{U,V} : X_{U,V} \to X_{V,U} \); in addition, if \( W \) is a third affine open subset of \( S \), and if \( \theta_{U,W} \), \( \theta_{V,W} \), and \( \theta_{U,W} \) are the restrictions of \( \theta_{U,V} \), \( \theta_{V,W} \), and \( \theta_{U,W} \) to the inverse images of \( U \cap V \cap W \) in \( X_V \), \( X_W \), and \( X_W \) respectively under the structure morphisms, then we have \( \theta_{U,W} \circ \theta_{V,W} = \theta_{U,W} \). As a result, there exists a prescheme \( X \), a cover \( (T_U) \) of \( X \) by affine open sets, and for every \( U \) an \( \mathcal{O}_U \)-algebra \( \phi_U : X_U \to T_U \), such that \( \phi_U \) maps \( f^{-1}_U(U \cap V) \) to \( T_U \cap T_V \), and we have \( \theta_{U,V} \circ \phi_U = \phi_V \) (I, 2.3.1). The morphism \( g_U = f_U \circ \phi_U^{-1} \) makes \( T_U \) an \( S \)-prescheme, and the morphisms \( g_U \) and \( g_V \) coincide on \( T_U \cap T_V \), hence \( X \) is an \( S \)-prescheme. In addition, it is clear by definition that \( X \) is affine over \( S \) and that \( \mathcal{A}(T_U) = B|U \), hence \( \mathcal{A}(X) = B \).

\[ \square \]

We say that the \( S \)-scheme \( X \) defined in this way is associated to the \( \mathcal{O}_S \)-algebra \( B \), or is the spectrum of \( B \), and we denote it by Spec(\( B \)).

**Corollary (1.3.2).** — Let \( X \) be a prescheme affine over \( S \), \( f : X \to S \) the structure morphism. For every affine open \( U \subset S \), the induced prescheme on \( f^{-1}(U) \) is the affine scheme with ring \( \Gamma(U, \mathcal{A}(X)) \).

**Proof.** As we can suppose that \( X \) is associated to an \( \mathcal{O}_S \)-algebra by Propositions (1.2.6) and (1.3.1), the corollary follows from the construction of \( X \) described in Proposition (1.3.1).

\[ \square \]

**Example (1.3.3).** — Let \( S \) be the affine plane over a field \( K \), where the point 0 has been doubled (I, 5.5.11); with the notation of (I, 5.5.11), \( S \) is the union of two affine open sets \( Y_1 \) and \( Y_2 \); if \( f \) is the open immersion \( Y_1 \to S \), then \( f^{-1}(Y_2) = Y_1 \cap Y_2 \) is not an affine open set in \( Y_1 \) (loc. cit.), hence we have an example of an affine scheme which is not affine over \( S \).

**Corollary (1.3.4).** — Let \( S \) be an affine scheme; for an \( S \)-prescheme \( X \) to be affine over \( S \), it is necessary and sufficient for \( X \) to be affine scheme.

**Corollary (1.3.5).** — Let \( X \) be a prescheme affine over a prescheme \( S \), and let \( Y \) be an \( X \)-prescheme. For \( Y \) to be affine over \( S \), it is necessary and sufficient for \( Y \) to be affine over \( X \).

**Proof.** We immediately reduce to the case where \( S \) is an affine scheme, and then we can reduce to the case where \( X \) is an affine scheme (1.3.4); the two conditions of the statement then give that \( Y \) is an affine scheme (1.3.4).

\[ \square \]
(1.3.6). Let $X$ be a prescheme affine over $S$. To define a prescheme $Y$ affine over $X$, it is equivalent, by Corollary (1.3.5), to give a prescheme $Y$ affine over $S$, and an $S$-morphism $g : Y \to X$; in other words (Proposition (1.3.1) and (1.2.7)), it is equivalent to give a quasi-coherent $\mathcal{O}_S$-algebra $\mathcal{B}$ and a homomorphism $\mathcal{A}(X) \to \mathcal{B}$ of $\mathcal{O}_S$-algebras (which can be considered as defining on $\mathcal{B}$ an $\mathcal{A}(X)$-algebra structure). If $f : X \to S$ is the structure morphism, then we have $\mathcal{B} = f_*(\mathcal{O}_X)$.

**Corollary (1.3.7).** Let $X$ be a prescheme affine over $S$; for $X$ to be of finite type over $S$, it is necessary and sufficient for the quasi-coherent $\mathcal{O}_S$-algebra $\mathcal{A}(X)$ to be of finite type (I, 9.6.2).

**Proof.** By definition (I, 9.6.2), we can reduce to the case where $S$ is affine; then $X$ is an affine scheme (1.3.4), and if $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, then $\mathcal{A}(X)$ is the $\mathcal{O}_S$-algebra $B$; as $\Gamma(U, \mathcal{B}) = B$, the corollary follows from (I, 9.6.2) and (I, 6.3.3). □

**Corollary (1.3.8).** Let $X$ be a prescheme affine over $S$; for $X$ to be reduced, it is necessary and sufficient for the quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}(X)$ to be reduced (I, 4.1.4).

**Proof.** The question is local on $S$; by Corollary (1.3.2), the corollary follows from (I, 5.1.1) and (I, 5.1.4). □

1.4. Quasi-coherent sheaves over a prescheme affine over $S$

**Proposition (1.4.1).** Let $X$ be a prescheme affine over $S$, $Y$ an $S$-scheme, and $\mathcal{F}$ (resp. $\mathcal{G}$) a quasi-coherent $\mathcal{O}_X$-module (resp. an $\mathcal{O}_Y$-module). Then the map $(h, u) \mapsto (\mathcal{A}(h), \mathcal{A}(u))$ from the set of morphisms $(Y, \mathcal{G}) \to (X, \mathcal{F})$ to the set of di-homomorphisms $(\mathcal{A}(X), \mathcal{A}(\mathcal{F})) \to (\mathcal{A}(Y), \mathcal{A}(\mathcal{G}))$ (1.1.2) and (1.1.3) is bijective.

**Proof.** The proof follows exactly as that of Proposition (1.2.7) by using (I, 2.2.5) and (I, 2.2.4), and the details are left to the reader. □

**Corollary (1.4.2).** If, in addition to the hypotheses of Proposition (1.4.1), we suppose that $Y$ is affine over $S$, then for $(h, u)$ to be an isomorphism, it is necessary and sufficient for $(\mathcal{A}(h), \mathcal{A}(u))$ to be a di-isomorphism.

**Proposition (1.4.3).** For every pair $(\mathcal{B}, \mathcal{M})$ consisting of a quasi-coherent $\mathcal{O}_S$-algebra $\mathcal{B}$ and a quasi-coherent $\mathcal{B}$-module $\mathcal{M}$ (considered as an $\mathcal{O}_S$-module or as a $\mathcal{B}$-module, which are equivalent (I, 9.6.1)), there exists a pair $(X, \mathcal{F})$ consisting of a prescheme $X$ affine over $S$ and of a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, such that $\mathcal{A}(X) = \mathcal{B}$ and $\mathcal{A}(\mathcal{F}) = \mathcal{M}$; in addition, this couple is determined up to unique isomorphism.

**Proof.** The uniqueness follows from Proposition (1.4.1) and Corollary (1.4.2); the existence is proved as in Proposition (1.3.1), and we leave the details to the reader. □

We denote by $\mathcal{M}$ the $\mathcal{O}_X$-module $\mathcal{F}$, and we say that it is associated to the quasi-coherent $\mathcal{B}$-module $\mathcal{M}$; for every affine open $U \subset S$, $\mathcal{M}|_{p^{-1}(U)}$ (where $p$ is the structure morphism $X \to S$) is canonically isomorphic to $(\Gamma(U, \mathcal{M}))^{-1}$.

**Corollary (1.4.4).** On the category of quasi-coherent $\mathcal{B}$-modules, $\mathcal{M}$ is an additive covariant exact functor in $\mathcal{M}$, which commutes with inductive limit and direct sums.

**Proof.** We immediately reduce to the case where $S$ is affine, and the corollary then follows from (I, 1.3.5), (I, 1.3.9), and (I, 1.3.11). □

**Corollary (1.4.5).** Under the hypotheses of Proposition (1.4.3), for $\mathcal{M}$ to be an $\mathcal{O}_X$-module of finite type, it is necessary and sufficient for $\mathcal{M}$ to be a $\mathcal{B}$-module of finite type.

**Proof.** We immediately reduce to the case where $S = \text{Spec}(A)$ is an affine scheme. Then $\mathcal{B} = \mathcal{B}$, where $B$ is an $A$-algebra of finite type (I, 9.6.2), and $\mathcal{M} = \mathcal{M}$, where $M$ is a $B$-module (I, 1.3.13); over the prescheme $X$, $\mathcal{O}_X$ is associated to the ring $B$ and $\mathcal{M}$ to the $B$-module $M$; for $\mathcal{M}$ to be of finite type, it is therefore necessary and sufficient for $M$ to be of finite type (I, 1.3.13), hence our assertion. □

**Proposition (1.4.6).** Let $Y$ be a prescheme affine over $S$, $X$ and $X'$ two preschemes affine over $Y$ (hence also over $S$ (1.3.5)). Let $\mathcal{B} = \mathcal{A}(Y)$, $\mathcal{A} = \mathcal{A}(X)$, and $\mathcal{A}' = \mathcal{A}(X')$. Then $X \times_Y X'$ is affine over $Y$ (thus also over $S$), and $\mathcal{A}(X \times_Y X')$ canonically identifies with $\mathcal{A} \otimes_\mathcal{B} \mathcal{A}'$. 


PROOF. By (I, 9.6.1), $\mathcal{A} \otimes_B \mathcal{A}'$ is a quasi-coherent $\mathcal{B}$-algebra, thus also a quasi-coherent $\mathcal{O}_S$-algebra (I, 9.6.1); let $Z$ be the spectrum of $\mathcal{A} \otimes_B \mathcal{A}'$ (1.3.1). The canonical $\mathcal{B}$-homomorphisms $\mathcal{A} \to \mathcal{A} \otimes_B \mathcal{A}'$ and $\mathcal{A} \to \mathcal{A} \otimes_B \mathcal{A}'$ correspond (1.2.7) to $\mathcal{Y}$-morphisms $Z \to X$ and $p' : Z \to X'$. To see that the triple $(Z, p, p')$ is a product $X \times_Y X'$, we can reduce to the case where $S$ is an affine scheme with ring $C$ (I, 3.2.6.4). But then $Y, X, X'$ are affine schemes (1.3.4) whose rings $B, A, A'$ are $C$-algebras such that $B = B, \mathcal{A} = A, \mathcal{A}' = A'$. We then know (I, 1.3.13) that $\mathcal{A} \otimes_B \mathcal{A}'$ canonically identifies with the $\mathcal{O}_S$-algebra $(A \otimes_B A')^\sim$, hence the ring $A(Z)$ identifies with $A \otimes_B A'$ and the morphisms $p$ and $p'$ correspond to the canonical homomorphisms $A \to A \otimes_B A'$ and $A' \to A \otimes_B A'$. The proposition then follows from (I, 3.2.2).

Corollary (1.4.7). — Let $\mathcal{F}$ (resp. $\mathcal{F}'$) be a quasi-coherent $\mathcal{O}_X$-module (resp. $\mathcal{O}_X$-module); then $\mathcal{A}(\mathcal{F} \otimes_Y \mathcal{F}')$ canonically identifies with $\mathcal{A}(\mathcal{F}) \otimes_{\mathcal{A}(Y)} \mathcal{A}(\mathcal{F}')$.

PROOF. We know that $\mathcal{F} \otimes_Y \mathcal{F}'$ is quasi-coherent over $X \times_Y X'$ (I, 9.1.2). Let $g : Y \to S, f : X \to Y$, and $f' : X' \to Y$ be the structure morphisms, such that the structure morphism $h : Z \to S$ is equal to $g \circ f \circ p$ and to $g \circ f' \circ p'$. We define a canonical homomorphism

$$\mathcal{A}(\mathcal{F}) \otimes_{\mathcal{A}(Y)} \mathcal{A}(\mathcal{F}') \to \mathcal{F} \otimes \mathcal{F}'$$

in the following way: for every open $U \subset S$, we have canonical homomorphisms $\Gamma(f^{-1}(g^{-1}(U)), \mathcal{F}) \to \Gamma(h^{-1}(U), p^*(\mathcal{F}))$ and $\Gamma(f'^{-1}(g^{-1}(U)), \mathcal{F}') \to \Gamma(h^{-1}(U), p'^*(\mathcal{F}'))$ (0, 4.4.3), thus we obtain a canonical homomorphism $\Gamma(f^{-1}(g^{-1}(U)), \mathcal{F}) \otimes_{\mathcal{A}(Y)} \Gamma(f'^{-1}(g^{-1}(U)), \mathcal{F}') \to \Gamma(h^{-1}(U), p^*(\mathcal{F})) \otimes_{\mathcal{A}(Y)} \Gamma(h^{-1}(U), p'^*(\mathcal{F}'))$.

To see that we have defined an isomorphism of $\mathcal{A}(Z)$-modules, we can reduce to the case where $S$ (and as a result $X, X', Y$, and $X \times_Y X'$) are affine scheme, and (with the notation of Proposition (1.4.6)), $\mathcal{F} = \overline{M}, \mathcal{F}' = \overline{M}'$, where $M$ (resp. $M'$) is an $A$-module (resp. an $A'$-module). Then $\mathcal{F} \otimes_Y \mathcal{F}'$ identifies with the sheaf on $X \times_Y X'$ associated to the $(A \otimes_B A')$-module $M \otimes_B M'$ (I, 9.1.3), and the corollary follows from the canonical identification of the $\mathcal{O}_S$-modules $(M \otimes_B M')^\sim$ and $\overline{M} \otimes_B \overline{M}'$ (where $M, M'$, and $B$ are considered as $C$-modules) (I, 1.3.12 and I, 1.6.3).

If we apply Corollary (1.4.7) in particular to the case where $X = Y$ and $\mathcal{F}' = \mathcal{O}_X$, then we see that the $\mathcal{A}'$-module $\mathcal{F}(f^*(\mathcal{F}))$ identifies with $\mathcal{A}(\mathcal{F}) \otimes_B \mathcal{A}'$.

(1.4.8). In particular, when $X = X' = Y$ (X being affine over $S$), we see that if $\mathcal{F}$ and $\mathcal{G}$ are two quasi-coherent $\mathcal{O}_X$-modules, then we have

$$\mathcal{A}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) = \mathcal{A}(\mathcal{F}) \otimes_{\mathcal{A}(X)} \mathcal{A}(\mathcal{G})$$

up to canonical functorial isomorphism. If in addition $\mathcal{F}$ admits a finite presentation, then it follows from (I, 1.6.3) and (I, 1.6.12) that

$$\mathcal{A}(\mathcal{H}om_X(\mathcal{F}, \mathcal{G})) = \mathcal{H}om_{\mathcal{A}(X)}(\mathcal{A}(\mathcal{F}), \mathcal{A}(\mathcal{G}))$$

up to canonical isomorphism.

Remark (1.4.9). — If $X$ and $X'$ are two preschemes affine over $S$, then the sum $X \sqcup X'$ is also affine over $S$, as the sum of two affine schemes is an affine scheme.

Proposition (1.4.10). — Let $S$ be a prescheme, $B$ a quasi-coherent $\mathcal{O}_S$-algebra, and $X = \text{Spec}(B)$. For a quasi-coherent sheaf of ideals $\mathcal{F}$ of $B$, $\overline{\mathcal{F}}$ is quasi-coherent sheaf of ideals of $\mathcal{O}_X$, and the closed subscheme $Y$ of $X$ defined by $\overline{\mathcal{F}}$ is canonically isomorphic to $\text{Spec}(B / \mathcal{F})$.

PROOF. It follows immediately from (I, 4.2.3) that $Y$ is affine over $S$; by Proposition (1.3.1), we reduce to the case where $S$ is affine, and the proposition then follows immediately from (I, 4.1.2).

We can also express the result of Proposition (1.4.10) by saying that if $h : B \to B'$ is a surjective homomorphism of quasi-coherent $\mathcal{O}_S$-algebras, $\mathcal{A}(h) : \text{Spec}(B') \to \text{Spec}(B)$ is a closed immersion.

Proposition (1.4.11). — Let $S$ be a prescheme, $B$ a quasi-coherent $\mathcal{O}_S$-algebra, and $X = \text{Spec}(B)$. For every quasi-coherent sheaf of ideals $\mathcal{H}$ of $\mathcal{O}_S$, we have (denoting by $f$ the structure morphism $X \to S$) $f^*(\mathcal{H}) \mathcal{O}_X = (\mathcal{H} \mathcal{O}_S)^\sim$ up to canonical isomorphism.
Proof. Since the questions is local on $S$, we can reduce to the case where $S = \text{Spec}(A)$ is affine, and in this case the proposition is none other than (I, 1.6.9).

1.5. Change of base prescheme

Proposition (1.5.1). — Let $X$ be a prescheme affine over $S$. For every extension $g : S' \to S$ of the base prescheme, $X' = X_{(S')} = X \times_S S'$ is affine over $S'$.

Proof. If $f'$ is the projection $X' \to S'$, then it suffices to prove that $f'^{-1}(U')$ is an affine open subset for every affine open subset $U'$ of $S'$ such that $g(U')$ is contained in an affine open subset $U$ of $S$ (1.2.1); we can thus reduce to the case where $S$ and $S'$ are affine, and it suffices to prove that $X'$ is then an affine scheme (1.3.4). But then (1.3.4) $X$ is an affine scheme, and if $A$, $A'$, and $B$ are the rings of $S$, $S'$, and $X$ respectively, then we know that $X'$ is the affine scheme with ring $A' \otimes_A B$ (I, 3.2.2).

Corollary (1.5.2). — Under the hypotheses of Proposition (1.5.1), let $f : X \to S$ be the structure morphism, $f' : X' \to S'$ and $g' : X' \to X$ the projections, such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{g'} & X' \\
\downarrow{f} & & \downarrow{f'} \\
S & \xleftarrow{g} & S'
\end{array}
\]

is commutative. For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, there exists a canonical isomorphism of $\mathcal{O}_{S'}$-modules

\[(1.5.2.1) \quad u : g^*(f'_*(\mathcal{F})) \simeq f'_*(g'^*(\mathcal{F})).\]

In particular, there exists a canonical isomorphism from $\mathcal{A}(X')$ to $g^*(\mathcal{A}(X))$.

Proof. To define $u$, it suffices to define a homomorphism

\[v : f'_*(\mathcal{F}) \to g'_*(g'^*(f'_*(\mathcal{F}))) = f'_*(g'^*(\mathcal{F}))\]

and to set $u = v|$. (0, 4.4.3). We take $v = f'_*(\rho)$, where $\rho$ is the canonical homomorphism $\mathcal{F} \to g'_*(g'^*(\mathcal{F}))$ (0, 4.4.3). To prove that $u$ is an isomorphism, we can reduce to the case where $S$ and $S'$, hence $X$ and $X'$, are affine; with the notation of Proposition (1.5.1), we then have $\mathcal{F} = M$, where $M$ is a $B$-module. We then note immediately that $g^*(f'_*(\mathcal{F}))$ and $f'_*(g'^*(\mathcal{F}))$ are both equal to the $\mathcal{O}_{S'}$-module associated to the $A'$-module $A' \otimes_A M$ (where $M$ is considered as an $A$-module), and that $u$ is the homomorphism associated to the identity ((I, 1.6.7), (I, 1.6.5), (I, 1.6.7)).

Remark (1.5.3). — We do not have that Corollary (1.5.2) remains true when $X$ is not assumed affine over $S$, even when $S' = \text{Spec}(k(s)) (s \in S)$ and $S' \to S$ is the canonical morphism (I, 2.4.5)—in which case $X'$ is none other than the fibre $f^{-1}(s)$ (I, 3.6.2). In other words, when $X$ is not affine over $S$, the operation “direct image of quasi-coherent sheaves” does not commute with the operation of “passing to fibres”. However, we will see in Chapter III (III, 4.2.4) a result in this sense, of an “asymptotic” nature, valid for coherent sheaves on $X$ when $f$ is proper (5.4) and $S$ is Noetherian.

Corollary (1.5.4). — For every prescheme $X$ affine over $S$ and every $s \in S$, the fibre $f^{-1}(s)$ (where $f$ denoted the structure morphism $X \to S$) is an affine scheme.

Proof. It suffices to apply Proposition (1.5.1) with $S' = \text{Spec}(k(s))$ and to use Corollary (1.3.4).

Corollary (1.5.5). — Let $X$ be an $S$-prescheme, $S'$ a prescheme affine over $S$; then $X' = X_{(S')}$ is a prescheme affine over $X$. In addition, if $f : X \to S$ is the structure morphism, then there is a canonical isomorphism of $\mathcal{O}_X$-algebras $\mathcal{A}(X') \simeq f^*(\mathcal{A}(S'))$, and for every quasi-coherent $\mathcal{A}(S')$-module $\mathcal{M}$, a canonical di-isomorphism $f^*(\mathcal{M}) \simeq \mathcal{A}(f^*(\mathcal{M}))$, denoting by $f' = f_{(S')}$ the structure morphism $X' \to S'$.

Proof. It suffices to swap the roles of $X$ and $S'$ in (1.5.1) and (1.5.2).
(1.5.6). Now let $S, S'$ be two preschemes, $q : S' \to S$ a morphism, $\mathcal{B}$ (resp. $\mathcal{B}'$) a quasi-coherent $\mathcal{O}_S$-algebra (resp. $\mathcal{O}_{S'}$-algebra), $u : \mathcal{B} \to \mathcal{B}'$ a $q$-morphism (that is, a homomorphism $\mathcal{B} \to q_*(\mathcal{B}')$ of $\mathcal{O}_S$-algebras). If $X = \text{Spec}(\mathcal{B})$, $X' = \text{Spec}(\mathcal{B}')$, then we canonically obtain a morphism

$$v = \text{Spec}(u) : X' \to X$$

such that the diagram

$$(1.5.6.1) \quad \begin{array}{ccc} X' & \xrightarrow{v} & X \\ \downarrow \textstyle q \downarrow \quad \downarrow \textstyle q \\ S' & \to & S \end{array}$$

is commutative (the vertical arrows being the structure morphisms). Indeed, the data of $u$ is equivalent to that of a homomorphism of quasi-coherent $\mathcal{O}_{S'}$-algebras $u^* : q^*(\mathcal{B}) \to \mathcal{B}' (0, 4.4.3)$; this thus canonically defines an $S'$-morphism

$$w : \text{Spec}(\mathcal{B}') \to \text{Spec}(q^*(\mathcal{B}))$$

such that $\mathcal{A}(w) = u^* (1.2.7)$. On the other hand, it follows from (1.5.2) that $\text{Spec}(q^*(\mathcal{B}))$ canonically identifies with $X \times_S S'$; the morphism $v$ is the composition $X' \xrightarrow{w} X \times_S S' \xrightarrow{p_1} X$ of $w$ with the first projection, and the commutativity of (1.5.6.1) follows from the definitions. Let $U$ (resp. $U'$) be an affine open of $S$ (resp. $S'$) such that $q(U') \subset U$, $A = \Gamma(U, \mathcal{O}_S)$, $A' = \Gamma(U', \mathcal{O}_{S'})$ their rings, $B = \Gamma(U, \mathcal{B})$, $B' = \Gamma(U', \mathcal{B}')$; the restriction of $u$ to a $(q|U')$-morphism: $\mathcal{B}|U \to \mathcal{B}'|U'$ corresponds to a di-homomorphism of algebras $B \to B'$; if $V, V'$ are the inverse images of $U, U'$ in $X, X'$ respectively, under the structure morphisms, then the morphism $V' \to V$, the restriction of $v$, corresponds (I, 1.7.3) to the above di-homomorphism.

(1.5.7). Under the same hypotheses as in (1.5.6), let $\mathcal{M}$ be a quasi-coherent $\mathcal{B}$-module; there is then a canonical isomorphism of $\mathcal{O}_X$-modules

$$(1.5.7.1) \quad v^*(\mathcal{M}) \simeq (q^*(\mathcal{M}) \otimes q^*(\mathcal{B}'))^\sim.$$ 

Indeed, the canonical isomorphism (1.5.2.1) gives a canonical isomorphism from $p_1^*(\mathcal{M})$ to the sheaf on $\text{Spec}(q^*(\mathcal{B}))$ associated to the $q^*(\mathcal{B})$-module $q^*(\mathcal{M})$, and it then suffices to apply (1.4.7).

1.6. Affine morphisms

(1.6.1). We say that a morphism $f : X \to Y$ of preschemes is affine if it defines $X$ as a prescheme affine over $Y$. The properties of preschemes affine over another translates as follows in this language:

Proposition (1.6.2). —

(i) A closed immersion is affine.

(ii) The composition of two affine morphisms is affine.

(iii) If $f : X \to Y$ is an affine $S$-morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is affine for every base change $S' \to S$.

(iv) If $f : X \to Y$ and $f' : X' \to Y'$ are two affine $S$-morphisms, then

$$f \times_S f' : X \times_S X' \to Y \times_S Y'$$

is affine.

(v) If $f : X \to Y$ and $g : Y \to Z$ are two morphisms such that $g \circ f$ is affine and $g$ is separated, then $f$ is affine.

(vi) If $f$ is affine, then so if $f_{\text{red}}$.

Proof. By (I, 5.5.12), it suffices to prove (i), (ii), and (iii). But (i) is none other than Example (1.2.2), and (ii) is none other than Corollary (1.3.5); finally, (iii) follows from Proposition (1.5.1), since $X_{(S')}$ identifies with the product $X \times_Y Y_{(S')}$ (I, 3.3.11).

Corollary (1.6.3). — If $X$ is an affine scheme and $Y$ is a scheme, then every morphism $f : X \to Y$ is affine.

Proposition (1.6.4). — Let $Y$ be a locally Noetherian prescheme, $f : X \to Y$ a morphism of finite type. For $f$ to be affine, it is necessary and sufficient for $f_{\text{red}}$ to be.
PROOF. By (1.6.2, vi), we see only need to prove the sufficiency of the condition. It suffices to prove that if \( Y \) is affine and Noetherian, then \( X \) is affine; but \( Y_{\text{red}} \) is then affine, so the same is true for \( X_{\text{red}} \) by hypothesis. Now \( X \) is Noetherian, so the conclusion follows from (I, 6.1.7). \( \square \)

1.7. Vector bundle associated to a sheaf of modules

(1.7.1). Let \( A \) be a ring, \( E \) an \( A \)-module. Recall that we call the symmetric algebra on \( E \) and denote by \( S(E) \) (or \( S_A(E) \)) the quotient algebra of the tensor algebra \( T(E) \) by the two-sided ideal generated by the elements \( x \otimes y - y \otimes x \), where \( x \) and \( y \) vary over \( E \). The algebra \( S(E) \) is characterized by the following universal property: if \( \sigma \) is the canonical map \( E \to S(E) \) (obtained by composing \( E \to T(E) \) with the canonical map \( T(E) \to S(E) \)), then every \( A \)-linear map \( E \to B \), where \( B \) is a commutative \( A \)-algebra, factors uniquely as \( E \to S(E) \to B \), where \( g \) is an \( A \)-homomorphism of algebras. We immediately deduce from this characterization that for two \( A \)-modules \( E \) and \( F \), we have

\[
S(E \oplus F) = S(E) \otimes S(F)
\]

up to canonical isomorphism; in addition, \( S(E) \) is a covariant functor in \( E \), from the category of \( A \)-modules to that of commutative \( A \)-algebras; finally, the above characterization also shows that if \( E = \lim_{\lambda} E_\lambda \) then we have \( S(E) = \lim_{\lambda} S(E_\lambda) \) up to canonical isomorphism. By abuse of language, a product \( \sigma(x_1)\sigma(x_2) \cdots \sigma(x_n) \), where \( x_i \in E \), is often denoted by \( x_1x_2 \cdots x_n \) if no confusion follows. The algebra \( S(E) \) is graded, \( S_n(E) \) being the set of linear combinations of \( n \) elements of \( E \) (\( n \geq 0 \)); the algebra \( S(A) \) is canonically isomorphic to the polynomial algebra \( A[T] \) is an indeterminate, and the algebra \( S(A^n) \) with the polynomial algebra in \( n \) indeterminates \( A[T_1, \ldots, T_n] \).

(1.7.2). Let \( \phi \) be a ring homomorphism \( A \to B \). If \( F \) is a \( B \)-module, then the canonical map \( F \to S(F) \) gives a canonical map \( F_\phi \to S(F)_\phi \), which thus factors as \( F_\phi \to S(F_\phi) \to S(F)_\phi \); the canonical homomorphism \( S(F_\phi) \to S(F)_\phi \) is surjective, but not necessarily bijective. If \( E \) is an \( A \)-module, then every \( A \)-homomorphism \( E \to F \) (that is to say, every \( A \)-homomorphism \( E \to F_\phi \)) thus canonically gives an \( A \)-homomorphism of algebras \( S(E) \to S(F_\phi) \to S(F)_\phi \), that is to say a \( A \)-homomorphism of algebras \( S(E) \to S(F) \).

With the same notations, for every \( A \)-module \( E \), \( S(E \otimes_A B) \) canonically identifies with the algebra \( S(E) \otimes A B \); this follows immediately from the universal property of \( S(E) \) (1.7.1).

(1.7.3). Let \( R \) be a multiplicative subset of the ring \( A \); apply (1.7.2) to the ring \( B = R^{-1}A \), and remembering that \( R^{-1}E = E \otimes_A R^{-1}A \), we see that we have \( S(R^{-1}E) = R^{-1}S(E) \) up to canonical isomorphism. In addition, if \( R' \supset R \) is a second multiplicative subset of \( A \), then the diagram

\[
\begin{array}{ccc}
R^{-1}E & \longrightarrow & R'^{-1}E \\
\downarrow & & \downarrow \\
S(R^{-1}E) & \longrightarrow & S(R'^{-1}E)
\end{array}
\]

is commutative.

(1.7.4). Now let \( (S, \mathcal{O}) \) be a ringed space, and let \( \mathcal{E} \) be a \( \mathcal{O} \)-module over \( S \). If to any open \( U \subset S \) we associate the \( \Gamma(U, \mathcal{O}) \)-module \( S(\Gamma(U, \mathcal{O})) \), then we define (see the functorial nature of \( S(E) \) (1.7.2)) a presheaf of algebras; we say that the associated sheaf, which we denote by \( S(\mathcal{E}) \) or \( S_{\mathcal{O}}(\mathcal{E}) \) is the symmetric \( \mathcal{O} \)-algebra on the \( \mathcal{O} \)-module \( \mathcal{E} \). It follows immediately from (1.7.1) that \( S(\mathcal{E}) \) is a solution to a universal problem: every homomorphism of \( \mathcal{O} \)-modules \( \mathcal{E} \to \mathcal{B} \), where \( \mathcal{B} \) is an \( \mathcal{O} \)-algebra, factors uniquely as \( \mathcal{E} \to S(\mathcal{E}) \to \mathcal{B} \), the second arrow being a homomorphism of \( \mathcal{O} \)-algebras. There is thus a bijective correspondence between homomorphisms \( \mathcal{E} \to \mathcal{B} \) of \( \mathcal{O} \)-modules and homomorphisms \( S(\mathcal{E}) \to \mathcal{B} \) of \( \mathcal{O} \)-algebras. In particular, every homomorphism \( \mu : \mathcal{E} \to \mathcal{F} \) of \( \mathcal{O} \)-modules defines a homomorphism \( S(\mu) : S(\mathcal{E}) \to S(\mathcal{F}) \) of \( \mathcal{O} \)-algebras, and \( S(\mathcal{E}) \) is thus a covariant functor in \( \mathcal{E} \).

By (1.7.2) and the commutativity of \( S \) with inductive limits, we have \( (S(\mathcal{E}))_x = S(\mathcal{E}_x) \) for every point \( x \in S \). If \( \mathcal{E}, \mathcal{F} \) are two \( \mathcal{O} \)-modules, then \( S(\mathcal{E} \oplus \mathcal{F}) \) canonically identifies with \( S(\mathcal{E}) \otimes_{\mathcal{O}} S(\mathcal{F}) \), as we see for the corresponding presheaves.

We also note that \( S(\mathcal{E}) \) is a graded \( \mathcal{O} \)-algebra, the infinite direct sum of the \( S_n(\mathcal{E}) \), where the \( \mathcal{O} \)-module \( S_n(\mathcal{E}) \) is the sheaf associated to the presheaf \( U \mapsto S_n(\Gamma(U, \mathcal{E})) \). If we take in particular
which are localized at the points of $p$ with $t$ (1.7.9).

Let $(T, R)$ be a second ringed space, $f$ a morphism $(S, \mathcal{A}) \to (T, R)$. If $\mathcal{F}$ is a $R$-module, then $S(f^*(\mathcal{F}))$ canonically identifies with $f^*(S(\mathcal{F}))$; indeed, if $f = (\psi, \theta)$, then by definition (0, 4.3.1),

\[ S(f^*(\mathcal{F})) = S(\psi^*(\mathcal{F}) \otimes_{\psi^*(\mathcal{R})} \mathcal{A}) = S(\psi^*(\mathcal{F})) \otimes_{\psi^*(\mathcal{R})} \mathcal{A} \]

(1.7.2); for every open $U$ of $S$ and every section $h$ of $S(\psi^*(\mathcal{F}))$ over $U$, $h$ coincides, in a neighborhood $V$ of every point $s \in U$, with an element of $S(\Gamma(V, \psi^*(\mathcal{F})))$; if we refer to the definition of $\psi^*(\mathcal{F})$ (0, 3.7.1) and take into account that every element of $S(E)$ for a module $E$ is a linear combination of a finite number of products of elements of $E$, then we see that there is a neighborhood $W$ of $\psi(s)$ in $T$, a section $h'$ of $S(\mathcal{F})$ over $W$, and a neighborhood $V' \subset V \cap \psi^{-1}(W)$ of $s$ such that $h$ coincides with $t \mapsto h'(\psi(t))$ over $V'$; hence out assertion.

**Proposition (1.7.6).** — Let $A$ be a ring, $S = \text{Spec}(A)$ its prime spectrum, $\mathcal{E} = \mathfrak{M}$ the $O_S$-module associated to an $A$-module $M$; then the $O_S$-algebra $S(\mathcal{E})$ is associated to the $A$-algebra $S(M)$.

**Proof.** For every $f \in A$, $S(M_f) = (S(M))_f$ (1.7.3), and the proposition thus follows from Definition (I, 1.3.4). □

**Corollary (1.7.7).** — If $S$ is a prescheme, $\mathcal{E}$ a quasi-coherent $O_S$-module, then the $O_S$-algebra $S(\mathcal{E})$ is quasi-coherent. If in addition $\mathcal{E}$ is of finite type, then each of the $O_S$-modules $S_n(\mathcal{E})$ is of finite type.

**Proof.** The first assertion is an immediate consequence of (1.7.6) and of (I, 1.4.1); the second follows from the fact that if $E$ is an $A$-module of finite type, then $S_n(E)$ is an $A$-module of finite type; we then apply (I, 1.3.13). □

**Definition (1.7.8).** — Let $\mathcal{E}$ be a quasi-coherent $O_S$-module. We call the vector bundle over $S$ defined by $\mathcal{E}$ and denote by $V(\mathcal{E})$ the spectrum (1.3.1) of the quasi-coherent $O_S$-algebra $S(\mathcal{E})$.

By (1.2.7), for every $S$-prescheme $X$, there is a canonical bijective correspondence between the $S$-morphisms $X \to V(\mathcal{E})$ and the homomorphisms of $O_S$-algebras $S(\mathcal{E}) \to S(\mathcal{A})(X)$, thus also between these $S$-morphisms and the homomorphisms of $O_S$-modules $\mathcal{E} \to \mathcal{A}(X) \to \mathcal{F}(X)$ (where $f$ is the structure morphism $X \to S$). In particular:

(1.7.9). Take for $X$ a subscheme induced by $S$ on an open $U \subset S$. Then the $S$-morphisms $U \to V(\mathcal{E})$ are none other than the $U$-sections (I, 2.5.5) of the $U$-prescheme induced by $V(\mathcal{E})$ on the open $p^{-1}(U)$ (where $p$ is the structure morphism $V(\mathcal{E}) \to S$). From what we have just seen, these $U$-sections bijectively correspond to homomorphisms of $O_U$-modules $\mathcal{E} \to j_*(O_U)$ (where $j$ is the canonical injection $U \to S$), or equivalently (0, 4.4.3) with the $(O_S|U)$-homomorphisms $j^*(\mathcal{E}) = \mathcal{E}|U \to O_S|U$. In addition, it is immediate that the restriction to an open $U' \subset U$ of an $S$-morphism $U \to V(\mathcal{E})$ corresponds to the restriction to $U'$ of the corresponding homomorphism $\mathcal{E}|U \to O_S|U$. We conclude that the sheaf of germs of $S$-sections of $V(\mathcal{E})$ is canonically identified with the dual $\mathcal{E}^\vee$ of $\mathcal{E}$.

In particular, if we set $X = U = S$, then the zero homomorphism $\mathcal{E} \to O_S$ corresponds to a canonical $S$-section of $V(\mathcal{E})$, called the zero $S$-section (cf. (8.3.3)).

(1.7.10). Now take $X$ to be the spectrum $\{ \xi \}$ of a field $K$; the structure morphism $f : X \to S$ then corresponds to a monomorphism $k(s) \to K$, where $s = f(\xi)$ (I, 2.4.6); the $S$-morphisms $\{ \xi \} \to V(\mathcal{E})$ are none other than the geometric points of $V(\mathcal{E})$ with values in the extension $K$ of $k(s)$ (I, 3.4.5), points which are localized at the points of $p^{-1}(s)$. The set of these points, which we can call the rational geometric fibre over $K$ of $V(\mathcal{E})$ over the point $s$, is identified by (1.7.8) with the set of homomorphisms of $O_S$-modules $\mathcal{E} \to f_* (O_X)$, or, equivalently (0, 4.4.3) with the set of homomorphisms of $O_X$-modules $f^*(\mathcal{E}) \to O_X = K$. But we have by definition (0, 4.3.1) $f^*(\mathcal{E}) = E \otimes_{O_S} K = E \otimes_{k(s)} K$, setting $E = E \otimes_{O_S} K$; the geometric fibre of $V(\mathcal{E})$ rational over $K$ over $s$ thus identifies with the dual of the $K$-vector space $E \otimes_{k(s)} K$; if $E$ or $K$ is of finite dimension over $k(s)$, then this dual also identifies with $(E^\vee)^{\vee} \otimes_{k(s)} K$, denoting by $(E^\vee)^{\vee}$ the dual of the $k(s)$-vector space $E^\vee$.
(i) \( \mathbf{V}(\mathcal{E}) \) is a contravariant functor in \( \mathcal{E} \) from the category of quasi-coherent \( \mathcal{O}_S \)-modules to the category of affine \( S \)-schemes.

(ii) If \( \mathcal{E} \) is an \( \mathcal{O}_S \)-module of finite type, then \( \mathbf{V}(\mathcal{E}) \) is of finite type over \( S \).

(iii) If \( \mathcal{E} \) and \( \mathcal{F} \) are two quasi-coherent \( \mathcal{O}_S \)-modules, then \( \mathbf{V}(\mathcal{E} \oplus \mathcal{F}) \) canonically identifies with \( \mathbf{V}(\mathcal{E}) \times_S \mathbf{V}(\mathcal{F}) \).

(iv) Let \( g : S' \to S \) be a morphism; for every quasi-coherent \( \mathcal{O}_S \)-module \( \mathcal{E} \), \( \mathbf{V}(g^*(\mathcal{E})) \) canonically identifies with \( \mathbf{V}(\mathcal{E}) \times_S \mathbf{V}(g^* S') \).

(v) A surjective homomorphism \( \mathcal{E} \to \mathcal{F} \) of quasi-coherent \( \mathcal{O}_S \)-modules corresponds to a closed immersion \( \mathbf{V}(\mathcal{F}) \to \mathbf{V}(\mathcal{E}) \).

Proof. (i) is an immediate consequence of (1.2.7), taking into account that every homomorphism of \( \mathcal{O}_S \)-modules \( \mathcal{E} \to \mathcal{F} \) canonically defines a homomorphism of \( \mathcal{O}_S \)-algebras \( S(\mathcal{E}) \to S(\mathcal{F}) \).

(ii) follows immediately from the definition (I, 6.2.1) and the fact that if \( E \) is an \( A \)-module of finite type, then \( S(E) \) is an \( A \)-algebra of finite type. To prove (iii), it suffices to start with the canonical isomorphism \( S(\mathcal{E} \oplus \mathcal{F}) \simeq S(\mathcal{E}) \otimes_{\mathcal{O}_S} S(\mathcal{F}) \) (1.7.4) and to apply (1.4.6). Similarly, to prove (iv), it suffices to start with the canonical isomorphism \( S(g^*(\mathcal{E})) \simeq g^*(S(\mathcal{E})) \) (1.7.5) and to apply (1.5.2).

Finally, to establish (v), it suffices to remark that if the homomorphism \( \mathcal{E} \to \mathcal{F} \) is surjective, then so is the corresponding homomorphism \( S(\mathcal{E}) \to S(\mathcal{F}) \) of \( \mathcal{O}_S \)-algebras, and the conclusion follows from (1.4.10). \( \square \)

(1.7.12). Take in particular \( \mathcal{E} = \mathcal{O}_S \); the prescheme \( \mathbf{V}(\mathcal{O}_S) \) is the affine \( S \)-scheme, spectrum of the \( \mathcal{O}_S \)-algebra \( \mathcal{O}_S(S) \) which identifies with the \( \mathcal{O}_S \)-algebra \( \mathcal{O}_S[T] = \mathcal{O}_S \otimes_{\mathcal{O}_A} \mathbb{Z}[T] \) (\( T \) indeterminate); this is evident when \( S = \text{Spec}(\mathbb{Z}) \), by virtue of (1.7.6), and we pass from there to the general case by considering the structure morphism \( S \to \text{Spec}(\mathbb{Z}) \) and using (1.7.11, iv). Because of this result, we set \( \mathbf{V}(\mathcal{O}_S) = S[T] \), and we thus have the formula

\[
S[T] = S \otimes_{\mathbb{Z}} \mathbb{Z}[T].
\]

The identification of the sheaf of germs of \( S \)-sections of \( S[T] \) with \( \mathcal{O}_S \), already seen in (I, 3.2.15), here in a more general context, as a special case of (1.7.9).

(1.7.13). For every \( S \)-prescheme \( X \), we have seen (1.7.8) that \( \text{Hom}_S(X, S[T]) \) canonically identifies with \( \text{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, \mathcal{O}(X)) \), which is canonically isomorphic to \( \Gamma(S, \mathcal{O}(X)) \), and as a result is equipped with the structure of a ring; in addition, to every \( S \)-morphism \( h : X \to Y \) there corresponds a morphism \( \Gamma(\mathcal{O}(h)) : \Gamma(S, \mathcal{O}(Y)) \to \Gamma(S, \mathcal{O}(X)) \) for the ring structures (1.1.2). When we equip \( \text{Hom}_S(X, S[T]) \) with a ring structure as defined, then we can see that \( \text{Hom}(X, S[T]) \) can be considered as a contravariant functor in \( X \), from the category of \( S \)-preschemes to that of rings. On the other hand, \( \text{Hom}_S(X, \mathbf{V}(\mathcal{E})) \) is likewise identified with \( \text{Hom}_{\mathcal{O}_S}(\mathcal{E}, \mathcal{O}(X)) \) (where \( \mathcal{O}(X) \) is considered as an \( \mathcal{O}_S \)-module); as a result, we can canonically equip it with a module structure over the ring \( \text{Hom}_S(X, S[T]) \), and we see as above that the pair

\[
(\text{Hom}_S(X, S[T]), \text{Hom}(X, \mathbf{V}(\mathcal{E})))
\]

is a contravariant functor in \( X \), with values in the category whose elements are the pairs \((A, M)\) consisting of a ring \( A \) and an \( A \)-module \( M \), the morphisms being di-homomorphisms.

We will interpret these facts by saying that \( S[T] \) is an \( S \)-scheme of rings and that \( \mathbf{V}(\mathcal{E}) \) is an \( S \)-scheme of modules on the \( S \)-scheme of rings \( S[T] \) (cf. Chapter 0, §8).

(1.7.14). We will see that the structure of an \( S \)-scheme of modules defined on the \( S \)-scheme \( \mathbf{V}(\mathcal{E}) \) allows us to reconstruct the \( \mathcal{O}_S \)-module \( \mathcal{E} \) up to unique isomorphism: for this, we will show that \( \mathcal{E} \) is canonically isomorphic to an \( \mathcal{O}_S \)-submodule of \( S(\mathcal{E}) = \mathcal{O}(\mathbf{V}(\mathcal{E})) \), defined by means of this structure. Indeed (1.7.4) the set \( \text{Hom}_{\mathcal{O}_S}(\mathcal{E}, \mathcal{O}(X)) \) of homomorphisms of \( \mathcal{O}_S \)-algebras is canonically identified with \( \text{Hom}_{\mathcal{O}_S}(\mathcal{E}, \mathcal{O}(X)) \), the set of homomorphisms of \( \mathcal{O}_S \)-modules: if \( h \) and \( h' \) are two elements of this latter set, \( s_i \) \((1 \leq i \leq n)\) sections of \( \mathcal{E} \) over an open \( U \subset S \), \( t \) a section of \( \mathcal{O}(X) \) over \( U \), then we have by definition

\[
(h + h')(s_1s_2 \cdots s_n) = \prod_{i=1}^{n} (h(s_i) + h'(s_i))
\]
and

\[(t \cdot h)(s_1 s_2 \cdots s_n) = t^n \prod_{i=1}^{n} h(s_i).\]

This being so, if \(z\) is a section of \(S(\mathcal{E})\) over \(U\), then \(h \mapsto h(z)\) is a map from \(\text{Hom}_S(X, V(\mathcal{E})) = \text{Hom}_{\mathcal{O}_U}(S(\mathcal{E}), \mathcal{A}(X))\) to \(\Gamma(U, \mathcal{A}(X))\). We will show that \(\mathcal{E}\) is identified with a submodule of \(S(\mathcal{E})\) such that, for every open \(U \subset S\), every section \(z\) of this \(\mathcal{O}_S\)-submodule of \(U\), and every \(S\)-prescheme \(X\), the map \(h \mapsto h(z)\) from \(\text{Hom}_{\mathcal{O}_U}(S(\mathcal{E})|U, \mathcal{A}(X)|U)\) to \(\Gamma(U, \mathcal{A}(X))\) is a homomorphism of \((\Gamma(U, \mathcal{A}(X))-\text{modules}).\)

It is immediate that \(\mathcal{E}\) has this property; to show the converse, we can reduce to proving that when \(S = \text{Spec}(A), \mathcal{E} = M\), a section \(z\) of \(S(\mathcal{E})\) over \(S\) that (for \(U = S\)) has the property stated above is necessarily a section of \(\mathcal{E}\); we then have \(z = \sum_{n=0}^\infty z_n\), where \(z_n \in S_n(M)\), and it is a question of proving that \(z_n = 0\) for \(n \neq 1\). Set \(B = S(M)\) and take for \(X\) the prescheme \(\text{Spec}(B[T])\), where \(T\) is an indeterminate. The set \(\text{Hom}_{\mathcal{O}_U}(S(\mathcal{E}), \mathcal{A}(X))\) identifies with the set of ring homomorphisms \(h : B \to B[T]\) (I, 1.3.13), and from what we saw above, we have \((T \cdot h)(z) = \sum_{n=0}^\infty T^n h(z_n)\); the hypothesis on \(z\) implies that we have \(\sum_{n=0}^\infty T^n h(z_n) = T \cdot \sum_{n=0}^\infty h(z_n)\) for every homomorphism \(h\). In particular we take for \(h\) the canonical injection, then \(\sum_{n=0}^\infty T^n z_n = T \cdot \sum_{n=0}^\infty z_n\), which implies the conclusion \(z_n = 0\) for \(n \neq 1\).

**Proposition (1.7.15).** — Let \(Y\) be a prescheme whose underlying space is Noetherian, or a quasi-compact scheme. Every affine \(Y\)-scheme \(X\) of finite type over \(Y\) is \(Y\)-isomorphic to a closed \(Y\)-subscheme of a \(Y\)-scheme of the form \(V(\mathcal{E})\), where \(\mathcal{E}\) is a quasi-coherent \(\mathcal{O}_Y\)-module of finite type.

**Proof.** The quasi-coherent \(\mathcal{O}_Y\)-algebra \(\mathcal{A}(X)\) is of finite type (1.3.7). The hypotheses imply that \(\mathcal{A}(X)\) is generated by a quasi-coherent \(\mathcal{O}_Y\)-submodule of finite type \(\mathcal{E}\) (I, 9.6.5); by definition, this implies that the canonical homomorphism \(S(\mathcal{E}) \to \mathcal{A}(X)\) canonically extending the injection \(\mathcal{E} \to \mathcal{A}(X)\) is surjective; the conclusion then follows from (1.4.10). \(\square\)

### §2. Homogeneous prime spectra

#### 2.1. Generalities on graded rings and modules

**Notation (2.1.1).** — Given a positively-graded ring \(S\), we denote by \(S_n\) the subset of \(S\) consisting of homogeneous elements of degree \(n\) \((n \geq 0)\), by \(S_+\) the (direct) sum of the \(S_n\) for \(n > 0\); we have \(1 \in S_0\), \(S_0\) is a subring of \(S, S_+\) is a graded ideal of \(S\), and \(S\) is the direct sum of \(S_0\) and \(S_+\). If \(M\) is a graded module over \(S\) (with positive or negative degrees), we similarly denote by \(M_n\) the \(S_0\)-module consisting of homogeneous elements of \(M\) of degree \(n\) (with \(n \in \mathbb{Z}\)).

For every integer \(d > 0\), we denote by \(S^{(d)}\) the direct sum of the \(S_{nd}\); by considering the elements of \(S_{nd}\) as homogeneous of degree \(n\), the \(S_{nd}\) define on \(S^{(d)}\) a graded ring structure.

For every integer \(k\) such that \(0 < k < d - 1\), we denote by \(M^{(d,k)}\) the direct sum of the \(M_{nd+k}\) \((n \in \mathbb{Z})\); this is a graded \(S^{(d)}\)-module when we consider the elements of \(M_{nd+k}\) as homogeneous of degree \(n\). We write \(M^{(d)}\) in place of \(M^{(d,0)}\).

With the above notation, for every integer \(n\) (positive or negative), we denote by \(M(n)\) the graded \(S\)-module defined by \((M(n))_k = M_{n+k}\) for every \(k \in \mathbb{Z}\). In particular, \(S(n)\) will be a graded \(S\)-module such that \((S(n))_k = S_{n+k}\), by agreeing to set \(S_n = 0\) for \(n < 0\). We say that a graded \(S\)-module \(M\) is free if it is isomorphic, considered as a graded module, to a direct sum of modules of the form \(S(n)\); as \(S(n)\) is a monogeneous \(S\)-module, generated by the element \(1\) of \(S\) considered as an element of degree \(-n\), it is equivalent to say that \(M\) admits a basis over \(S\) consisting of homogeneous elements.

We say a graded \(S\)-module \(M\) admits a finite presentation if there exists an exact sequence \(P \to Q \to M \to 0\), where \(P\) and \(Q\) are finite direct sums of modules of the form \(S(n)\) and the homomorphisms are of degree \(0\) (cf. (2.1.2)).

(2.1.2). Let \(M\) and \(N\) be two graded \(S\)-modules; we define on \(M \otimes_Z N\) a graded \(S\)-module structure in the following way. On the tensor product \(M \otimes_Z N\), we can define a graded \(Z\)-module structure (where \(Z\) is graded by \(Z_0 = Z, Z_n = 0\) for \(n \neq 0\)) by setting \((M \otimes_Z N)_q = \bigoplus_{m+n=q} M_m \otimes_Z N_n\) (as \(M\) and \(N\) are respectively direct sums of the \(M_m\) and the \(N_n\), we know that we can canonically identify \(M \otimes_Z N\) with the direct sum of all the \(M_m \otimes_Z N_n\)). This being so, we have \(M \otimes_S N = (M \otimes_Z N)/P\), where \(P\) is the \(Z\)-submodule of \(M \otimes_Z N\) generated by the elements \((xs) \otimes y - x \otimes (sy)\) for \(x \in M, s \in S, y \in N\).
y \in N, s \in S; it is clear that \( P \) is a graded \( \mathbb{Z} \)-submodule of \( M \otimes \mathbb{Z} N \), and we see immediately that we obtain a graded \( S \)-module structure on \( M \otimes \mathbb{Z} N \) by passing to the quotient.

For two graded \( S \)-modules \( M \) and \( N \), recall that a homomorphism \( u : M \to N \) of \( S \)-modules is said to be of degree \( k \) if \( u(M_i) \subset N_{i+k} \) for all \( j \in \mathbb{Z} \). If \( H_n \) denotes the set of all the homomorphisms of degree \( n \) from \( M \) to \( N \), then we denote by \( \text{Hom}_S(M, N) \) the (direct) sum of the \( H_n \) \((n \in \mathbb{Z})\) in the \( S \)-module \( H \) of all the homomorphisms (of \( S \)-modules) from \( M \) to \( N \); in general, \( \text{Hom}_S(M, N) \) is not equal to the latter. However, we have \( H = \text{Hom}_S(M, N) \) when \( M \) is of finite type; indeed, we can then suppose that \( M \) is generated by a finite number of homogeneous elements \( x_i \) \((1 \leq i \leq n)\), and every homomorphism \( u \in H \) can be written in a unique way as \( \sum_{k \in \mathbb{Z}} u_k \), where for each \( k \), \( u_k(x_i) \) is equal to the homogeneous component of degree \( k + \deg(x_i) \) of \( u(x_i) \) \((1 \leq i \leq n)\), which implies that \( u_k = 0 \) except for a finite number of indices; we have by definition that \( u_k \in H_k \), hence the conclusion.

We say that the elements of degree \( 0 \) of \( \text{Hom}_S(M, N) \) are the homomorphisms of graded \( S \)-modules.

It is clear that \( S_n H_n \subset H_{m+n} \), so the \( H_n \) define on \( \text{Hom}_S(M, N) \) a graded \( S \)-module structure.

It follows immediately from these definitions that we have

\begin{align}
M(m) \otimes_S N(n) &= (M \otimes_S N)(m+n), \\
\text{Hom}_S(M(m), N(n)) &= (\text{Hom}_S(M, N))(n-m),
\end{align}

for two graded \( S \)-modules \( M \) and \( N \).

Let \( S \) and \( S' \) be two graded rings; a homomorphism of graded rings \( \phi : S \to S' \) is a homomorphism of rings such that \( \phi(S_n) \subset S'_n \) for all \( n \in \mathbb{Z} \) (in other words, \( \phi \) must be a homomorphism of degree \( 0 \) of graded \( \mathbb{Z} \)-modules). The data of such a homomorphism defines on \( S' \) a graded \( S' \)-module structure; equipped with this structure and its graded ring structure, we say that \( S' \) is a graded \( S' \)-algebra.

If \( M \) is also a graded \( S \)-module, then the tensor product \( M \otimes_S S' \) of graded \( S \)-modules is equipped in a natural way with a graded \( S' \)-module structure, the grading being defined as above.

**Lemma 2.1.3.** — Let \( S \) be a ring graded in positive degrees. For a subset \( E \) of \( S_+ \) consisting of homogeneous elements to generate \( S_+ \) as an \( S \)-module, it is necessary and sufficient for \( E \) to generate \( S \) as an \( S_0 \)-algebra.

**Proof.** The condition is evidently sufficient; we show that it is necessary. Let \( E_n \) (resp. \( E^n \)) be the set of elements of \( E \) equal to \( n \) (resp. \( \leq n) \); it suffices to show, by induction on \( n > 0 \), that \( S_n \) is the \( S_0 \)-module generated by the elements of degree \( n \) which are products of elements of \( E^n \). This is evident for \( n = 1 \) by virtue of the hypothesis; the latter also shows that \( S_n = \sum_{i=0}^{n-1} S_{i} E_{n-i} \), and the induction argument is then immediate.

\( \square \)

**Corollary 2.1.4.** — For \( S_+ \) to be an ideal of finite type, it is necessary and sufficient for \( S \) to be an \( S_0 \)-algebra of finite type.

**Proof.** We can always assume that a finite system of generators of the \( S_0 \)-algebra \( S \) (resp. of the \( S \)-ideal \( S_+ \)) consists of homogeneous elements, by replacing each of the generators considered by its homogeneous components.

\( \square \)

**Corollary 2.1.5.** — For \( S \) to be Noetherian, it is necessary and sufficient for \( S_0 \) to be Noetherian and for \( S \) to be an \( S_0 \)-algebra of finite type.

**Proof.** The condition is evidently sufficient; it is necessary, since \( S_0 \) is isomorphic to \( S/S_+ \) and \( S_+ \) must be an ideal of finite type (2.1.4).

\( \square \)

**Lemma 2.1.6.** — Let \( S \) be a ring graded in positive degrees, which is an \( S_0 \)-algebra of finite type. Let \( M \) be a graded \( S \)-module of finite type. Then:

\( i \) The \( M_n \) are \( S_0 \)-modules of finite type, and there exists an integer \( n_0 \) such that \( M_n = 0 \) for \( n \leq n_0 \).

(ii) There exists an integer \( n_1 \) and an integer \( h > 0 \) such that, for every integer \( n \geq n_1 \), we have \( M_{n+h} = S_h M_n \).

(iii) For every pair of integers \( (d, k) \) such that \( d > 0, 0 \leq k \leq d - 1 \), \( M^{(d,k)} \) is an \( S^{(d)} \)-module of finite type.

(iv) For every integer \( d > 0 \), \( S^{(d)} \) is an \( S_0 \)-algebra of finite type.
(v) There exists an integer \( h > 0 \) such that \( S_{nh} = (S_h)^m \) for all \( m > 0 \).

(vi) For every integer \( n > 0 \), there exists an integer \( m_0 \) such that \( S_m \subset S_+^n \) for all \( m \geq m_0 \).

**Proof.** We can assume that \( S \) is generated (as an \( S_0 \)-algebra) by homogeneous elements \( f_i \) of degrees \( h_i \) \((1 \leq i \leq r)\), and \( M \) is generated (as an \( S \)-module) by homogeneous elements \( x_j \) of degrees \( k_j \) \((1 \leq j \leq s)\). It is clear that \( M_n \) is formed by linear combinations, with coefficients in \( S_0 \), of elements \( f_1^{a_1} \cdots f_r^{a_r} x_j \) such that the \( a_i \) are integers \( \geq 0 \) satisfying \( k_j + \sum_i a_i h_i = n \); for each \( j \), there are only finitely many systems \((a_i)\) satisfying this equation, since the \( h_i \) are \( > 0 \), hence the first assertion of (i); the second is evident. On the other hand, let \( h \) be the l.c.m. of the \( h_i \) and set \( g_i = f_i^{h/h_i} \) \((1 \leq i \leq r)\) such that all the \( g_i \) are of degree \( h \); let \( z_i \) be the elements of \( M \) of the form \( f_1^{a_1} \cdots f_r^{a_r} x_j \) with \( 0 \leq a_i < h/h_i \) for \( 1 \leq i \leq r \); there are finitely many of these elements, so let \( n_1 \) be the largest of their degrees. It is clear that for \( n \geq n_1 \), every element of \( M_{n+h} \) is a linear combination of the \( z_i \) whose coefficients are monomials of degree \( \leq 0 \) with respect to the \( g_i \); so we have \( M_{n+h} = S_p M_n \) which establishes (ii). In a similar way, we see (for all \( d > 0 \)) that an element of \( M^{(d,k)} \) is a linear combinations, with coefficients in \( S_0 \), of elements of the form \( g_i^{d} f_1^{\alpha_1} \cdots f_r^{\alpha_r} x_j \) with \( 0 \leq \alpha_i < d \), \( g \) being a homogeneous element of \( S \); hence (iii); (iv) then follows from (iii) and from Lemma (2.1.3), by taking \( M = S_+ \), since \((S_+)^{(d)} = (S(d))_+ \). The assertion of (v) is deduced from (ii) by taking \( M = S \).

Finally, for a given \( n \), there are finitely many systems \((a_i)\) such that \( a_i \geq 0 \) and \( \sum a_i h_i < n \), so if \( m_0 \) is the largest value of the sum \( \sum a_i h_i \) of these systems, then we have \( S_m \subset S_+^n \) for \( m > m_0 \), which proves (vi). \( \square \)

**Corollary (2.1.7).** — If \( S \) is Noetherian, then so is \( S^{(d)} \) for every integer \( d > 0 \).

**Proof.** This follows from (2.1.5) and (2.1.6), (iv). \( \square \)

(2.1.8). Let \( p \) be a graded prime ideal of the graded ring \( S \); \( p \) is thus a direct sum of the subgroups \( p_n = p \cap S_n \). Suppose that \( p \) does not contain \( S_+ \). Then if \( f \in S_+ \) is not in \( p \), the relation \( f^n x \in p \) is equivalent to \( x \in p \); in particular, if \( f \in S_d \) \((d > 0)\), for all \( x \in S_{m-nd} \), then the relation \( f^n x \in p_m \) is equivalent to \( x \in p_{m-nd} \).

**Proposition (2.1.9).** — Let \( n_0 \) be an integer \( > 0 \); for all \( n \geq n_0 \), let \( p_n \) be a subgroup of \( S_n \). For there to exist a graded prime ideal \( p \) of \( S \) not containing \( S_+ \) and such that \( p \cap S_n = p_n \) for all \( n \geq n_0 \), it is necessary and sufficient for the following conditions to be satisfied:

1. \( S_m p_n \subset p_{m+n} \) for all \( m \geq 0 \) and all \( n \geq n_0 \).
2. For \( m \geq n_0 \), \( n \geq n_0 \), \( f \in S_m \), \( g \in S_n \), the relation \( f g \in p_{m+n} \) implies \( f \in p_m \) or \( g \in p_n \).
3. \( p_n \neq S_n \) for at least one \( n \geq n_0 \).

In addition, the graded prime ideal \( p \) is then unique.

**Proof.** It is evident that the conditions (1st) and (2nd) are necessary. In addition, if \( p \not\supset S_+ \), then there exists at least one \( k > 0 \) such that \( p \cap S_k \neq S_k \); if \( f \in S_k \) is not in \( p \), the relation \( p \cap S_n = S_n \) implies \( p \cap S_{n-k} = S_{n-k} \) according to (2.1.8); therefore, if \( p \cap S_n = S_n \) for a certain value of \( n \), we would have \( p \supset S_+ \) contrary to the hypothesis, which proves that (3rd) is necessary. Conversely, suppose that the conditions (1st), (2nd), and (3rd) are satisfied. Note that if for an integer \( d \geq n_0 \), \( f, g \in S_d \) is not in \( p_d \), then, if \( p \) exists, \( p_{m+n} \) for \( m < n_0 \) is necessarily equal to the sum of the \( x \in S_m \) such that \( f^r x \in p_{m+rd} \) except for a finite number of values of \( r \). This already proves that if \( p \) exists, then it is unique. It remains to show that if we define the \( p_m \) for \( m < n_0 \) by the previous condition, then \( p = \bigcap_{m=0}^{n_0} p_m \) is a prime ideal. First, note that by virtue of (2nd), for \( m \geq n_0 \), \( p_m \) is also defined as the set of the \( x \in S_m \) such that \( f^r x \in p_{m+rd} \) except for a finite number of values of \( r \) (so \( g \in S_m, x \in p_m \) then we have \( f^r g x \in p_{m+rd} \) except for a finite number of values of \( r \), so \( g \in p_{m+rd} \)), which proves that \( p \) is an ideal of \( S \). To establish that this ideal is prime, in other words that the ring \( S/p \), graded by the subgroups \( S_n/p_n \), is an integral domain, it suffices (by considering the components of higher degree of two elements of \( S/p \)) to prove that if \( x \in S_m \) and \( y \in S_n \) are such that \( x \notin p_m \) and \( y \notin p_m \), then \( x y \notin p_{m+n} \). If not, for \( r \) large enough, we would have \( f^{2r} x y \in p_{m+n+2rd} \), but we have \( f^{r} y \notin p_{m+rd} \) for all \( r > 0 \); it then follows from (2nd) that, except for a finite number of values of \( r \), we have \( f^{r} x \in p_{m+rd} \), and we conclude that \( x \in p_m \) contrary to the hypothesis. \( \square \)
We note that if, in the graded ring $S$, the radical of $0$ in $S$ is established in a similar way.

$(2.1.10)$ We say that a subset $\mathfrak{J}$ of $S_+$ is an ideal of $S_+$ if it is an ideal of $S$, and $\mathfrak{J}$ is a graded prime ideal of $S_+$ if it is the intersection of $S_+$ and a graded prime ideal of $S$ not containing $S_+$ (this prime ideal is also unique according to Proposition $(2.1.9)$). If $\mathfrak{J}$ is an ideal of $S_+$, the radical of $\mathfrak{J}$ in $S_+$ is the set of elements of $S_+$ which have a power in $\mathfrak{J}$, in other words the set $\mathfrak{p}_+(\mathfrak{J}) = \mathfrak{p}(\mathfrak{J}) \cap S_+$. In particular, the radical of $0$ in $S_+$ is then called the nilradical of $S_+$ and denoted by $\mathfrak{N}_+$: this is the set of nilpotent elements of $S_+$. If $\mathfrak{J}$ is an graded ideal of $S_+$, then its radical $\mathfrak{p}_+(\mathfrak{J})$ is a graded ideal: by passing to the quotient ring $S/\mathfrak{J}$, we can reduce to the case $\mathfrak{J} = 0$, and it remains to see that if $x = x_h + x_{h+1} + \cdots + x_k$ is nilpotent, then so are the $x_i \in S_i (1 \leq h \leq i \leq k)$; we can assume $x_k \neq 0$ and the component of highest degree of $x^n$ is then $x_k^n$, hence $x_k$ is nilpotent, and we then argue by induction on $k$. We say that the graded ring $S$ is essentially reduced if $\mathfrak{N}_+ = 0$, in other words, if $S_+$ does not contain nilpotent elements $\neq 0$.

$(2.1.11)$ We note that if, in the graded ring $S$, an element $x$ is a zero-divisor, then so is its component of highest degree. We say that a ring $S$ is essentially integral if the ring $S_+$ (without the unit element) does not contain a zero-divisor and is $\neq 0$: it suffices that a homogeneous element $\neq 0$ in $S_+$ is not a zero-divisor in this ring. It is clear that if $p$ is a graded prime ideal of $S_+$, then $S/p$ is essentially integral.

Let $S$ be an essentially integral graded ring, and let $x_0 \in S_0$: if there then exists a homogeneous element $f \neq 0$ of $S_+$ such that $x_0 f = 0$, then we have $x_0 S_+ = 0$, since we have $(x_0 g) f = (x_0 f) g = 0$ for all $g \in S_+$, and the hypothesis thus implies $x_0 g = 0$. For $S$ to be integral, it is necessary and sufficient for $S_0$ to be integral and the annihilator of $x_0$ in $S_0$ to be 0.

2.2. Rings of fractions of a graded ring

$(2.2.1)$ Let $S$ be a graded ring, in positive degrees, $f$ a homogeneous element of $S$, of degree $d > 0$; then the ring of fractions $S' = S_f$ is graded, taking for $S'_n$ the set of the $x/f^k$, where $x \in S_{n+kd}$ with $k \geq 0$ (we observe here that $n$ can take arbitrary negative values); we denote the subring $S'_0 = (S_f)_0$ of $S'$ consisting of elements of degree 0 by the notation $S(f)_0$.

If $f \in S_d$, then the monomials $(f/1)^h$ in $S_f$ ($h$ a positive or negative integer) form a free system over the ring $S(f)$, and the set of their linear combinations is none other than the ring $(S(d)/f)_f$, which is thus isomorphic to $S(f)[T, T^{-1}] = S(f) \otimes_{\mathbb{Z}} \mathbb{Z}[T, T^{-1}]$ (where $T$ is an indeterminate). Indeed, if we have a relation $\sum_{h=-a}^b z_h(f/1)^h = 0$ with $z_h = x_h/f^m$, where the $x_h$ are in $S_{md}$, then this relation is equivalent by definition to the existence of a $k > -a$ such that $\sum_{h=-a}^b f^{h+k} x_h = 0$, and as the degrees of the terms of this sum are distinct, we have $f^{h+k} x_h = 0$ for all $h$, hence $z_h = 0$ for all $h$.

If $M$ is a graded $S$-module, then $M' = M_f$ is a graded $S_f$-module, $M'_0$ being the set of the $z/f^k$ with $z \in M_{n+kd}$ ($k \geq 0$); we denote by $M(f)$ the set of the homogenous elements of degree 0 of $M'$, it is immediate that $M(f)$ is an $S(f)$-module and that we have $(M'(d))_f = M(f) \otimes_{S(f)} (S(d)/f)$. We say that a graded ring $S$ is essentially reduced if $\mathfrak{N}_+ = 0$, in other words, if $S_+$ does not contain nilpotent elements $\neq 0$.

**Lemma (2.2.2).** Let $d$ and $e$ be integers $> 0$, $f \in S_d$, $g \in S_e$. There exists a canonical ring isomorphism

$$S(fg) \simeq (S(f))_{g^d/f^e};$$

if we canonically identify these two rings, then there exists a canonical module isomorphism

$$M(fg) \simeq (M(f))_{g^d/f^e}.$$

**Proof.** Indeed, $fg$ divides $f^g g^d$, and this latter element divides $(fg)^{de}$, so the graded rings $S(fg)$ and $S(f^g g^d)$ are canonically identified; on the other hand, $S(f^g g^d)$ also identifies with $(S(f'))_{g^d/f^e}$, and as $f'/1$ is invertible in $S_{p'}$, $S(f^g g^d)$ also identifies with $(S(p'))_{g^d/f^e}$. The element $g^d/f^e$ is of degree 0 in $S_{p'}$, we immediately conclude that the subring of $(S(p'))_{g^d/f^e}$ consisting of elements of degree 0 is $(S(f'))_{g^d/f^e}$, and as we evidently have $S(f') = S(f)$, this proves the first part of the proposition; the second is established in a similar way. \[\square\]
(2.2.3). Under the hypotheses of (2.2.2), it is clear that the canonical homomorphism \( S_f \to S_{fg} \)
(0, 1.4.1), which sends \( x/f^k \) to \( g^k x/(fg)^k \), is of degree 0, thus gives by restriction a canonical homomorphism \( S(f) \to S_{(fg)} \), such that the diagram

\[
\begin{array}{ccc}
S(f) & \xrightarrow{\sim} & (S(f))_{g^d/f^d} \\
\downarrow & & \downarrow \\
S_{(fg)} & \to & (S_{(fg)})_{g^d/f^d}
\end{array}
\]

is commutative. We similarly define a canonical homomorphism \( M(f) \to M_{(fg)} \).

**Lemma (2.2.4).** — If \( f \) and \( g \) are two homogeneous elements of \( S_+ \), then the ring \( S_{(fg)} \) is generated by the union of the canonical images of \( S(f) \) and \( S(g) \).

**Proof.** By virtue of Lemma (2.2.2), it suffices to see that \( 1/(g^d/f^e) = f^{d+e}/(fg)^d \) belongs to the canonical image of \( S(g) \) in \( S_{(fg)} \), which is evident by definition.

**Proposition (2.2.5).** — Let \( d \) be an integer > 0 and let \( f \in S_d \). Then there exists a canonical ring isomorphisms \( S_{(f)} \simeq S^{(d)}/(f - 1)S^{(d)}; \) if we identify these two rings by this isomorphism, then there exists a canonical module isomorphism \( M_{(f)} \simeq M^{(d)}/(f - 1)M^{(d)} \).

**Proof.** The first of these isomorphisms is defined by sending \( x/f^n \), where \( x \in S_{nd} \), to the element \( \overline{x} \), the class of \( x \) mod. \( (f - 1)S^{(d)} \); this map is well-defined, because we have the congruence \( f^h x \equiv x \) mod. \( (f - 1)S^{(d)} \) for all \( x \in S^{(d)} \), so if \( f^h x = 0 \) for an \( h > 0 \), then we have \( \overline{x} = 0 \). On the other hand, if \( x \in S_{nd} \) is such that \( x = (f - 1)y \) with \( y = y_{hd} + y_{(h+1)d} + \cdots + y_{kd} \) with \( y_{jd} \in S_{jd} \) and \( y_{hd} \neq 0 \), then we necessarily have \( h = n \) and \( x = -y_{hd} \), as well as the relations \( y_{(j+1)d} = fy_{jd} \) for \( h \leq j \leq k - 1 \), \( fy_{kd} = 0 \), which ultimately gives \( f^{k-n} x = 0 \); we send every class \( \overline{x} \) mod. \( (f - 1)S^{(d)} \) of an element \( x \in S_{nd} \) to the element \( x/f^n \) of \( S_{(f)} \), since the preceding remark shows that this map is well-defined. It is immediate that these two maps thus defined are ring homomorphisms, each the reciprocal of the other. We proceed exactly the same way for \( M \).

**Corollary (2.2.6).** — If \( S \) is Noetherian, then so is \( S_{(f)} \) for \( f \) homogeneous of degree > 0.

**Proof.** This follows immediately from Corollary (2.1.7) and Proposition (2.2.5).

(2.2.7). Let \( T \) be a multiplicative subset of \( S_+ \) consisting of homogeneous elements; \( T_0 = T \cup \{1\} \) is then a multiplicative subset of \( S_+ \); as the elements of \( T_0 \) are homogeneous, the ring \( T_0^{-1}S \) is still graded in the evident way; we denote by \( S(T) \) the subring of \( T_0^{-1}S \) consisting of elements of order 0, that is to say, the elements of the form \( x/h \), where \( h \in T \) and \( x \) is homogeneous of degree equal to that of \( h \). We know (0, 1.4.5) that \( T_0^{-1}S \) is canonically identified with the inductive limit of the rings \( S_f \), where \( f \) varies over \( T \) (with respect to the canonical homomorphisms \( S_f \to S_{fg} \)); as this identification respects the degrees, it identifies \( S(T) \) with the inductive limit of the \( S(f) \) for \( f \in T \). For every graded \( S \)-module \( M \), we similarly define the module \( M(T) \) (over the ring \( S(T) \)) consisting of elements of degree 0 of \( T_0^{-1}M \), and we see that this module is the inductive limit of the \( M(f) \) for \( f \in T \).

If \( p \) is a graded prime ideal of \( S_+ \), then we denote by \( S_{(p)} \) and \( M_{(p)} \) the ring \( S(T) \) and the module \( M(T) \) respectively, where \( T \) is the set of homogeneous elements of \( S_+ \) which do not belong to \( p \).
2.3. Homogeneous prime spectrum of a graded ring

(2.3.1). Given a graded ring $S$, in positive degrees, we call the **homogeneous prime spectrum** of $S$ and denote it by $\text{Proj}(S)$ the set of graded prime ideals of $S_+$ (2.1.10), or equivalently the set of graded prime ideals of $S$ not containing $S_+$; we will define a scheme structure having $\text{Proj}(S)$ as the underlying set.

(2.3.2). For every subset $E$ of $S$, let $V_+(E)$ be the set of graded prime ideals of $S$ containing $S$ not containing $S_+$; this is thus the subset $V(E) \cap \text{Proj}(S)$ of $\text{Spec}(S)$. From (I, 1.1.2) we deduce:

(2.3.2.1) $V_+(0) = \text{Proj}(S)$, $V_+(S) = V_+(S_+) = \emptyset$,

(2.3.2.2) $V_+(\bigcup \lambda E_{\lambda}) = \bigcap \lambda V_+(E_{\lambda})$,

(2.3.2.3) $V_+(EE') = V_+(E) \cup V_+(E')$.

We do not change $V_+(E)$ by replacing $E$ with the graded ideal generated by $E$; in addition, if $\mathfrak{J}$ is a graded ideal of $S$, then we have

(2.3.2.4) $V_+(\mathfrak{J}) = V_+(\bigcup_{q\geq n}(\mathfrak{J} \cap S_q))$ for all $n > 0$: indeed, if $p \in \text{Proj}(S)$ contains the homogeneous elements of $\mathfrak{J}$ of degree $\geq n$, then as by hypothesis there exists a homogeneous element $f \in S_q$ not contained in $p$, for every $m \geq 0$ and every $x \in S_m \cap \mathfrak{J}$, we have $f^r x \in \mathfrak{J} \cap S_{m+rd}$ for all but finitely many values of $r$, so $f^r x \in p \cap S_{m+rd}$, which implies that $x \in p \cap S_m$ (2.1.9).

Finally, we have, for every graded ideal $\mathfrak{J}$ of $S$,

(2.3.2.5) $V_+(\mathfrak{J}) = V_+(\mathfrak{J} \cap S)$.

(2.3.3). By definition, the $V_+(E)$ are the closed subsets of $X = \text{Proj}(S)$ for the topology induced by the spectral topology of $\text{Spec}(S)$, which we also call the **spectral topology** on $X$. For all $f \in S$, we set

(2.3.3.1) $D_+(f) = D(f) \cap \text{Proj}(S) = \text{Proj}(S) - V_+(f)$

and so, for any two elements $f$ and $g$ of $S$ (I, 1.1.9.1),

(2.3.3.2) $D_+(fg) = D_+(f) \cap D_+(g)$.

Proposition (2.3.4). — The $D_+(f)$, as $f$ runs over the set of homogeneous elements of $S_+$, form a base for the topology of $X = \text{Proj}(S)$.

**Proof.** It follows from (2.3.2.2) and (2.3.2.4) that every closed subset of $X$ is the intersection of sets of the form $V_+(f)$, where $f$ is homogeneous of degree $> 0$. $\square$

(2.3.5). Let $f$ be a **homogeneous** element of $S_+$, of degree $d > 0$;

§3. HOMOGENEOUS SPECTRUM OF A SHEAF OF GRADED ALGEBRAS

3.1. Homogeneous spectrum of a quasi-coherent graded $\mathcal{O}_Y$-algebra

§4. PROJECTIVE BUNDLES; AMPLE SHEAVES

4.1. Definition of projective bundles

**Definition (4.1.1).** — Let $Y$ be a prescheme, $\mathcal{E}$ a quasi-coherent $\mathcal{O}_Y$-module, and $S_{\mathcal{O}_Y}(\mathcal{E})$ the symmetric $\mathcal{O}_Y$-algebra of $\mathcal{E}$ (1.7.4), which is quasi-coherent (1.7.7). We define the **projective bundle on $Y$ defined by $\mathcal{E}$**, denoted $P(\mathcal{E})$, to be the $Y$-scheme $P = \text{Proj}(S_{\mathcal{O}_Y}(\mathcal{E}))$. The $\mathcal{O}_P$-module $\mathcal{O}_P(1)$ is called the **fundamental sheaf on $P$**.

When $Y$ is affine of ring $A$, then we have $\mathcal{E} = \tilde{E}$ for some $A$-module $E$, and we then write $P(E)$ instead of $P(\tilde{E})$.

When we take $\mathcal{E} = \mathcal{O}_Y^m$, we write $P^n_Y$ instead of $P(\mathcal{E})$; if, further, $Y$ is affine of ring $A$, then we also write $P^n_A$ instead of $P_Y^n$. Since $S_{\mathcal{O}_Y}(\mathcal{E}_Y)$ is canonically identified with $\mathcal{O}_Y[T]$(1.7.4), $P_Y^m$ is canonically identified with $Y$ (3.1.7); Example (2.4.3) is then exactly $P_k^1$. 

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(4.1.2). Let \( \mathcal{E} \) and \( \mathcal{F} \) be quasi-coherent \( \mathcal{O}_Y \)-modules; let \( u : \mathcal{E} \to \mathcal{F} \) be an \( \mathcal{O}_Y \)-homomorphism; there is a canonically corresponding homomorphism \( S(u) : S_{\mathcal{O}_Y}(\mathcal{E}) \to S_{\mathcal{O}_Y}(\mathcal{F}) \) of graded \( \mathcal{O}_Y \)-algebras (1.7.4). If \( u \) is surjective, then so too is \( S(u) \), and thus (3.6.2, (i)) \( \text{Proj}(S(u)) \) is a closed immersion \( \mathbb{P}(\mathcal{F}) \to \mathbb{P}(\mathcal{E}) \), which we denote by \( \mathbb{P}(u) \). We can thus say that \( \mathbb{P}(\mathcal{E}) \) is a contravariant functor in \( \mathcal{E} \), with the condition that we only consider surjective \( \mathcal{O}_Y \)-modules.

Still supposing that \( u \) is surjective, and letting \( P = \mathbb{P}(\mathcal{E}), Q = \mathbb{P}(\mathcal{F}) \), and \( j = \mathbb{P}(u) \), we have, up to isomorphism, that

\[
j^*(\mathcal{O}_P(n)) = \mathcal{O}_Q(n) \quad \text{for all } n \in \mathbb{Z}
\]

by (3.6.3).

(4.1.3). Now let \( \psi : Y' \to Y \) be a morphism, and let \( \mathcal{E}' = \psi^*(\mathcal{E}) \); then \( S_{\mathcal{O}_Y'}(\mathcal{E}') = \psi^*(S_{\mathcal{O}_Y}(\mathcal{E})) \) (1.7.5); thus (3.5.3)

\[
\mathbb{P}(\psi^*(\mathcal{E}')) = \mathbb{P}(\mathcal{E}) \times_Y Y'
\]

up to canonical isomorphism; furthermore, we clearly have that

\[
\psi^*(S_{\mathcal{O}_Y}(\mathcal{E}))(n) = (S_{\mathcal{O}_Y'}(\mathcal{E}'))(n)
\]

for all \( n \in \mathbb{Z} \), whence, letting \( P = \mathbb{P}(\mathcal{E}) \) and \( P' = \mathbb{P}(\mathcal{E}') \), we have (3.5.4), up to isomorphism, that

\[
\mathcal{O}_P'(n) = \mathcal{O}_P(n) \otimes_Y \mathcal{O}_Y' \quad \text{for all } n \in \mathbb{Z}.
\]

**Proposition (4.1.4).** — Let \( \mathcal{L} \) be an invertible \( \mathcal{O}_Y \)-module. For every quasi-coherent \( \mathcal{O}_Y \)-module \( \mathcal{E} \), there exists a canonical \( Y \)-isomorphism \( i_{\mathcal{L}} : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E} \otimes \mathcal{L}) \); furthermore, if we let \( P = \mathbb{P}(\mathcal{E}) \) and \( Q = \mathbb{P}(\mathcal{E} \otimes \mathcal{L}) \), then \( i_{\mathcal{L}}^* \mathcal{O}_Q(n) \) is canonically isomorphic to \( \mathcal{O}_P(n) \otimes_Y \mathcal{L}^\otimes n \) for all \( n \in \mathbb{Z} \).

**Proof.** Note first of all that, if \( A \) is a ring, \( E \) an \( A \)-module, and \( L \) a free monogenous \( A \)-module, then we can canonically define a homomorphism of \( A \)-modules

\[
S_n(E \otimes L) \longrightarrow S_n(E) \otimes L^\otimes n
\]

by sending \((x_1 \otimes y_1) \ldots (x_n \otimes y_n)\) to the element

\[
(x_1 x_2 \ldots x_n) \otimes (y_1 \otimes y_2 \otimes \ldots \otimes y_n)
\]

\((x_i \in E, y_i \in L, \text{ for } i \leq i \leq n)\);

we can immediately see (by restricting to the case where \( L = A \)) that this homomorphism is in fact an isomorphism. We thus obtain a canonical isomorphism of graded \( A \)-algebras \( S_A(E \otimes L) \to \bigoplus_{n \geq 0} S_n(E) \otimes L^\otimes n \). By returning to the conditions of (4.1.4), the above remarks allow us to define a canonical isomorphism of graded \( \mathcal{O}_Y \)-algebras

\[
S_{\mathcal{O}_Y}(\mathcal{E} \otimes \mathcal{O}_Y \mathcal{L}) \simto \bigoplus_{n \geq 0} S_n(\mathcal{E}) \otimes \mathcal{O}_Y \mathcal{L}^\otimes n
\]

by defining this isomorphism as an isomorphism of presheaves, and taking into account (1.7.4), (I, 1.3.9), and (I, 1.3.12). The proposition then follows from (3.1.8, (iii)) and (3.2.10). \( \Box \)

(4.1.5). With the hypotheses of (4.1.1), let \( P = \mathbb{P}(\mathcal{E}) \), and denote by \( p \) the structure morphism \( P \to Y \).

Since, by definition, \( \mathcal{E} = (S_{\mathcal{O}_Y}(\mathcal{E}))_Y \), we have a canonical homomorphism \( a_1 : \mathcal{E} \to p_*(\mathcal{O}_P(1)) \) (3.3.2.2), and thus (0, 4.4.3) also a canonical homomorphism

\[
a_1^* : p^*(\mathcal{E}) \longrightarrow \mathcal{O}_P(1).
\]

**Proposition (4.1.6).** — The canonical homomorphism \( a_1^* \) is surjective.

**Proof.** We have seen, in (3.3.2), that \( a_1^* \) corresponds functorially to the canonical homomorphism \( \mathcal{E} \otimes \mathcal{O}_P \to (S_{\mathcal{O}_Y}(\mathcal{E}))_Y(1) \); since, by definition, \( \mathcal{E} \) generates \( S_{\mathcal{O}_Y}(\mathcal{E}) \), this homomorphism is surjective, whence the conclusion, by (3.2.4) \( \Box \)
4.2. Morphisms from a prescheme to a projective bundle

(4.2.1). Keeping the notation of (4.1.5), let $X$ be a $Y$-prescheme, $q : X \to Y$ the structure morphism, and let $r : X \to P$ be a $Y$-morphism such that the following diagram commutes:

$$
\begin{array}{ccc}
P & \xleftarrow{r} & X \\
\downarrow{p} & \swarrow{q} \\
Y & \xrightarrow{} &
\end{array}
$$

Since the functor $r^*$ is right exact (0, 4.3.1), we obtain, from the surjective homomorphism in (4.1.5.1), a surjective homomorphism

$$
r^*(a^1_f) : r^*(p^*(\mathcal{O})) \longrightarrow r^*(\mathcal{O}_P(1)).
$$

But $r^*(p^*(\mathcal{O})) = q^*(\mathcal{O})$, and $r^*(\mathcal{O}_P(1))$ is locally isomorphic to $r^*(\mathcal{O}_P) = \mathcal{O}_X$, or, in other words, the latter is an invertible sheaf $\mathcal{L}_r$ on $\mathcal{O}_X$, and so we have defined, given $r$, a canonical surjective $\mathcal{O}_X$-homomorphism $\varphi_r : q^*(\mathcal{O}) \to \mathcal{L}_r$.

When $Y = \text{Spec}(A)$ is affine, and $\mathcal{O} = \mathcal{E}$, we can further clarify this homomorphism in the following way: given $f \in E$, it follows from (2.6.3) that

$$
r^{-1}(D_+(f)) = X_{\varphi_r(f)}.
$$

Now let $V$ be an affine open subset of $X$ that is contained inside $r^{-1}(D_+(f))$, and let $B$ be its ring, which is an $A$-algebra; let $S = S_A(E)$; the restriction of $r$ to $V$ corresponds to an $A$-homomorphism $\omega : S_f \to B$, and we have that $q^*(\mathcal{O})|V = (E \otimes_A B)^{-}$ and $\mathcal{L}_r|V = L_r$, whence $L_r = (S(1))_f \otimes_{S(f)} B_{[\omega]}$ (I, 1.6.5). The restriction of $\varphi_r$ to $q^*(\mathcal{O})|V$ corresponds to the $B$-homomorphism $u : E \otimes_A B \to L_r$, which sends $x \otimes 1$ to $(x/1) \otimes f = (f/1) \otimes \omega(x/f)$. The canonical extension of $\varphi_r$ to a homomorphism of $\mathcal{O}_X$-algebras

$$
\psi_r : q^*(S(\mathcal{O})) = S(q^*(\mathcal{O})) \longrightarrow S(\mathcal{L}_r) = \bigoplus_{n \geq 0} \mathcal{L}_r^\otimes n
$$

is thus such that the restriction of $\psi_r$ to $q^*(S_n(\mathcal{O}))|V$ corresponds to the homomorphism $S_n(\mathcal{E} \otimes_A B) = S_n(E) \otimes_A B \to L_r^\otimes n$ that sends $s \otimes 1$ to $(f/1)^{\otimes n} \otimes \omega(s/f^n)$. (4.2.2). Conversely, suppose that we are given a morphism $q : X \to Y$, an invertible $\mathcal{O}_X$-module $\mathcal{L}$, and a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{E}$; to each homomorphism $\varphi : q^*(\mathcal{O}) \to \mathcal{L}$ there canonically corresponding homomorphism of quasi-coherent $\mathcal{O}_X$-algebras

$$
\psi : S(\mathcal{E}) = q^*(S(\mathcal{O})) = S(q^*(\mathcal{O})) \longrightarrow \bigoplus_{n \geq 0} \mathcal{L}^\otimes n
$$

and thus (3.7.1) a $Y$-morphism $r_{\mathcal{L}, \mathcal{E}} : G(\psi) \to \text{Proj}(S(\mathcal{O})) = \text{Proj}(\mathcal{E})$, which we denote by $r_{\mathcal{L}, \mathcal{E}} \varphi$. If $\varphi$ is surjective, then so too is $\psi$, and thus (3.7.4) $r_{\mathcal{L}, \mathcal{E}} \varphi$ is everywhere defined. Furthermore, with the notation of (4.2.1) and (4.2.2):

**Proposition (4.2.3).** Given a morphism $q : X \to Y$ and a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{E}$, maps $r \to (\mathcal{L}_r, \varphi_r)$ and $(\mathcal{L}, \varphi) \to r_{\mathcal{L}, \mathcal{E}} \varphi$ give a bijective correspondence between the set of $Y$-morphisms $r : X \to \text{Proj}(\mathcal{E})$ and the set of equivalence classes of pairs $(\mathcal{L}, \varphi)$ of an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a surjective homomorphism $\varphi : q^*(\mathcal{O}) \to \mathcal{L}$, where such pairs $(\mathcal{L}, \varphi)$ and $(\mathcal{L}', \varphi')$ are defined to be equivalent if there exists an $\mathcal{O}_X$-isomorphism $\tau : \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ such that $\varphi' = \tau \circ \varphi$.

**Proof.** Start first with a $Y$-morphism $r : X \to \text{Proj}(\mathcal{E})$, and construct $\mathcal{L}_r$ and $\varphi_r$ (4.2.1), and let $r' = r_{\mathcal{L}_r, \mathcal{E}} \varphi_r$; it follows immediately from (4.2.1) and (3.7.2) that the morphisms $r$ and $r'$ are identical (by taking the generator of $\mathcal{L}_r$ to be the element $(f/1) \otimes 1$ to define the homomorphisms $\nu_n$ of (3.7.2)). Conversely, take a pair $(\mathcal{L}, \varphi)$ and construct $r'' = r_{\mathcal{L}, \mathcal{E}} \varphi$, and then $\mathcal{L}'_{\nu_n}$ and $\varphi_{\nu_n}$; we will show that there exists a canonical isomorphism $\tau : \mathcal{L}_r \xrightarrow{\sim} \mathcal{L}$ such that $\varphi = \tau \circ \varphi_{\nu_n}$; to define it, we can restrict to the case where $Y = \text{Spec}(A)$ and $X = \text{Spec}(B)$ are affine, and (with the notation of (4.2.1)
and (3.7.2) associate to each element \((x/1) \otimes 1\) of \(L_n\) (where \(x \in E\) the element \(v_1(x)c\) of \(L\). We immediately see that \(\tau\) does not depend on the chosen generator \(c\) of \(L\); since \(v_1\) is surjective by hypothesis, to prove that \(\tau\) is an isomorphism it suffices to to show that, if \(x/1 = 0\) in \((S(1))(f)\), then \(v_1(x)/1 = 0\) in \(B\); but the first equality implies that \(f^nx = 0\) in \(S_{n+1}(E)\) for some \(n\), and this implies that \(v_{n+1}(f^nx) = g^n v_1(x) = 0\) in \(B\), whence the conclusion. Finally, it is immediate that, for any two equivalent pairs \((\mathcal{L}, \varphi)\) and \((\mathcal{L}', \varphi')\), we have \(r_{\mathcal{L}, \varphi} = r_{\mathcal{L}', \varphi'}\). \(\square\)

In particular, for \(X = Y\):

**Theorem (4.2.4).** — The set of \(Y\)-sections of \(\mathbf{P}(\mathcal{E})\) is in canonical bijective correspondence with the set of quasi-coherent sub-\(\mathcal{O}_Y\)-modules \(\mathcal{F}\) of \(\mathcal{E}\) such that \(\mathcal{E} / \mathcal{F}\) is invertible.

We note that this property corresponds to the classical definition of “projective space” as the set of hyperplanes of a vector space (the classical case corresponding to \(Y = \text{Spec}(K)\), where \(K\) is a field, and \(\mathcal{E} = \mathcal{E}\), where \(E\) is a finite-dimensional \(K\)-vector space; the \(\mathcal{F}\) having the property described in (4.2.4) then correspond to the hyperplanes of \(E\), and we already know that the \(Y\)-sections of \(\mathbf{P}(\mathcal{E})\) are then the \(K\)-rational points of \(\mathbf{P}(\mathcal{E})\) (I, 3.4.5)).

**Remark (4.2.5).** — Since there is a canonical bijective correspondence between \(Y\)-morphisms from \(X\) to \(P\) and their graph morphisms, \(X\)-sections of \(P \times_Y X\) (I, 3.3.14), we see that, conversely, (4.2.3) can be deduced from (4.2.4). Denote by \(\text{Hyp}_Y(X, \mathcal{E})\) the set of quasi-coherent sub-\(\mathcal{O}_X\)-modules \(\mathcal{F}\) of \(\mathcal{E} \otimes_Y \mathcal{O}_X = q^*(\mathcal{E})\) such that \(q^*(\mathcal{E}) / \mathcal{F}\) is an invertible \(\mathcal{O}_X\)-module. If \(g : X' \to X\) is a \(Y\)-morphism, then it follows from the fact that \(g^*\) is right exact that \(g^* q^*(\mathcal{E}) / \mathcal{F} = g^* q^*(\mathcal{E}) / g^*(\mathcal{F})\), and so the latter sheaf is invertible, and thus \(\text{Hyp}_Y(X, \mathcal{E})\) is a contravariant functor into the category of \(Y\)-preschemes. We can thus interpret the theorem (4.2.4) as defining a canonical isomorphism of functors \(\text{Hom}_Y(X, \mathbf{P}(\mathcal{E}))\) and \(\text{Hyp}_Y(X, \mathcal{E})\), where both functors are contravariant in the variable \(X\) and map into the category of \(Y\)-preschemes. This also gives a characterisation of the projective bundle \(P = \mathbf{P}(\mathcal{E})\) by the following universal property, which is much closer to the geometric intuition than the constructions from §§2–3: for every morphism \(q : X \to Y\) and every invertible \(\mathcal{O}_X\)-module \(\mathcal{L}\) that is a quotient of \(\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{O}_X\), there exists a unique \(Y\)-morphism \(r : X \to P\) such that \(\mathcal{L} = r^*(\mathcal{O}_P(1))\).

We will see later that we can similarly define, amongst other things, “Grassmannian” schemes.

**Corollary (4.2.6).** — Suppose that every invertible \(\mathcal{O}_Y\)-module is trivial (I, 2.4.8). Let \(V\) be the group \(\text{Hom}_{\mathcal{O}_Y}(\mathcal{E}, \mathcal{O}_Y)\), considered as a module over the ring \(A = \Gamma(Y, \mathcal{O}_Y)\), and let \(V^*\) be the subset of \(V\) consisting of surjective homomorphisms. Then the set of \(Y\)-sections of \(\mathbf{P}(\mathcal{E})\) is canonically identified with \(V^*/A^*\), where \(A^*\) is the group of units of \(A\).

In particular:

1. The corollary (4.2.6) applies whenever \(Y\) is a local scheme (I, 2.4.8). Let \(Y\) be an arbitrary prescheme, \(y\) a point of \(Y\), and \(Y' = \text{Spec}(k(y))\); then the fibre \(p^{-1}(y)\) of \(\mathbf{P}(\mathcal{E})\) can, by (4.1.3.1), be identified with \(\mathbf{P}(\mathcal{E}^y)\), where \(\mathcal{E}^y = \mathcal{E}_y \otimes_{\mathcal{O}_y} k(y) = \mathcal{E}_y / \mathcal{M}_y\mathcal{E}_y\) is considered as a vector space over \(k(y)\). More generally, if \(K\) is an extension of \(k(y)\), then \(p^{-1}(y) \otimes_{k(y)} K\) can be identified with \(\mathbf{P}(\mathcal{E}^y \otimes_{k(y)} K)\). The corollary (4.2.6) then shows that the set of geometric points of \(\mathbf{P}(\mathcal{E})\) with values in the extension \(K\) of \(k(y)\) (I, 3.4.5), which we can also call the rational geometric fibre over \(K\) of \(\mathbf{P}(\mathcal{E})\) over the point \(y\), can be identified with the projective space associated to the dual of the \(K\)-vector space \(\mathcal{E}^y \otimes_{k(y)} K\).

2. Suppose that \(Y\) is affine of ring \(A\), and, further, that every invertible \(\mathcal{O}_Y\)-module is trivial; further, take \(\mathcal{E} = \mathcal{O}^n_V\); then, in (4.2.6), \(V\) can be identified with \(A^n\) (I, 1.3.8), and \(V^*\) with the sets of systems \((f_i)_{1 \leq i \leq n}\) of elements of \(A\) that generate the ideal \(A\); any two such systems define the same \(Y\)-section of \(\mathbf{P}_A^{-1} = \mathbf{P}_A^{-1}\), or, in other words, the same point of \(\mathbf{P}_A^{-1}\) with values in \(A\), if and only if one of them can be obtained from the other by multiplication by an invertible element of \(A\).

These properties justify the terminology “projective bundle” for \(\mathbf{P}(\mathcal{E})\). We note that the definitions that we will similarly obtain for “projective space” is in fact dual to the classical definition; this is imposed upon us by the necessity of being able to define \(\mathbf{P}(\mathcal{E})\) for arbitrary quasi-coherent \(\mathcal{O}_Y\)-modules \(\mathcal{E}\), and not just locally free ones.
Remark (4.2.7). — We will see, in Chapter V, that, if $Y$ is connected and locally Noetherian, and if $\mathcal{E}$ is locally free, then, letting $P = P(\mathcal{E})$, every invertible $\mathcal{O}_P$-module is isomorphic to an $\mathcal{O}_P$-module of the form $\mathcal{L}^l \otimes \mathcal{O}_P(m)$, with $\mathcal{L}$ some invertible $\mathcal{O}_Y$-module, well defined up to isomorphism, and $m$ some well defined integer. In other words, $H^1(Y, \mathcal{O}_Y^*)$ is isomorphic to $Z \times H^1(Y, \mathcal{O}_Y^*)$ (0, 5.4.7).

We will also see ((III, 2.1.14), taking (0, 5.4.10) into account) that $p_*(\mathcal{L} \otimes m)$ is isomorphic to $\mathcal{L}^l \otimes \mathcal{O}_Y(S_{\mathcal{O}_Y}(\mathcal{E}))_m$ if $m \geq 0$. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module, then every $Y$-morphism $P(\mathcal{E}) \to P(\mathcal{F})$ is determined by the data of an invertible $\mathcal{O}_Y$-module, an integer $m \geq 0$, and an $\mathcal{O}_Y$-homomorphism $\psi : \mathcal{F} \to \mathcal{L}^l \otimes \mathcal{O}_Y(S_{\mathcal{O}_Y}(\mathcal{E}))_m$ such that the corresponding homomorphism $\psi^2$ of $\mathcal{O}_Y(\mathcal{E})$-modules is surjective. We will also see that, if the $Y$-morphism in question is an isomorphism, then $m = 1$ and $\mathcal{F}$ is isomorphic to $\mathcal{E} \otimes \mathcal{L}^l$ (the converse of (4.1.4)).

This will allow us to determine the sheaf of germs of automorphisms of $P(\mathcal{E})$ as the quotient of the sheaf of groups $\text{Aut}(\mathcal{E})$ (which is locally isomorphic to $\text{GL}(n, \mathcal{O}_Y)$ is $\mathcal{E}$ is of rank $n$) by $\mathcal{O}_Y^*$.

(4.2.8). Keeping the notation of (4.2.1), let $u : X' \to X$ be a morphism; if the $Y$-morphism $r : X \to P$ corresponds to the homomorphism $\varphi : q^*(\mathcal{E}) \to \mathcal{L}$, then the $Y$-morphism $r \circ u$ corresponds to $u^*(\varphi) : u^*(q^*(\mathcal{E})) \to u^*(\mathcal{L})$, as follows immediately from the definitions.

(4.2.9). Let $\mathcal{E}$ and $\mathcal{F}$ be quasi-coherent $\mathcal{O}_Y$-modules, $v : \mathcal{E} \to \mathcal{F}$ a surjective homomorphism, and $j = P(v)$ the corresponding closed immersion $P(\mathcal{F}) \to P(\mathcal{E})$ (4.1.2). If the $Y$-morphism $r : X \to P(\mathcal{F})$ corresponds to the homomorphism $\varphi : q^*(\mathcal{F}) \to \mathcal{L}$, then the $Y$-morphism $j \circ r$ corresponds to $q^*(\mathcal{E}) \to q^*(\mathcal{F}) \to \mathcal{L}$; this again follows from the definition given in (4.2.1).

(4.2.10). Let $\psi : Y' \to Y$ be a morphism, and let $\mathcal{E}' = \psi^*(\mathcal{E})$. If the $Y$-morphism $r : X \to P$ corresponds to the homomorphism $\varphi : q^*(\mathcal{E}) \to \mathcal{L}$, then the $Y'$-morphism $r_{(Y')} : X_{(Y')} \to P' = P(\mathcal{E}')$ corresponds to $q_{(Y')}(\mathcal{E})' = q^*(\mathcal{E}) \otimes \mathcal{O}_{Y'} \to \mathcal{L} \otimes \mathcal{O}_{Y'}$. Indeed, by (4.1.3.1), we have the commutative diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{p_{(Y')}} & P' = P_{(Y')} \\
\downarrow & & \downarrow r_{(Y')} \\
Y & \xrightarrow{p} & P \xleftarrow{r} X
\end{array}
\]

From (4.1.3.1), we have

\[
(r_{(Y')})^*(\mathcal{O}_P(1)) = (r_{(Y')})^*(u^*(\mathcal{O}_P(1))) = v^* (r^*(\mathcal{O}_P(1))) = v^*(\mathcal{L}) = \mathcal{L} \otimes \mathcal{O}_{Y'};
\]

we also know that $u^*(\mathcal{O}_P^2)$ is exactly the canonical homomorphism $\alpha^2_1 : (p_{(Y')})^*(\mathcal{E}') \to \mathcal{O}_P(1)$; we can see this by explicitly calculating the canonical homomorphisms $\alpha^2_1$ to $P$ and $P'$ as in (4.1.6). Whence our claim.

4.3. The Segre morphism

(4.3.1). Let $Y$ be a prescheme, and $\mathcal{E}$ and $\mathcal{F}$ quasi-coherent $\mathcal{O}_Y$-modules; let $P_1 = P(\mathcal{E})$ and $P_2 = P(\mathcal{F})$, and denote the structure morphisms by $p_1 : P_1 \to Y$ and $p_2 : P_2 \to Y$. Let $Q = P_1 \times_Y P_2$, and let $q_1 : Q \to P_1$ and $q_2 : Q \to P_2$ be the canonical projections; then the $\mathcal{O}_Q$-module $\mathcal{L} = \mathcal{O}_{P_1}(1) \otimes_{\mathcal{O}_Y} \mathcal{O}_{P_2}(1) = q_1^*(\mathcal{O}_{P_1}(1)) \otimes_{\mathcal{O}_Q} q_2^*(\mathcal{O}_{P_2}(1))$ is invertible, since it is the tensor product of two invertible $\mathcal{O}_Q$-modules (0, 5.4.4). Also, if $r = p_1 \circ q_1 = p_2 \circ q_2$ is the structure morphism $Q \to Y$, then $r^*(\mathcal{E} \otimes \mathcal{F}) = q_1^*(p_1^*(\mathcal{E})) \otimes_{\mathcal{O}_Q} q_2^*(p_2^*(\mathcal{F}))$ (0, 4.3.3); the canonical surjective homomorphisms (4.1.5) $p_1^*(\mathcal{E}) \to \mathcal{O}_{P_1}(1)$ and $p_2^*(\mathcal{F}) \to \mathcal{O}_{P_2}(1)$ thus give, by taking the tensor product, a canonical homomorphism

\[
(4.3.1.1) \quad s : r^*(\mathcal{E} \otimes \mathcal{F}) \to \mathcal{L}
\]

which is evidently surjective; from this we obtain (4.2.2) a canonical morphism, called the Segre morphism:

\[
(4.3.1.2) \quad \zeta : P(\mathcal{E}) \times_Y P(\mathcal{F}) \to P(\mathcal{E} \otimes \mathcal{F}).
\]
We can study the morphism \( \zeta \) more explicitly in the case where \( Y = \text{Spec}(A) \) is affine, and \( \mathcal{E} = \widetilde{E} \) and \( \mathcal{F} = \widetilde{F} \), where \( E \) and \( F \) are \( A \)-modules, whence \( \mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F} = (E \otimes_A F)^\sim \) (I, 1.3.12); let \( R = S_A(E), S = S_A(F), \) and \( T = S_A(E \otimes_A F) \); let \( f \in E \) and \( g \in F \), and consider the affine open \( D_+(f) \times_Y D_+(g) = \text{Spec}(B) \) of \( Q \), where \( B = R(f) \otimes_A S(g) \); the restriction of \( \mathcal{L} \) to this affine open is \( \widetilde{L} \), where

\[
L = (R(1))(f) \otimes_A (S(1))(g)
\]

and the element \( c = (f/1) \otimes (g/1) \) is a generator of \( L \) considered as a free \( B \)-module (2.5.7). The homomorphism (4.3.1.1) corresponds to the homomorphism

\[
(\alpha \otimes \beta b \mapsto b((x/1) \otimes (y/1))
\]

from \( (E \otimes_A F) \otimes_A B \) to \( L \). With the notation of (3.7.2), we thus have that \( v_1(x \otimes y) = (x/f) \otimes (y/g) \); the restriction of \( \zeta \) to \( D_+(f) \times_Y D_+(g) \) is a morphism from this affine scheme to \( D_+(f \otimes g) \), corresponding to the ring homomorphism \( \omega : T_{(f \otimes g)} \to R(f) \otimes_A S(g) \) defined by

\[
\omega((x \otimes y)/(f \otimes g)) = (x/f) \otimes (y/g)
\]

for \( x \in E \) and \( y \in F \).

(4.3.2). It follows from (4.2.3) that we have a canonical isomorphism

\[
\tau : \zeta^{-1}(\mathcal{E}_p(1)) \sim \mathcal{F}_p(1) \otimes_Y \mathcal{O}_p(1)
\]

where we let \( P = \text{Spec}(\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F}) \). We will show that, for \( x \in \Gamma(Y, \mathcal{E}) \) and \( y \in \Gamma(Y, \mathcal{F}) \), we have

\[
\tau(a_1(x \otimes y)) = a_1(x) \otimes a_1(y).
\]

Indeed, we can restrict to the case where \( Y \) is affine, and we then have, with the notation of (4.3.1) and (2.6.2), that \( a_1^\otimes \mathcal{E}(x \otimes y) = (x \otimes y)/1 \) in \( (T(1))(f \otimes g) \), that \( a_1^\otimes \mathcal{E}(x) = x/1 \) in \( (R(1))(f) \), and that \( a_1^\otimes \mathcal{E}(y) = y/1 \) in \( (S(1))(g) \). The definition of \( \tau \) given in (4.2.3) and the calculation of \( v_1 \) done in (4.3.1) then immediately prove the claim (4.3.2.2). From this we obtain the equation

\[
\zeta^{-1}(P_{x \otimes y}) = (P_1)x \times_Y (P_2)y
\]

with the notation of (3.4). Indeed, taking (3.3.3) into account, the equation (4.3.2.2) (by restricting to the affine case, with the help of (I, 3.2.7) and (I, 3.2.3)) leaves us only to prove the following lemma:

**Lemma 4.3.2.4.** — Let \( B \) and \( B' \) be \( A \)-algebras, and let \( Y = \text{Spec}(A), Z = \text{Spec}(B), \) and \( Z' = \text{Spec}(B') \); then \( D(t \otimes t') = D(t) \times_Y D(t') \) for any \( t \in B, t' \in B' \).

**Proof.** Indeed, if \( p : Z \times_Y Z' = Z \) and \( p' : Z \times_Y Z' = Z' \) are the canonical projections, then it follows from (I, 1.2.2.2) that \( p^{-1}(D(t)) = D(t \otimes 1) \) and \( p'^{-1}(D(t')) = D(1 \otimes t') \); the conclusion follows from (I, 3.2.7) and (I, 1.1.9.1), since \( (t \otimes 1)(1 \otimes t') = t \otimes t' \).

**Proposition 4.3.3.** — The Segre morphism is a closed immersion.

**Proof.** Since the question is local on \( Y \), we can restrict to the case where \( Y \) is affine. With the notation of (4.3.1) and (4.3.1), the \( D_+(f \otimes g) \) then form a basis for the topology of \( P \), since the \( f \otimes g \) generate \( T \) when \( f \) runs over \( E \) and \( g \) runs over \( F \). By (4.3.2.3), we also know that \( \zeta^{-1}(D_+(f \otimes g)) = D_+(f) \times_Y D_+(g) \). It thus suffices (I, 4.2.4) to prove that the restriction of \( \zeta \) to \( D_+(f) \times_Y D_+(g) \) is a closed immersion into \( D_+(f \otimes g) \). But, with the same notation, the equation (4.3.1.3) shows that \( \omega \) is surjective, which completes the proof.

(4.3.4). The Segre morphism is functorial in \( \mathcal{E} \) and \( \mathcal{F} \), if we consider only surjective homomorphisms of quasi-coherent \( \mathcal{O}_Y \)-modules. Indeed, we must then show that, if \( \mathcal{E} \to \mathcal{E}' \) is a surjective \( \mathcal{O}_Y \)-homomorphism, then the diagram

\[
\begin{array}{ccc}
P(\mathcal{E}') \times P(\mathcal{F}) & \xrightarrow{j \times 1} & P(\mathcal{E}) \times P(\mathcal{F}) \\
\zeta \downarrow & & \zeta \\
P(\mathcal{E} \otimes \mathcal{F}) & \longrightarrow & P(\mathcal{E} \otimes \mathcal{F})
\end{array}
\]
commutes, where $j$ denotes the canonical closed immersion $P(\mathcal{E}') \to P(\mathcal{E})$. Let $P_i' = P(\mathcal{E}')$ and keep the notation from (4.3.1); then $j \times 1$ is a closed immersion (I, 4.3.1) and, up to isomorphism,

$$(j \times 1)^*(\mathcal{O}_{P_1}(1) \otimes \mathcal{O}_{P_2}(1)) = j^*(\mathcal{O}_{P_1}(1)) \otimes \mathcal{O}_{P_2}(1) = \mathcal{O}_{P_1'}(1) \otimes \mathcal{O}_{P_2}(1)$$

by (4.1.21) and (I, 9.1.5); our claim then follows from (4.2.8) and (4.2.9).

(4.3.5). With the notation of (4.3.1), let $\psi : Y' \to Y$ be a morphism, and let $\mathcal{E}' = \psi^*(\mathcal{E})$ and $\mathcal{F}' = \psi^*(\mathcal{F})$; then the Segre morphism $P(\mathcal{E}') \times P(\mathcal{F}') \to P(\mathcal{E}' \otimes \mathcal{F}')$ can be identified with $s_Y(Y')$. Indeed, keeping the notation of (4.3.1), let $P'_1 = P(\mathcal{E}')$ and $P'_2 = P(\mathcal{F}')$; we know (4.1.3.1) that $P'_i$ can be identified with $(P_i)_{Y'}$ ($i = 1, 2$), and so the structure morphism $P'_1 \times P'_2 \to Y'$ can be identified with $r_{Y'}$. Also $\mathcal{E}' \otimes \mathcal{F}'$ can be identified with $\psi^*(\mathcal{E} \otimes \mathcal{F})$, and so $P(\mathcal{E}' \otimes \mathcal{F}')$ can be identified with $(P(\mathcal{E} \otimes \mathcal{F}))(Y')$. Finally, $\mathcal{O}_{P_1'}(1) \otimes \mathcal{O}_{P_2'}(1) = \mathcal{L}'$ can be identified with $L \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'}$ by (4.3.1.3) and (I, 9.1.11). The canonical homomorphism $(r_{Y'})^*(\mathcal{E}' \otimes \mathcal{F}') \to \mathcal{L}'$ can then be identified with $s_Y(Y')$, and our claim follows from (4.2.10).

**Remark (4.3.6).** The prescheme given by the sum of $P(\mathcal{E})$ and $P(\mathcal{F})$ is even canonically isomorphic to a closed subsheaf of $P(\mathcal{E} \oplus \mathcal{F})$. Indeed, the surjective homomorphisms $\mathcal{E} \oplus \mathcal{F} \to \mathcal{E}$ and $\mathcal{E} \oplus \mathcal{F} \to \mathcal{F}$ correspond to closed immersions $P(\mathcal{E}) \to P(\mathcal{E} \oplus \mathcal{F})$ and $P(\mathcal{F}) \to P(\mathcal{E} \oplus \mathcal{F})$; everything then reduces to showing that the underlying spaces of the closed subschemes of $P(\mathcal{E} \oplus \mathcal{F})$ obtained in this way have empty intersection. Since the question is local on $Y$, we can adopt the notation of (4.3.1); but $S_n(E)$ and $S_n(F)$ can be identified with submodules of $S_n(E \oplus F)$ with intersection consisting only of 0; if $p$ is a graded prime ideal of $S(E)$ such that $p \cap S_n(E) \neq S_n(E)$ for any $n \geq 0$, then there exists a corresponding graded prime ideal of $S(E \oplus F)$ whose intersection with $S_n(E)$ is $p \cap S_n(E)$; but who also contains $S_n(F)$, as we immediately see; thus no point in $\text{Proj}(S(E))$ can have the same image in $\text{Proj}(S(E \oplus F))$ as any point in $\text{Proj}(S(F))$.

### 4.4. Immersions into projective bundles; very ample sheaves

**Proposition (4.4.1).** Let $Y$ be a quasi-compact scheme, or a prescheme whose underlying space is Noetherian, $q : X \to Y$ a morphism of finite type, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module.

(i) Let $\mathcal{J}$ be a positively-graded quasi-coherent $\mathcal{O}_Y$-algebra, and $\psi : q^*(\mathcal{J}) \to \bigoplus_{n \geq 0} \mathcal{L}^\otimes n$ a homomorphism of graded algebras. For $r_{\mathcal{J},q}$ to be everywhere defined and an immersion, it is necessary and sufficient for there to exist an integer $n \geq 0$ and a quasi-coherent sub-$\mathcal{O}_Y$-module of finite type $\mathcal{E}$ of $\mathcal{J}$ such that the homomorphism $\psi' = q_n \circ q^*(j) : q^*(\mathcal{E}) \to \mathcal{L}^\otimes n = \mathcal{L}$ (where $j$ is the injection $\mathcal{E} \to \mathcal{J}$) is surjective and such that the morphism $r_{\mathcal{J},q} : X \to P(\mathcal{E})$ is an immersion.

(ii) Let $\mathcal{J}$ be a quasi-coherent $\mathcal{O}_Y$-module, and $\psi : q^*(\mathcal{J}) \to \mathcal{L}$ a surjective homomorphism. For the morphism $r_{\mathcal{J},q}$ to be an immersion $X \to P(\mathcal{J})$, it is necessary and sufficient for there to exist a quasi-coherent sub-$\mathcal{O}_Y$-module of finite type $\mathcal{E}$ of $\mathcal{J}$ such that the homomorphism $\psi' = \psi \circ q(j) : q^*(\mathcal{E}) \to \mathcal{L}$ (where $j$ is the canonical injection $\mathcal{E} \to \mathcal{J}$) is surjective and such that the morphism $r_{\mathcal{J},q} : X \to P(\mathcal{E})$ is an immersion.

**Proof.**

(i) The fact that $r_{\mathcal{J},q}$ is everywhere defined and is an immersion is equivalent, by (3.8.5), to the existence of some $n \geq 0$ and $\mathcal{E}$ such that, if $\mathcal{J}'$ is the subalgebra of $\mathcal{J}$ generated by $\mathcal{E}$, the homomorphism $q^*(\mathcal{E}) \to \mathcal{L}^\otimes n$ is surjective and the morphism $r_{\mathcal{J},q} : X \to \text{Proj}(\mathcal{J}')$ is everywhere defined and is an immersion. We already have a canonical surjective homomorphism $S(\mathcal{E}) \to \mathcal{J}'$ to which there exists a corresponding closed immersion $\text{Proj}(\mathcal{J}') \to P(\mathcal{E})$ (3.6.2); whence the conclusion.

(ii) Since $\mathcal{J}$ is the inductive limit of its quasi-coherent submodules of finite type $\mathcal{E}_i$ (I, 9.4.9), $S(\mathcal{J})$ is the inductive limit of the $S(\mathcal{E}_i)$; the conclusion then follows from (3.8.4), by observing that we can take all the $\mathcal{E}_i$ in the proof of (3.8.4) to be equal to 1: indeed, supposing that $Y$ is affine, if $r = r_{\mathcal{J},q}$ is an immersion, then $r(X)$ is a quasi-compact subspace of $P(\mathcal{J})$ that we can cover by finitely many open subsets of $P(\mathcal{J})$ of the form $D_+(f)$, with $f \in F$, such that $D_+(f) \cap r(X)$ is closed. 

\[\square\]
Definition (4.4.2). — Let \( Y \) be a prescheme, and \( q : X \to Y \) a morphism. We say that an invertible \( \mathcal{O}_X \)-module \( \mathcal{L} \) is very ample for \( q \), or relative to \( q \) (or very ample for (or relative to) \( Y \), or simply very ample, if \( q \) is clear from the context) if there exists a quasi-coherent \( \mathcal{O}_Y \)-module \( \mathcal{E} \) and a \( Y \)-immersion \( i \) from \( X \) to \( P = \text{Proj}(\mathcal{E}) \) such that \( \mathcal{L} \) is isomorphic to \( i^*(\mathcal{O}_P(1)) \).

It is equivalent (4.2.3) to say that there exists a quasi-coherent \( \mathcal{O}_Y \)-module \( \mathcal{E} \) and a surjective homomorphism \( \varphi : q^*(\mathcal{E}) \to \mathcal{L} \) such that \( r_{\mathcal{L}, \mathcal{E}} : X \to \text{Proj}(\mathcal{E}) \) is an immersion.

We note that the existence of a very ample (for \( Y \) \( \mathcal{O}_X \)-module implies that \( q \) is separated ((3.1.3) and (I, 5.5.1, (i) and (ii))).

Corollary (4.4.3). — Suppose that there exists a graded quasi-coherent \( \mathcal{O}_Y \)-algebra \( \mathcal{S} \), generated by \( \mathcal{S}_1 \), and a \( Y \)-immersion \( i : X \to \text{Proj}(\mathcal{S}) \) such that \( \mathcal{L} \) is isomorphic to \( i^*(\mathcal{O}_P(1)) \); then \( \mathcal{L} \) is very ample relative to \( q \).

Proof. If \( \mathcal{S} = \mathcal{S}_1 \), then the canonical homomorphism \( S(\mathcal{S}) \to \mathcal{S} \) is surjective, and so, by compositing with the corresponding closed immersion \( \text{Proj}(\mathcal{S}) \to \text{Proj}(\mathcal{S}_1) \) (3.6.2) and the immersion \( i \), we obtain an immersion \( j : X \to \text{Proj}(\mathcal{S}) = \text{Proj}(\mathcal{S}_1) \) such that \( \mathcal{L} \) is isomorphic to \( j^*(\mathcal{O}_{\text{Proj}(\mathcal{S}_1)}(1)) \).

Proposition (4.4.4). — Let \( q : X \to Y \) be a quasi-compact morphism, and \( \mathcal{L} \) an invertible \( \mathcal{O}_X \)-module. Then the following properties are equivalent:

(a) \( \mathcal{L} \) is very ample relative to \( q \).

(b) \( q_* (\mathcal{L}) \) is quasi-coherent, the canonical homomorphism \( \sigma : q^*(q_* (\mathcal{L})) \to \mathcal{L} \) is surjective, and the morphism \( r_{\mathcal{L}, \mathcal{S}} : X \to \text{Proj}(q_* (\mathcal{L})) \) is an immersion.

Proof. Since \( q \) is quasi-compact, we know that \( q_* (\mathcal{L}) \) is quasi-coherent if \( q \) is separated (I, 9.2.2).

We know (3.4.7) that the existence of a surjective homomorphism \( \varphi : q^*(\mathcal{E}) \to \mathcal{L} \) (with \( \mathcal{E} \) a quasi-coherent \( \mathcal{O}_Y \)-module) implies that \( \sigma \) is surjective; furthermore, given the factorisation \( q^*(\mathcal{E}) \to q^*(q_* (\mathcal{L})) \to \mathcal{L} \) of \( \varphi \), there is a canonically corresponding factorisation

\[
q^*(S(\mathcal{E})) \to q^*(S(q_* (\mathcal{L}))) \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}
\]

and so (3.8.3) the hypothesis that \( r_{\mathcal{L}, \mathcal{E}} \) is an immersion implies that so too is \( j = r_{\mathcal{L}, \mathcal{E}} \); furthermore (4.2.4), \( \mathcal{L} \) is isomorphic to \( j^*(\mathcal{O}_{\text{Proj}(\mathcal{E})}(1)) \), where \( \text{Proj}(\mathcal{E}) = \text{Proj}(\mathcal{E}_1) \). We thus see that (a) and (b) are equivalent.

Corollary (4.4.5). — Suppose that \( q \) is quasi-compact. For \( \mathcal{L} \) to be very ample relative to \( Y \), it is necessary and sufficient for there to exist an open cover \( (U_a) \) of \( Y \) such that \( \mathcal{L}|_{q^{-1}(U_a)} \) is very ample relative to \( U_a \) for every \( a \).

Proof. Indeed, condition (b) of (4.4.4) is local on \( Y \).

Proposition (4.4.6). — Let \( Y \) be a quasi-compact scheme, or a prescheme whose underlying space is Noetherian, \( q : X \to Y \) a morphism of finite type, and \( \mathcal{L} \) an invertible \( \mathcal{O}_X \)-module. Then conditions (a) and (b) of (4.4.4) are equivalent to the following:

(a') There exists a quasi-coherent \( \mathcal{O}_Y \)-module \( \mathcal{E} \) of finite type and a surjective homomorphism \( \varphi : q^*(\mathcal{E}) \to \mathcal{L} \) such that \( r_{\mathcal{L}, \mathcal{E}} \) is an immersion.

(b') There exists a coherent sub-\( \mathcal{O}_Y \)-module \( \mathcal{E} \) of \( q_* (\mathcal{L}) \) of finite type that has the properties stated in condition (a').

Proof. It is clear that (a') or (b') imply (a); also (a) implies (a'), by (4.4.1), and similarly (b) implies (b').

Corollary (4.4.7). — Suppose that \( Y \) is a quasi-compact scheme, or a Noetherian prescheme. If \( \mathcal{L} \) is very ample for \( q \), then there exists a graded quasi-coherent \( \mathcal{O}_Y \)-algebra \( \mathcal{S} \) such that \( \mathcal{S} \) is of finite type and generates \( \mathcal{S} \), and also a dominant open \( Y \)-immersion \( i : X \to \text{Proj}(\mathcal{S}) \) such that \( \mathcal{L} \) is isomorphic to \( i^*(\mathcal{O}_P(1)) \).
Proof. Indeed, condition (b) of (4.4.6) is satisfied; the structure morphism \( p : \mathcal{P}(\mathcal{E}) = P' \rightarrow Y \) is then separated and of finite type (3.1.3), and so \( P' \) is a quasi-compact scheme (resp. a Noetherian prescheme) if \( Y \) is a quasi-compact scheme (resp. a Noetherian prescheme). Let \( Z \) be the closure (I, 9.5.11) of the subscheme \( X' \) of \( P' \) associated to the immersion \( j = r_{\mathcal{L}, \mathcal{Q}} \) from \( X \) into \( P' \); it is clear that \( j \) factors as a dominant open immersion \( i : X \rightarrow Z \) followed by the canonical injection \( Z \rightarrow P' \). But \( Z \) can be identified with a prescheme \( \text{Proj}(\mathcal{I}) \), where \( \mathcal{I} \) is a graded \( \mathcal{O}_Y \)-algebra equal to the quotient of \( \mathcal{S}(\mathcal{E}) \) by a graded quasi-coherent sheaf of ideals (3.6.2), and it is clear that \( \mathcal{I}_X \) is of finite type and generates \( \mathcal{J} \); furthermore, \( \mathcal{O}_Z(1) \) is the inverse image of \( \mathcal{O}_{P'}(1) \) by the canonical injection (3.6.3), and so \( \mathcal{L} = f^*(\mathcal{O}_Z(1)) \).

Proposition (4.4.8). — Let \( q : X \rightarrow Y \) be a morphism, \( \mathcal{L} \) a very ample (relative to \( q \)) \( \mathcal{O}_X \)-module, and \( \mathcal{L}' \) an invertible \( \mathcal{O}_X \)-module, such that there exists a quasi-coherent \( \mathcal{O}_Y \)-module \( \mathcal{E}' \) and a surjective homomorphism \( q^*(\mathcal{E}') \rightarrow \mathcal{L}' \). Then \( \mathcal{L}' \otimes \mathcal{L} \) is very ample relative to \( q \).

Proof. The hypothesis implies the existence of a \( Y \)-morphism \( r' : X \rightarrow P = \mathcal{P}(\mathcal{E}')[2] \) such that \( \mathcal{L}' = r'^*(\mathcal{O}_P(1)) \) (4.2.1). There is, by hypothesis, a quasi-coherent \( \mathcal{O}_Y \)-module \( \mathcal{E} \) and a \( Y \)-immersion \( r : X \rightarrow P = \mathcal{P}(\mathcal{E}) \) such that \( \mathcal{L} = r^*(\mathcal{O}_P(1)) \). Let \( Q = \mathcal{P}(\mathcal{E} \otimes \mathcal{E}') \), and consider the Segre morphism \( \zeta : P \times_Y P' \rightarrow Q \) (4.3.1). Since \( r \) is an immersion, so too is \( (r, r')_Y : X \rightarrow P \times_Y P' \) (I, 5.3.14); but since \( \zeta \) is an immersion (4.3.3), so too is \( r'' : X \rightarrow P \times_Y P' \zeta \rightarrow Q \). Also, \( (4.2.1) \zeta(\mathcal{O}_Q(1)) \) is isomorphic to \( \mathcal{O}_P(1) \otimes \mathcal{O}_P(1) \), and so (I, 9.1.4) \( r'^*(\mathcal{O}_Q(1)) \) is isomorphic to \( \mathcal{L} \otimes \mathcal{L}' \), which proves the proposition.

Corollary (4.4.9). — Let \( q : X \rightarrow Y \) be a morphism.

(1) Let \( \mathcal{L} \) be an invertible \( \mathcal{O}_X \)-module, and \( \mathcal{K} \) an invertible \( \mathcal{O}_Y \)-module. For \( \mathcal{L} \) to be very ample relative to \( q \), it is necessary and sufficient for \( \mathcal{L} \otimes q^*(\mathcal{K}) \) to be so.

(2) If \( \mathcal{L} \) and \( \mathcal{L}' \) are very ample (relative to \( q \)) \( \mathcal{O}_X \)-modules, then so too is \( \mathcal{L} \otimes \mathcal{L}' \); in particular, \( \mathcal{L} \otimes \mathcal{L}^n \) is very ample relative to \( q \) for all \( n > 0 \).

Proof. Claim (ii) is an immediate consequence of (4.4.8), as well as the necessity of condition (i); conversely, if \( \mathcal{L} \otimes q^*(\mathcal{K}) \) is very ample, then so too is \( (\mathcal{L} \otimes q^*(\mathcal{K})) \otimes q^*(\mathcal{K}^{-1}) \), by the above, and the latter \( \mathcal{O}_X \)-module is isomorphic to \( \mathcal{L} \otimes (0, 5.4.3) \) and \( (0, 5.4.5) \).

Proposition (4.4.10). —

(i) For every prescheme \( Y \), every invertible \( \mathcal{O}_Y \)-module \( \mathcal{L} \) is very ample relative to the identity morphism \( 1_Y \).

(i bis) Let \( f : X \rightarrow Y \) be a morphism, and \( j : X' \rightarrow X \) an immersion. If \( \mathcal{L} \) is a very ample (relative to \( f \)) \( \mathcal{O}_X \)-module, then \( f^*(\mathcal{L}) \) is very ample relative to \( f \circ j \).

(ii) Let \( Z \) be a quasi-compact prescheme, \( f : X \rightarrow Y \) a morphism of finite type, \( g : Y \rightarrow Z \) a quasi-compact morphism, \( \mathcal{L} \) a very ample (relative to \( f \)) \( \mathcal{O}_Y \)-module, and \( \mathcal{K} \) a very ample (relative to \( g \)) \( \mathcal{O}_Y \)-module. Then there exists some integer \( n_0 > 0 \) such that \( \mathcal{L} \otimes f^*(\mathcal{K}^n) \) is very ample relative to \( g \circ f \) for all \( n \geq n_0 \).

(iii) Let \( f : X \rightarrow Y \) and \( g : Y' \rightarrow Y \) be morphisms, and let \( X' = X_{(Y')} \). If \( \mathcal{L} \) is a very ample (relative to \( f \)) \( \mathcal{O}_X \)-module, then \( \mathcal{L}' = \mathcal{L} \otimes \mathcal{O}_Y \) is a very ample (relative to \( f Y' \)) \( \mathcal{O}_Y \)-module.

(iv) Let \( f_j : X_j \rightarrow Y_j \) (\( i = 1, 2 \)) be \( S \)-morphism. If \( \mathcal{L}_j \) is a very ample (relative to \( f_j \)) \( \mathcal{O}_X \)-module (\( i = 1, 2 \)), then \( \mathcal{L} \otimes \mathcal{L}_2 \) is very ample relative to \( f_1 \times S f_2 \).

(v) Let \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) be morphisms. If an \( \mathcal{O}_X \)-module \( \mathcal{L} \) is very ample relative to \( g \circ f \), then it is also very ample relative to \( f \).

(vi) Let \( f : X \rightarrow Y \) be a morphism, and \( j \) the canonical injection \( X_{\text{red}} \rightarrow X \). If an \( \mathcal{O}_X \)-module \( \mathcal{L} \) is very ample relative to \( f \), then \( f^*(\mathcal{L}) \) is very ample relative to \( f_{\text{red}} \).

Proof. Property (i bis) follows immediately from the definition (4.4.2), and it is immediate that (vi) follows formally from (i bis) and (v), by an argument copied from the proof of (I, 5.5.12), which we leave to the reader. To prove (v), we consider, as in (I, 5.5.12), the factorisation \( X \xrightarrow{f_1} X \times \check{Y} \xrightarrow{p_2} Y \), where \( p_2 = (g \circ f) \times 1_Y \). It follows from the hypothesis and from (i) and (iv) that \( \mathcal{L} \otimes \mathcal{O}_Y \mathcal{O}_Y \) is very ample for \( p_2 \); but also \( \mathcal{L} = f^*(\mathcal{L} \otimes \mathcal{O}_2 \mathcal{O}_Y) \) (I, 9.1.4), and \( f_1 \) is an immersion (I, 5.3.11); we can thus apply (i bis).
To prove (i), we apply the definition (4.4.2) with $\mathcal{E} = \mathcal{L}$, and note that then $P(\mathcal{E})$ can be identified with $Y$ (4.1.4).

Now we prove (iii). There exists a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{E}$ and a $\mathcal{Y}$-immersion $i : X \to P(\mathcal{E}) = P$ such that $\mathcal{L} = i^*(\mathcal{O}_P(1))$; if we let $\mathcal{E}' = g^*(\mathcal{E})$, then $\mathcal{E}'$ is a quasi-coherent $\mathcal{O}_Y$-module, and we have that $P' = P(\mathcal{E}') = P(\mathcal{Y})$ (4.1.3.1), that $i(\mathcal{Y})$ is an immersion from $X(\mathcal{Y})$ into $P'$ (I, 4.3.2), and that $\mathcal{L}'$ is isomorphic to $(i(\mathcal{Y}))^*(\mathcal{O}_P(1))$ (4.2.10).

To prove (iv), note that, by hypothesis, a $\mathcal{Y}$-immersion $r_i : X_i \to P_i = P(\mathcal{E}_i)$, where $\mathcal{E}_i$ is a quasi-coherent $\mathcal{O}_{X_i}$-module, and $\mathcal{L}_i = r_i^*(\mathcal{O}_{P_i}(1))$ (i = 1, 2), $r_1 \times_S r_2$ is an $S$-immersion of $X_1 \times_S X_2$ into $P_1 \times_S P_2$ (I, 4.3.1), and the inverse image of $\mathcal{O}_{P_1}(1) \otimes_S \mathcal{O}_{P_2}(1)$ under this immersion is $\mathcal{L}_1 \otimes_S \mathcal{L}_2$. Now let $T = Y_1 \times_S Y_2$, and let $p_1$ and $p_2$ be the projections from $T$ to $Y_1$ and $Y_2$, respectively. If we let $P'_i = P(p_i^*(\mathcal{E}_i))$ (i = 1, 2), then $P'_1 \times T P'_2 = (P_1 \times Y_1) \times T (P_2 \times Y_2)$ is isomorphic to a quotient of an $\mathcal{O}_Y$-module $\mathcal{M}$, and an analogous calculation (based in particular on (I, 9.1.9.1) and (I, 9.1.2)) shows that, in the above identification, $\mathcal{O}_{P'_1}(1) \otimes T \mathcal{O}_{P'_2}(1)$ can be identified with $\mathcal{O}_{P_1} \otimes_S \mathcal{O}_{P_2}$. We can thus consider $r_1 \times_S r_2$ as a $T$-immersion from $X_1 \times_S X_2$ into $P'_1 \times T P'_2$, with the inverse image of $\mathcal{O}_{P'_1}(1) \otimes T \mathcal{O}_{P'_2}(1)$ under this immersion being $\mathcal{L}_1 \otimes_S \mathcal{L}_2$. We then finish the argument as in (4.4.8) by using the Segre morphism.

It remains only to prove (ii). We can first of all restrict to the case where $Z$ is an affine scheme, since, in general, there exists a finite cover $(U_i)$ of $Z$ by affine opens; if the proposition were proven for $\mathcal{X}(g^{-1}(U_i))$, $\mathcal{L}|_{f^{-1}(g^{-1}(U_i))}$, and an integer $n_i$, then it would suffice to take $n_0$ to be the largest of the $n_i$ to prove the proposition for $\mathcal{X}$ and $\mathcal{L}$ (4.4.5). The hypothesis implies that $f$ and $g$ are separated morphisms, and so $X$ and $Y$ are quasi-compact schemes.

There is an immersion $r : X \to P = P(\mathcal{E})$, where $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_Y$-module of finite type, and $\mathcal{L} = r^*(\mathcal{O}_P(1))$, by (4.4.6). We will see that there exists a very ample (relative to the composed morphism $Z \to Y \to X$) $\mathcal{O}_P$-module $\mathcal{M}$ such that $\mathcal{O}_P(1)$ is isomorphic to $\mathcal{M} \otimes \mathcal{X} \otimes \mathcal{E}(-m)$ for some integer $m$. For $n \geq m + 1$, $\mathcal{O}_P(1) \otimes \mathcal{X} \otimes \mathcal{E}(-n)$ will then be very ample for $Z$, by hypothesis and by (iv) applied to the morphisms $h : P \to Y$ and $1_Y$; since $r$ is an immersion and $\mathcal{L} \otimes r^*(\mathcal{X} \otimes \mathcal{E}(-n)) = r^*(\mathcal{O}_P(1) \otimes \mathcal{X} \otimes \mathcal{E}(-n))$, the conclusion will then follow from (i bis). To prove our claim concerning $\mathcal{O}_P(1)$, we will use the following lemma:

**Lemma (4.4.10.1).** — Let $Z$ be a quasi-compact scheme, or a prescheme whose underlying space is Noetherian, and let $g : Y \to Z$ be a quasi-compact morphism, $\mathcal{X}$ a very ample (with respect to $g$) invertible $\mathcal{O}_Y$-module, and $\mathcal{E}$ a quasi-coherent $\mathcal{O}_Y$-module of finite type. Then there exists an integer $m_0$ such that, for all $m \geq m_0$, $\mathcal{E}$ is isomorphic to a quotient of an $\mathcal{O}_Y$-module of the form $g^*(\mathcal{F}) \otimes \mathcal{X} \otimes \mathcal{E}(-m)$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Z$-module of finite type (depending on $m$).

This lemma will be proven in (4.5.10.1); the reader can verify that (4.4.10) is not used anywhere in (4.5).

Assuming this lemma, there exists a closed immersion $j_1$ from $P$ to $P_1 = P(g^*(\mathcal{F}) \otimes \mathcal{X} \otimes \mathcal{E}(-m))$ such that $\mathcal{O}_P(1)$ is isomorphic to $j_1^*(\mathcal{O}_{P_1}(1))$ (4.2.12). Now, there exists an isomorphism from $P_1$ to $P_2 = P(g^*(\mathcal{F}))$, sending $\mathcal{O}_{P_2}(1) \otimes Y \mathcal{X} \otimes \mathcal{E}(-m)$ to $\mathcal{O}_{P_2}(1) (4.1.4)$; we thus have a closed immersion $j_2 : P \to P_2$ such that $\mathcal{O}_P(1)$ is isomorphic to $j_2^*(\mathcal{O}_{P_2}(1)) \otimes Y \mathcal{X} \otimes \mathcal{E}(-m)$. Finally, $P_2$ can be identified with $P_3 = P_2 \times_Z Y$, where $P_3 = P(\mathcal{F})$, and $\mathcal{O}_{P_2}(1)$ with $\mathcal{O}_{P_3}(1) \otimes Z \mathcal{O}_Y (4.1.3)$. By definition, $\mathcal{O}_{P_2}(1)$ is very ample for $Z$; since so too is $\mathcal{X}$, we conclude, from (iv), that $\mathcal{O}_{P_3}(1) \otimes Y \mathcal{X}$ is very ample for $Z$; so too is $\mathcal{M} = j_3^*(\mathcal{O}_{P_3}(1) \otimes Y \mathcal{X})$ by (i bis), and $\mathcal{O}_P(1)$ is isomorphic to $\mathcal{M} \otimes Y \mathcal{X} \otimes \mathcal{E}(-m - 1)$, which finishes the proof.

**Proposition (4.11).** — Let $f : X \to Y$ and $f' : X' \to Y$ be morphisms, $X''$ the sum prescheme $X \sqcup X'$, and $f''$ the morphism $X'' \to Y$ that agrees with $f$ (resp. $f'$) on $X$ (resp. $X'$). Let $\mathcal{L}$ (resp. $\mathcal{L}'$) be an invertible $\mathcal{O}_X$-module (resp. invertible $\mathcal{O}_{X'}$-module), and let $\mathcal{L}''$ be the invertible $\mathcal{O}_{X''}$-module that agrees with $\mathcal{L}$ (resp. $\mathcal{L}'$) on $X$ (resp. $X'$). For $\mathcal{L}''$ to be very ample relative to $f''$, it is necessary and sufficient for $\mathcal{L}$ to be very ample relative to $f$ and for $\mathcal{L}'$ to be very ample relative to $f'$.
Proof. We can immediately restrict to the case where $Y$ is affine. If $\mathcal{L}''$ is very ample then so too are $\mathcal{L}'$ and $\mathcal{L}'$, by (4.4.10, (i bis)). Conversely, if $\mathcal{L}$ and $\mathcal{L}'$ are very ample, then it follows immediately from the definition (4.4.2) and from (4.3.6) that $\mathcal{L}''$ is very ample. $\square$

4.5. Ample sheaves

(4.5.1). Given a prescheme $X$ and an invertible $\mathcal{O}_X$-module $\mathcal{L}$, we define, for every $\mathcal{O}_X$-module $\mathcal{F}$ (when there will be no confusion possible over $\mathcal{L}$) $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{-n}$ ($n \in \mathbb{Z}$); we also define $S = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{-n})$ (a graded subring of the ring $\Gamma_*(\mathcal{L})$ defined in (0, 5.4.6)). If we consider $X$ as a $\mathbb{Z}$-prescheme, and we denote by $p$ the structure morphism $X \to \text{Spec}(\mathbb{Z})$, then there is a bijective correspondence between homomorphisms $p^*(\widehat{S}) \to \bigoplus_{n \geq 0} \mathcal{L}^{-n}$ of graded $\mathcal{O}_X$-algebras and endomorphisms of the graded ring $S$ (I, 2.2.5); the homomorphism $\epsilon : p^*(\widehat{S}) \to \bigoplus_{n \geq 0} \mathcal{L}^{-n}$ that then corresponds to the identity automorphism of $S$ is said to be canonical. There is a corresponding (3.1.7) morphism $G(\epsilon) \to \text{Proj}(S)$ that is also said to be canonical.

Theorem (4.5.2). — Let $X$ be a quasi-compact scheme or a prescheme whose underlying space is Noetherian, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module, and $S$ the graded ring $\bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{-n})$. Then the following conditions are equivalent:

(a) When $f$ runs over the set of homogeneous elements of $S_+$, the $X_f$ form a base of the topology of $X$.

(a') When $f$ runs over the set of homogeneous elements of $S_+$, the $X_f$ that are affine form a cover of $X$.

(b) The canonical morphism $G(\epsilon) \to \text{Proj}(S)$ (4.5.1) is everywhere defined and is a dominant open immersion.

(b') The canonical morphism $G(\epsilon) \to \text{Proj}(S)$ is everywhere defined and is a homeomorphism from the underlying space of $X$ to a subspace of $\text{Proj}(S)$.

(c) For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, if we denote by $\mathcal{F}_n$ the sub-$\mathcal{O}_X$-module of $\mathcal{F}(n)$ generated by the sections of $\mathcal{F}(n)$ over $X$, then $\mathcal{F}$ is the sum of the sub-$\mathcal{O}_X$-modules $\mathcal{F}_n(−n)$ over the integers $n > 0$.

(c') Property (c) holds for every quasi-coherent sheaf of ideals of $\mathcal{O}_X$.

Furthermore, if $(f_a)$ is a family of homogeneous elements of $S_+$ such that the $X_{f_a}$ are affine, then the restriction to $\bigcup_a X_{f_a}$ of the canonical morphism $X \to \text{Proj}(S)$ is an isomorphism from $\bigcup_a X_{f_a}$ to $\bigcup_a (\text{Proj}(S))_{f_a}$.

Proof. It is clear that (b) implies (b'), and (b') implies (a) by (3.7.3.1) (taking into account the fact that $\epsilon$ is the identity). Condition (a) implies (a'), since every $x \in X$ has an affine neighbourhood $U$ such that $\mathcal{L}|U$ is isomorphic to $\mathcal{O}_X|U$; if $f \in \Gamma(X, \mathcal{L}^{-n})$ is such that $x \in X_f \subseteq U$, then $X_f$ is also the set of $x' \in U$ such that $(f|U)(x') \neq 0$, and it is thus an affine open subset (I, 1.3.6). To prove that (a') implies (b), it suffices to prove the last claim of the theorem, and to further prove that, if $X = \bigcup_a X_{f_a}$ then condition (iv) of (3.8.2) is satisfied. This latter point follows immediately from (I, 9.3.1, (ii)). As for the last claim of (4.5.2), since $X_{f_a}$ is the inverse image of $(\text{Proj}(S))_{f_a}$ under $G(\epsilon) \to \text{Proj}(S)$, it suffices to apply (I, 9.3.2). Thus (a), (a'), (b), and (b') are all equivalent.

To show that (a') implies (c), note that, if $X_f$ is affine (with $f \in S_k$), then $\mathcal{F}|X_f$ is generated by its sections over $X_f$ (I, 1.3.9); on the other hand (I, 9.3.1, (ii)), such a section $s$ is of the form $(t|X_f) \otimes (f|X_f)^{-m}$, where $t \in \Gamma(X, \mathcal{F}(km))$; by definition, $t$ is also a section of $\mathcal{F}_{km}$, so $s$ is indeed a section of $\mathcal{F}_{km}(−km)$ over $X_f$, which proves (c). It is clear that (c) implies (c'), so it remains only to show that (c') implies (a). But let $U$ be an open neighbourhood of $x \in X$, and let $\mathcal{F}$ be a quasi-coherent sheaf of ideals of $\mathcal{O}_X$ defining a closed subscheme of $X$ that has $X − U$ as its underlying space (I, 5.2.1). Hypothesis (c') implies that there exists an integer $n > 0$ and a section $f$ of $\mathcal{F}(n)$ over $X$ such that $f(x) \neq 0$. But we clearly have $f \in S_n$, and $x \in X_f \subseteq U$, which proves (a).

When $X$ is a prescheme whose underlying space is Noetherian, the equivalent conditions of (4.5.2) imply that $X$ is a scheme, since it is isomorphic to a subscheme of the scheme $S = \text{Proj}(A)$, by (4.5.2, (b)).

Definition (4.5.3). — We say that an invertible $\mathcal{O}_X$-module $\mathcal{L}$ is ample if $X$ is a quasi-compact scheme and if the equivalent conditions of (4.5.2) are satisfied.
It evidently follows from criterion (a) of (4.5.2) that, if $\mathcal{L}$ is an ample $\mathcal{O}_X$-module, then, for every open subset $U$ of $X$, $\mathcal{L}|U$ is an ample $(\mathcal{O}_X|U)$-module.

It follows from the proof of (4.5.2) that the affine $X_f$ form a base of the topology of $X$. Furthermore:

**Corollary (4.5.4).** — Let $\mathcal{L}$ be an ample $\mathcal{O}_X$-module. For every finite subspace $Z$ of $X$ and every neighbourhood $U$ of $Z$, there exists an integer $n$ and some $f \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $X_f$ is an affine neighbourhood of $Z$ contained in $U$.

**Proof.** By (4.5.2, (b)), it suffices to prove that, for every finite subset $Z'$ of Proj($S$) and every open neighbourhood $U$ of $Z'$, there exists a homogeneous element $f \in S_+$ such that $Z \subset \text{Proj}(S)|_f \subset U$ (2.4.1). But, by definition, the closed set $Y$, complement of $U$ in Proj($S$), is of the form $V_+(\mathfrak{I})$, where $\mathfrak{I}$ is a graded ideal of $S$ that does not contain $S_+$ (2.3.2); also, the points of $Z'$ are, by definition, graded ideals $p_i$ of $S_+$ that do not contain $\mathcal{I}$ (2.3.1). There thus exists an element $f \in \mathfrak{I}$ that does not belong to any of the $p_i$ (Bourbaki, Alg. comm., chap. II, §1, no. 1, prop. 2), and, since the $p_i$ are graded, the argument made loc. cit. shows that we can even take $f$ to be homogeneous; this element then satisfies the claim.

**Proposition (4.5.5).** — Suppose that $X$ is a quasi-compact scheme or a prescheme whose underlying space is Noetherian. Then conditions (a) to (c') of (4.5.2) are equivalent to the following:

(d) For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type, there exists an integer $n_0$ such that, for all $n \geq n_0$, $\mathcal{F}(n)$ is generated by its sections over $X$.

(d') For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type, there exist integers $n > 0$ and $k > 0$ such that $\mathcal{F}$ is isomorphic to a quotient of the $\mathcal{O}_X$-module $\mathcal{L}^{\otimes (n-1)} \otimes \mathcal{O}_X^k$.

(d'') Property (d') holds for every quasi-coherent sheaf of ideals of $\mathcal{O}_X$ of finite type.

**Proof.** Since $X$ is quasi-compact, if a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is of finite type such that $\mathcal{F}(n)$ (which is of finite type) is generated by its sections over $X$, then $\mathcal{F}(n)$ is generated by a finite number of these sections (0, 5.2.3), and so (d) implies (d'), and it is clear that (d') implies (d''). Since every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is the inductive limits of its sub-$\mathcal{O}_X$-modules of finite type (I, 9.4.9), to satisfy condition (c') of (4.5.2), it suffices to do so for a quasi-coherent sheaf of ideals of $\mathcal{O}_X$ that is of finite type, and (d'') thus implies (c'). It remains only to show that, if $\mathcal{L}$ is ample, then property (d) is satisfied. Consider a finite cover of $X$ by $X_{f_i}$ ($f_i \in S_{n_i}$), that we can assume to be affine; by replacing the $f_i$ with suitable powers (which does not alter the $X_{f_i}$), we can assume that all the $n_i$ are equal to one single integer $m$. The sheaf $\mathcal{F}|X_{f_i}$, being of finite type, by hypothesis, is generated by a finite number of its sections $h_{i_j}$ over $X_{f_i}$ (I, 1.3.13); so there exists an integer $k_0$ such that the section $h_{i_j} \otimes f^{k_0}_i$ extends to a section of $\mathcal{F}(k_0m)$ over $X$ for every pair $(i, j)$ (I, 9.3.1). 

A fortiori, the $h_{i_j} \otimes f^{k_0}_i$ extend to sections of $\mathcal{F}(km)$ over $X$ for every $k \geq k_0$ and, for these values of $k$, $\mathcal{F}(km)$ is thus generated by its sections over $X$. For every $p$ such that $0 < p < m$, $\mathcal{F}(p)$ is also of finite type, and so there exists an integer $k_p$ such that $\mathcal{F}(p)(km) = \mathcal{F}(p + km)$ is generated by its sections over $X$ for all $k \geq k_p$. Taking $n_0$ to be the largest of the $k_p$, we thus conclude that $\mathcal{F}(n)$ is generated by its sections over $X$ for all $n \geq n_0$, since such an $n$ is of the form $n = km + p$, with $k \geq k_p$ and $0 \leq p < m$.

**Proposition (4.5.6).** — Let $X$ be a quasi-compact scheme, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module.

(i) Let $n > 0$ be an integer. For $\mathcal{L}$ to be ample, it is necessary and sufficient for $\mathcal{L}^{\otimes n}$ to be ample.

(ii) Let $\mathcal{L}'$ be an invertible $\mathcal{O}_X$-module such that, for all $x \in X$, there exists an integer $n > 0$ and a section $s'$ of $\mathcal{L}'^{\otimes n}$ over $X$ such that $s'(x) \neq 0$. Then, if $\mathcal{L}$ is ample, so too is $\mathcal{L} \otimes \mathcal{L}'$.

**Proof.** Property (i) is an evident consequence of criterion (a) of (4.5.2), since $X_{\mathcal{L}^{\otimes n}} = X_f$. On the other hand, if $\mathcal{L}$ is ample, then, for every $x \in X$ and every neighbourhood $U$ of $x$, there exists some $m > 0$ and $f \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that $x \in X_f \subset U$ (4.5.2, (a)); if $f' \in \Gamma(X, \mathcal{L}'^{\otimes n})$ is such that $f'(x) \neq 0$, then $s(x) \neq 0$ for $s = f'^{\otimes n} \otimes f^{\otimes m} \in \Gamma(X, (\mathcal{L} \otimes \mathcal{L}')^{\otimes mn})$, and so $x \in X_s \subset X_f \subset U$, which proves that $\mathcal{L} \otimes \mathcal{L}'$ is ample (4.5.2, (a)).

**Corollary (4.5.7).** — The tensor product of two ample $\mathcal{O}_X$-modules is ample.

**Corollary (4.5.8).** — Let $\mathcal{L}$ be an ample $\mathcal{O}_X$-module, and $\mathcal{L}'$ an invertible $\mathcal{O}_X$-module; then there exists an integer $n_0 > 0$ such that $\mathcal{L}^{\otimes n} \otimes \mathcal{L}'$ is ample and generated by its sections over $X$ for $n \geq n_0$. 

Proof. It follows from (4.5.5) that there exists an integer $m_0$ such that $L^\otimes m \otimes L'$ is generated by its sections over $X$ for all $m \geq m_0$; by (4.5.6), we can then take $n_0 = m_0 + 1$.

Remark (4.5.9). — Let $P = H^1(X, \mathcal{O}_X)$ be the group of classes of invertible $\mathcal{O}_X$-modules (0, 5.4.7), and let $P^+$ be the subset of $P$ consisting of classes of ample sheaves. Suppose that $P^+$ is non-empty. Then it follows from (4.5.7) and (4.5.8) that

$$P^+ + P^+ \subset P^+ \quad \text{and} \quad P^+ - P^+ = P$$

or, in other words, $P^+ \cup \{0\}$ is the set of positive elements in $P$ for a preorder structure on $P$ that is compatible with its group structure, and is even archimedean, by (4.5.8). This is why we sometimes say “positive sheaf” instead of ample sheaf, and “negative sheaf” for the inverse of an ample sheaf (but we will not use this terminology).

Proposition (4.5.10). — Let $Y$ be an affine scheme, $q : X \to Y$ a quasi-compact separated morphism, and $L$ an invertible $\mathcal{O}_X$-module.

(i) If $L$ is very ample for $q$, then $L$ is ample.

(ii) Suppose further that the morphism $q$ is of finite type. Then, for $L$ to be ample, it is necessary and sufficient for it to possess one of the following properties:

(e) There exists $n_0 > 0$ such that, for every integer $n \geq n_0$, $L^\otimes n$ is very ample for $q$.

(e') There exists $n > 0$ such that $L^\otimes n$ is very ample for $q$.

Proof. The first claim follows from the definition (4.4.2) of a very ample $\mathcal{O}_X$-module: if $A$ is the ring of $Y$, then there exists an $A$-module $E$ and a surjective homomorphism

$$\psi : q^* ((S(E))) \longrightarrow \bigoplus_{n \geq 0} L^\otimes n$$

such that $i = r_{L^\otimes n}$ is an everywhere-defined immersion $X \to P = P(\mathcal{E})$ and such that $L = i^* (\mathcal{O}_P(1))$; since the $D_+(f)$ for $f$ homogeneous in $(S(E))_+$ form a base for the topology of $P$, and since $i^{-1}(D_+(f)) = X_{\psi^*(f)}$, by (3.7.3.1), we see that condition (a) of (4.5.2) is satisfied, and so $L$ is ample.

Now to prove that, if $q$ is of finite type and $L$ is ample, then condition (e) is satisfied. Firstly, it follows from criterion (b) of (4.5.2) and from (4.4.1, (i)) that there exists an integer $k_0$ such that $L^\otimes k_0$ is very ample relative to $q$. Also, by (4.5.5), there exists an integer $m_0$ such that, for all $m \geq m_0$, $L^\otimes m$ is generated by its sections over $X$. Let $n_0 = k_0 + m_0$; if $n \geq n_0$, then we can write $n = k_0 + m$ with $m \geq m_0$, whence $L^\otimes n = L^\otimes k_0 \otimes L^\otimes m$. Since $L^\otimes m$ is generated by its sections over $X$, it follows from (4.4.8) and (3.4.7) that $L^\otimes n$ is very ample relative to $q$. Finally, it is clear that (e) implies (e'), and (e') implies that $L$ is ample by (i) and by (4.5.6, (i))

(4.5.10.1). [Proof of Lemma (4.4.10.1)]. Let $\mathcal{E}(n) = \mathcal{E} \otimes \mathcal{H}^\otimes n$; since $g$ is separated (4.4.2), the proof of (3.4.8) applies, and shows that the canonical homomorphism $g^* (g_*(\mathcal{E}(n))) \to \mathcal{E}(n)$ is surjective for $n$ large enough. Furthermore, since $Z$ is quasi-compact, the proof of (3.4.6) means that if suffices to prove the claim in the case where $Z$ is affine. But $\mathcal{H}$ is then ample, by (4.5.10, (i)), and the conclusion follows from (4.5.5, (d')).

□

Corollary (4.5.11). — Let $Y$ be an affine scheme, $q : X \to Y$ a separated morphism of finite type, $L$ an ample $\mathcal{O}_X$-module, and $L'$ an invertible $\mathcal{O}_X$-module. Then there exists an integer $n_0$ such that, for all $n \geq n_0$, $L^\otimes n \otimes L'$ is very ample relative to $q$.

Proof. There exists $m_0$ such that, for $m \geq m_0$, $L^\otimes m \otimes L'$ is generated by its sections over $X$ (4.5.8); there also exists $k_0$ such that $L^\otimes k$ is very ample relative to $q$ for $k \geq k_0$. Then $L^\otimes (k+m_0) \otimes L'$ is very ample if $k \geq k_0$ ((4.4.8) and (3.4.7)).

□

Remark (4.5.12). — We do not know if the hypothesis that an $\mathcal{O}_X$-module $L$ is such that $L^\otimes n$ is very ample (relative to $q$) implies the same conclusion for $L^\otimes (n+1)$.

Proposition (4.5.13). — Let $X$ be a quasi-compact prescheme, $Z$ a closed prescheme of $X$ defined by a nilpotent quasi-coherent sheaf $\mathcal{J}$ of ideals of $\mathcal{O}_X$, and $j$ the canonical injection $Z \to X$. For an invertible $\mathcal{O}_X$-module $L$ to be ample, it is necessary and sufficient for $L' = j^* (L)$ to be an ample $\mathcal{O}_Z$-module.
PROOF. The condition is necessary. Indeed, for every section \( f \) of \( \mathcal{L} \otimes^n \) over \( X \), let \( f' \) be its canonical image \( f \otimes 1 \), which is a section of \( \mathcal{L} \otimes^n = \mathcal{L} \otimes^n \otimes_{\mathcal{O}_X} (\mathcal{O}_X / \mathcal{F}) \) over the space \( Z \) (which is identical to \( X \)); it is clear that \( X'_f = Z'_f \), and so the criterion (a) of (4.5.2) shows that \( \mathcal{L}' \) is ample.

To see that the condition is sufficient, note first of all that we can restrict to the case where \( f'^2 = 0 \), by considering the (finite) sequence of preschemes \( X_k = (X, \mathcal{O}_X / \mathcal{F}^{k+1}) \) with each prescheme being a closed subscheme of the next, defined by a square-zero sheaf of ideals. But \( X \) is a scheme, since \( X_{\text{red}} \) is a scheme by hypothesis ((4.5.3) and (I, 5.5.1)). Criterion (a) of (4.5.2) shows that it suffices to prove

**Lemma (4.5.13.1).** — Under the hypotheses of (4.5.13), suppose further that \( \mathcal{F} \) is square-zero; with \( \mathcal{L} \) being an invertible \( \mathcal{O}_X \)-module, let \( g \) be a section of \( \mathcal{L} \otimes^n \) over \( Z \) such that \( Z_g \) is affine. Then there exists an integer \( m > 0 \) such that \( g \otimes^m \) is the canonical image of a section \( f \) of \( \mathcal{L} \otimes^m \) over \( X \).

**PROOF.** We have the exact sequence of \( \mathcal{O}_X \)-modules

\[
0 \to \mathcal{F}(n) \to \mathcal{O}_X(n) = \mathcal{L} \otimes^n \to \mathcal{O}_Z(n) = \mathcal{L} \otimes^n \to 0
\]

since \( \mathcal{F}(n) \) is an exact functor in \( \mathcal{F} \); from this, we have the exact sequence of cohomology

\[
0 \to \Gamma(X, \mathcal{F}(n)) \to \Gamma(X, \mathcal{L} \otimes^n) \to \Gamma(X, \mathcal{O}_Z(n)) \xrightarrow{\partial} H^1(X, \mathcal{F}(n))
\]

that sends, in particular, \( g \) to an element \( \partial g \in H^1(X, \mathcal{F}(n)) \).

Note that, since \( \mathcal{F}^2 = 0 \), \( \mathcal{F} \) can be considered as a quasi-coherent \( \mathcal{O}_X \)-module, and, for all \( k, \mathcal{L} \otimes^k \otimes_{\mathcal{O}_X} \mathcal{F}(n) = \mathcal{F}(n+k) \); for every section \( s \in \Gamma(X, \mathcal{L} \otimes^k) \), tensor multiplication with \( s \) is thus a homomorphism \( \mathcal{F}(n) \to \mathcal{F}(n+k) \) of \( \mathcal{O}_Z \)-modules, which then gives a homomorphism

\[
H^1(X, \mathcal{F}(n))^s \to H^1(X, \mathcal{F}(n+k))
\]

of cohomology groups.

With this, we will see that

\[
(4.5.13.2)
\]

for \( m > 0 \) large enough. In fact, \( Z_g \) is an affine open subset of \( Z \), and so \( H^1(Z_g, \mathcal{F}(n)) = 0 \) when \( \mathcal{F}(n) \) is considered as an \( \mathcal{O}_Z \)-module (I, 5.1.9.2). In particular, if we set \( g' = g|_{Z_g} \), and if we consider its image under the map \( \partial : \Gamma(Z_g, \mathcal{L} \otimes^n) \to H^1(Z_g, \mathcal{F}(n)) \), then \( \partial g' = 0 \). To better explain this equation, note that, in dimension 1, the cohomology of a sheaf of abelian groups is the same as its Čech cohomology (G, II, 5.9); to calculate \( \partial g \), we must thus consider a fine-enough open cover \( (U_a) \) of \( X \), that we can suppose to be finite and consisting of affine opens, and take, for each \( a \), a section \( g_a \in \Gamma(U_a, \mathcal{L} \otimes^n) \) whose canonical image in \( \Gamma(U_a, \mathcal{L} \otimes^n) \) is \( g|_{U_a} \), and to consider the cocycle class \( (g_{a,b} - g_{b,a}) \), with \( g_{a,b} \) being the restriction of \( g_a \) to \( U_a \cap U_b \) (with this cocycle taking values in \( \mathcal{F}(n) \)). We can further suppose that \( \partial g \) is calculated in the same way, by means of a cover given by the \( U_a \cap Z_g \), and restrictions \( g_a|_{(U_a \cap Z_g)} \) (by replacing, if necessary, \( (U_a) \) by a finer cover); the equation \( \partial g = 0 \) then implies that there exists, for each \( a \), a section \( h_a \in \Gamma(U_a \cap Z_g, \mathcal{F}(n)) \) such that

\[
(g_a - g_b)|_{(U_a \cap U_b \cap Z_g)} = h_{a,b} - h_{b,a}, \quad \text{where } h_{a,b} \text{ denotes the restriction of } h_a \text{ to } U_a \cap U_b \cap Z_g
\]

(G, II, 5.11). Then there exists an integer \( m > 0 \) such that \( g \otimes^m \) is the restriction to \( U_a \cap Z_g \) of a section \( t_a \in \Gamma(U_a, \mathcal{F}(n+m)) \) for all \( a \) (I, 9.3.1); thus \( g \otimes^m \equiv (g_a - g_b) = t_a - t_b \) for every pair of indices, which proves (4.5.13.2).

Now note that, if \( s \in \Gamma(X, \mathcal{O}_Z(p)) \) and \( t \in \Gamma(X, \mathcal{O}_Z(q)) \), then, in the group \( H^1(X, \mathcal{F}(p + q)) \),

\[
(4.5.13.3)
\]

\[
\partial(s \otimes t) = (\partial s) \otimes t + s \otimes (\partial t).
\]

Indeed, we can again calculate the two members by considering an open cover \( (U_a) \) of \( X \), and, for each \( a \), a section \( s_a \in \Gamma(U_a, \mathcal{O}_X(p)) \) (resp. \( t_a \in \Gamma(U_a, \mathcal{O}_X(q)) \)) whose canonical image in \( \Gamma(U_a, \mathcal{O}_Z(p)) \) (resp. \( \Gamma(U_a, \mathcal{O}_Z(q)) \)) is \( s|_{U_a} \) (resp. \( t|_{U_a} \)); equation (4.5.13.3) then follows from the equations

\[
(s_a \otimes t_a) - (s_b \otimes t_b) = (s_a - s_b) \otimes t_a + s_b \otimes (t_a - t_b)
\]

with the same notation as above. By induction on \( k \), we thus have

\[
(4.5.13.4)
\]

\[
\partial(g \otimes^k) = (kg \otimes^{k-1}) \otimes (\partial g)
\]

and we thus conclude from (4.5.13.2) and (4.5.13.4) that \( \partial(g \otimes^m) = 0 \); thus \( g \otimes^m \) is the canonical image of a section \( f \) of \( \mathcal{L} \otimes^m \) over \( X \), which proves (4.5.13).
Corollary (4.5.14). — Let \( X \) be a Noetherian scheme, and \( j \) the canonical injection \( X_{\text{red}} \to X \). For an invertible \( \mathcal{O}_X \)-module \( \mathcal{L} \) to be ample, it is necessary and sufficient for \( j^*(\mathcal{L}) \) to be an ample \( \mathcal{O}_{X_{\text{red}}} \)-module.

**Proof.** This follows from (I, 6.1.6)

### 4.6. Relatively ample sheaves

**Definition (4.6.1).** — Let \( f : X \to Y \) be a quasi-compact morphism, and \( \mathcal{L} \) an invertible \( \mathcal{O}_X \)-module. We say that \( \mathcal{L} \) is **ample relative to \( f \)**, or relative to \( Y \), or \( f \)-ample, or \( Y \)-ample (or even simply ample if no confusion may arise with the notion defined in (4.5.3)) if there exists an affine open cover \( \{U_a\} \) of \( Y \) such that, if we set \( X_a = f^{-1}(U_a) \), then \( \mathcal{L}|_{X_a} \) is an ample \( \mathcal{O}_{X_a} \)-module for all \( a \).

The existence of an \( f \)-ample \( \mathcal{O}_X \)-module implies that \( f \) is necessarily separated ((4.5.3) and (I, 5.5)).

**Proposition (4.6.2).** — Let \( f : X \to Y \) be a quasi-compact morphism, and \( \mathcal{L} \) an invertible \( \mathcal{O}_X \)-module. If \( \mathcal{L} \) is very ample relative to \( f \), then it is ample relative to \( f \).

**Proof.** This follows from the local (on \( Y \)) character of the notion of a very ample sheaf (4.4.5), from the definition (4.6.1), and from criterion (4.5.10, (i)).

**Proposition (4.6.3).** — Let \( f : X \to Y \) be a quasi-compact morphism, and \( \mathcal{L} \) an invertible \( \mathcal{O}_X \)-module, and let \( \mathcal{I} \) be the graded \( \mathcal{O}_Y \)-algebra \( \bigoplus_{n \geq 0} f_*(\mathcal{I}^n) \). Then the following conditions are equivalent:

(a) \( \mathcal{L} \) is \( f \)-ample.

(b) \( \mathcal{I} \) is quasi-coherent, and the canonical homomorphism \( \sigma : f^*(\mathcal{I}) \to \bigoplus_{n \geq 0} \mathcal{L}^n \) (4.4.3) is such that the \( Y \)-morphism \( r_{\mathcal{L},\mathcal{I}} : G(\sigma) \to \text{Proj}(\mathcal{I}) = P \) is everywhere defined and is a dominant open immersion.

(b') The morphism \( f \) is separated, and the \( Y \)-morphism \( r_{\mathcal{L},\mathcal{I}} \) is everywhere defined and is a homeomorphism from the underlying space of \( X \) to a subspace of \( \text{Proj}(\mathcal{I}) \).

Furthermore, if any of the above are satisfied, then, for all \( n \in \mathbb{Z} \), the canonical homomorphism

\[
r_{\mathcal{L},\mathcal{I}}^*(\mathcal{O}_P(n)) \to \mathcal{L}^n
\]

defined in (3.7.9.1) is an isomorphism.

Finally, for every quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \), if we set \( \mathcal{M} = \bigoplus_{n \geq 0} f_*(\mathcal{F} \otimes \mathcal{L}^n) \), then the canonical homomorphism

\[
r_{\mathcal{L},\mathcal{I}}^*(\mathcal{M}) \to \mathcal{F}
\]

defined in (3.7.9.2) is an isomorphism.

**Proof.** We note that (a) implies that \( f \) is separated, and thus that \( \mathcal{I} \) is quasi-coherent (I, 9.2.2, (a)). Since \( r_{\mathcal{L},\mathcal{I}} \) being an everywhere defined immersion is of a local (on \( Y \)) character, to prove that (a) implies (b) we can suppose that \( Y \) is affine and \( \mathcal{L} \) ample; the claim then follows from (4.5.2, (b)). It is clear that (b) implies (b'); finally, to prove that (b') implies (a), it suffices to consider an affine open cover \( \{U_a\} \) of \( Y \) and to apply the criterion (4.5.2, (b')) to each sheaf \( \mathcal{L}|_{X} \) for all \( a \).

For the final two claims, we use the fact that \( \sigma^p \) is here the identity, and the clarification of the homomorphisms (3.7.9.1) and (3.7.9.2); from this, it immediately follows that (4.6.3.1) is an isomorphism. As for (4.6.3.2), we can restrict to the case where \( Y \) is affine, and thus \( \mathcal{L} \) ample; it is clear that the homomorphism (4.6.3.2) is injective, and criterion (4.5.2, (c)) shows that it is surjective, whence the conclusion.

**Corollary (4.6.4).** — Let \( \{U_a\} \) be an open cover of \( Y \). For \( \mathcal{L} \) to be ample relative to \( Y \), it is necessary and sufficient for \( \mathcal{L}|_{f^{-1}(U_a)} \) to be ample relative to \( U_a \) for all \( a \).

**Proof.** Condition (b) is in fact local on \( Y \).

**Corollary (4.6.5).** — Let \( \mathcal{K} \) be an invertible \( \mathcal{O}_Y \)-module. For \( \mathcal{L} \) to be \( Y \)-ample, it is necessary and sufficient for \( \mathcal{L} \otimes f^*(\mathcal{K}) \) to be \( Y \)-ample.

**Proof.** This is an evident consequence of (4.6.4), by taking the \( U_a \) to be such that \( \mathcal{K}|_{U_a} \) is isomorphic to \( \mathcal{O}_{Y}|_{U_a} \) for all \( a \).

**Corollary (4.6.6).** — Suppose that \( Y \) is affine; for \( \mathcal{L} \) to be \( Y \)-ample, it is necessary and sufficient for \( \mathcal{L} \) to be ample.
PROOF. This is an immediate consequence of the definition (4.6.1) and the criteria (4.6.3, (b)) and (4.5.2, (b)), since here $\text{Proj}(\mathcal{S}) = \text{Proj}(\Gamma(Y, \mathcal{S}))$ by definition.

Corollary (4.6.7). — Let $f : X \to Y$ be a quasi-compact morphism. Suppose that there exists a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{E}$, and a $Y$-morphism $g : X \to P = \mathbb{P}(\mathcal{E})$ that is a homeomorphism from the underlying space of $X$ to a subspace of $P$; then $\mathcal{L} = g^*(\mathcal{O}_P(1))$ is $Y$-ample.

PROOF. We can assume $Y$ to be affine; the corollary then follows from the criterion (4.5.2, (a)), from equation (3.7.3.1), and from (4.2.3).

Proposition (4.6.8). — Let $X$ be a quasi-compact scheme or a prescheme whose underlying space is Noetherian, and let $f : X \to Y$ be a quasi-compact separated morphism. For an invertible $\mathcal{O}_X$-module $\mathcal{L}$ to be $f$-ample, it is necessary and sufficient for one of the following equivalent conditions to be satisfied:

(i) For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type, there exists an integer $n_0 > 0$ such that, for all $n \geq n_0$, the canonical homomorphism $\sigma : f^*(\mathcal{F} \otimes \mathcal{L}^n) \to \mathcal{F} \otimes \mathcal{L}^n$ is surjective.

(ii) For every quasi-coherent sheaf $\mathcal{J}$ of ideals of $\mathcal{O}_X$ of finite type, there exists an integer $n > 0$ such that the canonical homomorphism $\sigma : f^*(\mathcal{J} \otimes \mathcal{L}^n) \to \mathcal{J} \otimes \mathcal{L}^n$ is surjective.

PROOF. Since $X$ is quasi-compact, so too is $f(X)$, and so there exists a finite cover $(U_i)$ of $f(X)$ consisting of affine open subsets $U_i$ of $Y$. To prove condition (c) when $\mathcal{L}$ is $f$-ample, we can replace $Y$ by the $U_i$, and $X$ by the $f^{-1}(U_i)$, since, if we obtain, for each $i$, an integer $n_i$ such that (c) holds true (for $U_i$, $f^{-1}(U_i)$, and $\mathcal{L}_{f^{-1}(U_i)}$) for all $n \geq n_i$, then it suffices to take $n_0$ to be the largest of the $n_i$ in order to obtain (c) for $Y$, $X$, and $\mathcal{L}$. But if $Y$ is affine, condition (c) follows from (4.5.5, (d)), taking (4.6.6) into account. It is trivial that (c) implies (c'). Finally, to prove that (c') implies that $\mathcal{L}$ is $f$-ample, we can again restrict to the case where $Y$ is affine: in fact, every quasi-coherent sheaf $\mathcal{J}$ of ideals of $\mathcal{O}_X$ of finite type is the restriction of a coherent sheaf of ideals of $\mathcal{O}_X$ of finite type ($I$, 9.4.7), and hypothesis (c') implies that $\mathcal{J} \otimes (\mathcal{L}_{f^{-1}(U_i)})$ is generated by its sections (taking (I, 9.2.2) and (3.4.7) into account); it thus suffices to apply criterion (4.5.5, (d')).

Proposition (4.6.9). — Let $f : X \to Y$ be a quasi-compact morphism, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module.

(i) Let $n > 0$ be an integer. For $\mathcal{L}$ to be $f$-ample, it is necessary and sufficient for $\mathcal{L}^n$ to be $f$-ample.

(ii) Let $\mathcal{L}'$ be an invertible $\mathcal{O}_X$-module, and suppose that there exists an integer $n > 0$ such that the canonical homomorphism $\sigma : f^*(\mathcal{L}'^n) \to \mathcal{L}'^n$ is surjective. Then, if $\mathcal{L}$ is $f$-ample, so too is $\mathcal{L} \otimes \mathcal{L}'$.

PROOF. We can in fact immediately restrict to the case where $Y$ is affine, and the proposition is then an immediate consequence of (4.5.6).

Corollary (4.6.10). — The tensor product of two $f$-ample $\mathcal{O}_X$-modules is $f$-ample.

Proposition (4.6.11). — Let $Y$ be a quasi-compact prescheme, $f : X \to Y$ a morphism of finite type, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. For $\mathcal{L}$ to be $f$-ample, it is necessary and sufficient for it to possess one of the following equivalent properties:

(d) There exists some $n_0 > 0$ such that, for every integer $n \geq n_0$, $\mathcal{L}^n$ is very ample relative to $f$.

(d') There exists some $n > 0$ such that $\mathcal{L}^n$ is very ample relative to $f$.

PROOF. If $\mathcal{L}$ is ample relative to $f$, then there exists a finite cover $(U_i)$ of $Y$ by affine open subsets such that the $\mathcal{L}|_{f^{-1}(U_i)}$ are ample. We thus conclude (4.5.10) that there exists an integer $n_0$ such that $\mathcal{L}^n|_{f^{-1}(U_i)}$ is very ample relative to $f^{-1}(U_i) \to U_i$ for all $n \geq n_0$ and every $i$, and so $\mathcal{L}^n$ is very ample relative to $f$ (4.5.5). Conversely, (d') already implies that $\mathcal{L}^n$ is $f$-ample (4.6.2), and thus so too is $\mathcal{L}$ (4.6.9, (i)).

Corollary (4.6.12). — Let $Y$ be a quasi-compact prescheme, $f : X \to Y$ a morphism of finite type, and $\mathcal{L}$ and $\mathcal{L}'$ invertible $\mathcal{O}_X$-modules. If $\mathcal{L}$ is $f$-ample, then there exists some $n_0$ such that $\mathcal{L}^n \otimes \mathcal{L}'$ is very ample relative to $f$ for all $n \geq n_0$.

PROOF. We argue as in (4.6.11), by using a finite affine open cover of $Y$ and (4.5.11).

Proposition (4.6.13). —

(i) For every prescheme $Y$, every invertible $\mathcal{O}_Y$-module $\mathcal{L}$ is ample relative to the identity morphism $1_Y$.
(i bis) Let \( f : X \to Y \) be a quasi-compact morphism, and \( j : X' \to X \) a quasi-compact morphism that is a homeomorphism from the underlying space of \( X' \) to a subspace of \( X \). If \( \mathcal{L} \) is an \( \mathcal{O}_X \)-module that is ample relative to \( f \), then \( j^*(\mathcal{L}) \) is ample relative to \( f \circ j \).

(ii) Let \( Z \) be a quasi-compact prescheme, \( f : X \to Y \) and \( g : Y \to Z \) quasi-compact morphisms, \( \mathcal{L} \) an \( \mathcal{O}_X \)-module that is ample relative to \( f \), and \( \mathcal{X} \) an \( \mathcal{O}_Y \)-module that is ample relative to \( g \). Then there exists an integer \( n_0 > 0 \) such that \( \mathcal{L} \otimes f^*(\mathcal{X}^{\otimes n}) \) is ample relative to \( g \circ f \) for all \( n \geq n_0 \).

(iii) Let \( f : X \to Y \) be a quasi-compact morphism, and \( g : Y' \to Y \) a morphism, and let \( X' = X_{(Y')} \). If \( \mathcal{L} \) is an \( \mathcal{O}_X \)-module that is ample relative to \( f \), then \( \mathcal{L}' = \mathcal{L} \otimes_Y \mathcal{O}_{(Y')} \) is an \( \mathcal{O}_{X'} \)-module that is ample relative to \( f_{(Y')} \).

(iv) Let \( f_i : X_i \to Y_i \) (\( i = 1, 2 \)) be quasi-compact \( S \)-morphisms. If \( \mathcal{L}_i \) is an \( \mathcal{O}_{X_i} \)-modules that is ample relative to \( f_i \) (\( i = 1, 2 \)), then \( \mathcal{L}_1 \otimes_S \mathcal{L}_2 \) is ample relative to \( f_1 \times_S f_2 \).

(v) Let \( f : X \to Y \) and \( g : Y \to Z \) be morphisms such that \( g \circ f \) is quasi-compact. If an \( \mathcal{O}_X \)-module \( \mathcal{L} \) is ample relative to \( g \circ f \), and if \( g \) is separated or the underlying space of \( X \) is locally Noetherian, then \( \mathcal{L} \) is ample relative to \( f \).

(vi) Let \( f : X \to Y \) be a quasi-compact morphism, and \( j \) the canonical injection \( X_{\text{red}} \to X \). If \( \mathcal{L} \) is an \( \mathcal{O}_X \)-module that is ample relative to \( f \), then \( j^*(\mathcal{L}) \) is ample relative to \( f_{\text{red}} \).

Proof. Note first of all that (v) and (vi) follow from (i), (i bis), and (iv) by the same argument as in (4.4.10), by using (4.6.4) instead of (4.4.5); we leave the details of the argument to the reader. Claim (i) is trivially a consequence of (4.4.10, (i)) and (4.6.2). To prove (i bis), (iii), and (iv), we will use the following lemma:

Lemma (4.6.13.1). —

(i) Let \( u : Z \to S \) be a morphism, \( \mathcal{L} \) an invertible \( \mathcal{O}_Z \)-module, and \( s \) a section of \( \mathcal{L} \) over \( S \), and let \( s' \) be the canonically corresponding section \( u^*(\mathcal{L}) = \mathcal{L}' \) over \( Z \). Then \( Z_{s'} = u^{-1}(S_s) \).

(ii) Let \( Z \) and \( Z' \) be \( S \)-preschemes, \( p \) and \( p' \) the projections of \( T = Z \times_S Z' \), \( \mathcal{L} \) (resp. \( \mathcal{L}' \)) an invertible \( \mathcal{O}_T \)-module (resp. invertible \( \mathcal{O}'_T \)-module), and let \( t \) (resp. \( t' \)) be a section of \( \mathcal{L} \) (resp. \( \mathcal{L}' \)) over \( Z \) (resp. \( Z' \)), and \( s \) (resp. \( s' \)) the corresponding section of \( p^*(\mathcal{L}) \) (resp. \( p'^*(\mathcal{L}') \)) over \( Z \times_S Z' \). Then \( T_{s \otimes s'} = Z_t \times_S Z'_{t'} \).

Proof. It follows from the definitions that we can restrict to the case where all the preschemes in question are affine. Furthermore, in (i), we can suppose that \( \mathcal{L} = \mathcal{O}_Z \); claim (i) then follows immediately from (I, 1.2.2.2). Similarly, in (ii), we can restrict to the case where \( \mathcal{L} = \mathcal{O}_Z \) and \( \mathcal{L}' = \mathcal{O}'_Z \), and then the claim reduces to Lemma (4.3.2.4).

We now prove (i bis). We can assume that \( Y \) is affine (4.6.4), and thus that \( \mathcal{L} \) is ample (4.6.6); if \( s \) runs over the set of the union of the \( \Gamma(X, \mathcal{L}^{\otimes n}) \) (\( n > 0 \)), then the \( X_s \) form a base for the topology of \( X \) (4.5.2, (a)), and so, by hypothesis, the \( j^{-1}(X_s) \) form a base of the topology of \( X' \); we thus conclude, by Lemma (4.6.13.1, (i)) and (4.5.2, (a)), that \( j^*(\mathcal{L}) \) is ample.

Next we prove (iii). We can again suppose that \( Y \) and \( Y' \) are affine (4.6.4), whence it follows that the projection \( h : X' \to X \) is affine (1.5.5). Since \( \mathcal{L} \) is ample (4.6.6), if \( s \) runs over the set of sections of the \( \mathcal{L}^{\otimes n} \) (\( n > 0 \)) over \( X \) such that \( X_s \) is affine, then the \( X_s \) cover \( X \) (4.5.2, (a')), and so the \( h^{-1}(X_s) \) are affine (1.2.5) and cover \( X' \); it thus follows again from Lemma (4.6.13.1, (i)) and (4.5.2, (a')) that \( \mathcal{L} \) is ample, since the morphism \( f_{(Y')} \) is quasi-compact (I, 6.6.4, (iii)).

To prove (iv), note first of all that \( f_1 \times_S f_2 \) is quasi-compact (I, 6.6.4, (iv)). We can further suppose that \( S, Y_1, \) and \( Y_2 \) are affine (4.6.4) and (I, 3.2.7), and thus that \( \mathcal{L}_i \) is ample (\( i = 1, 2 \)) (4.6.6). The open subsets \( (X_1)_{s_1} \times_S (X_2)_{s_2} \) form a cover of \( X_1 \times_S X_2 \), where \( s_1 \) runs over the sections of \( \mathcal{L}_1^{\otimes n_1} \) such that \( (X_1)_{s_1} \) is affine (4.5.2, (a')). Then, replacing \( s_1 \) and \( s_2 \) with suitable powers, which does not change the \( (X_1)_{s_1} \), we can assume that \( n_1 = n_2 \). We thus deduce, from (4.6.13.1, (i)) and (4.5.2, (a')), that \( \mathcal{L}_1 \otimes_S \mathcal{L}_2 \) is ample, whence the claim, since \( Y_1 \times_S Y_2 \) is affine (4.6.6).

It remains only to prove (ii). By the same argument as in (4.4.10), but here using (4.6.4), we can restrict to the case where \( Z \) is affine. Since \( \mathcal{X} \) is then affine, and \( Y \) quasi-compact, there exists a finite number of sections \( s_i \in \Gamma(Y, \mathcal{X}^{\otimes k}) \) such that the \( Y_{s_i} \) are affine and cover \( Y \) (4.5.2, (a')); replacing the \( s_i \) with suitable powers, we can further suppose that all the \( k_i \) are equal to one single integer \( k \). Let \( s_i' \) be the sections of \( f^*(\mathcal{X}^{\otimes k}) \) over \( X \) that canonically correspond to the \( s_i \), so that the \( X_{s_i'} = f^{-1}(Y_{s_i}) \) (4.6.1.13, (i)) cover \( X \). Since \( \mathcal{L}'|_{X_{s_i'}} \) is ample (4.6.4) and (4.6.6)), there exists, for each
a finite number of sections \( t_{ij} \in \Gamma(X, \mathcal{L} \otimes \Omega_{ij}) \) such that the \( X_{ij} \) are affine, contained in the \( X_{ij} \), and cover \( X_{ij} \) (4.5.2, (a’)); we can also suppose that all the \( n_{ij} \) are equal to one single integer \( n \). With this in mind, \( X \) is separated and quasi-compact, and so there exists an integer \( m > 0 \), and, for every \((i,j)\), a section

\[
u_{ij} \in \Gamma(X, \mathcal{L} \otimes \Omega \otimes X f^* (\mathcal{H} \otimes \Omega^m))
\]

such that \( t_{ij} \otimes \nu_{ij} \) is the restriction to \( X_{ij} \) of \( \nu_{ij} \) (I, 9.3.1); furthermore, \( X_{u_{ij}} = X_{ij} \), and so the \( X_{u_{ij}} \) are affine and cover \( X \). We can also suppose that \( m \) is of the form \( nr \); if we set \( n_0 = rk \), then we see (4.5.2, (a’)) that \( \mathcal{L} \otimes \mathcal{O}_X f^* (\mathcal{H} \otimes \Omega_{ij}) \) is ample. Furthermore, there exists \( h_0 > 0 \) such that \( \mathcal{H} \otimes \Omega_{ij} \) is generated by its sections over \( Y \) for all \( h \geq h_0 \) (4.5.5); a fortiori, \( f^* (\mathcal{H} \otimes \Omega_{ij}) \) is generated by its sections over \( X \) for all \( h \geq h_0 \), by definition of the inverse images ((0, 3.7.1) and (4.4.1)). We thus conclude that \( \mathcal{L} \otimes f^* (\mathcal{H} \otimes \Omega_{ij}) \) is ample for all \( h \geq h_0 \) (4.5.6), which finishes the proof. \( \square \)

Remark (4.6.14). — Under the conditions of (ii), we refrain from believing that \( \mathcal{L} \otimes f^* (\mathcal{H}) \) is ample for \( g \circ f \); in fact, since \( \mathcal{L} \otimes f^* (\mathcal{H}^{-1}) \) is also ample for \( f \) (4.6.5), we would conclude that \( \mathcal{L} \) is ample for \( g \circ f \); taking, in particular, \( g \) to be the identity morphism, every invertible \( \mathcal{O}_X \)-module would be ample for \( f \), which is not the case in general (see (5.1.6), (5.3.4), (i), and (5.3.1)).

Proposition (4.6.15). — Let \( f : X \to Y \) be a quasi-compact morphism, \( \mathcal{J} \) a quasi-coherent locally-nilpotent sheaf of ideals of \( \mathcal{O}_X \), \( Z \) the closed subscheme of \( X \) defined by \( \mathcal{J} \), and \( j : Z \to X \) the canonical injection. For an invertible \( \mathcal{O}_X \)-module \( \mathcal{L} \) to be ample for \( f \), it is necessary and sufficient for \( j^* (\mathcal{L}) \) to be ample for \( f \circ j \).

Proof. Since the question is local on \( Y \) (4.6.4), we can suppose \( Y \) to be affine; since \( X \) is then quasi-compact, we can suppose \( \mathcal{J} \) to be nilpotent. Taking (4.6.6) into account, the proposition is then exactly the same as (4.5.13). \( \square \)

Corollary (4.6.16). — Let \( X \) be a locally Noetherian prescheme, and \( f : X \to Y \) a quasi-compact morphism. For an invertible \( \mathcal{O}_X \)-module \( \mathcal{L} \) to be ample for \( f \), it is necessary and sufficient for its inverse image \( \mathcal{L}' \) under the canonical injection \( X_{red} \to X \) to be ample for \( f_{red} \).

Proof. We have already seen that the condition is necessary (4.6.13, (vii)); conversely, if it is satisfied, then we can restrict, to prove that \( \mathcal{L} \) is ample for \( f \), to the case where \( Y \) is affine (4.6.4); then \( Y_{red} \) is also affine, and \( \mathcal{L}' \) is ample (4.6.6), and so too is \( \mathcal{L} \) by (4.5.13), since \( \mathcal{L} \) is then Noetherian and \( X_{red} \) a closed subscheme of \( X \) defined by a quasi-coherent nilpotent sheaf of ideals (I, 6.1.6). \( \square \)

Proposition (4.6.17). — With the notation and hypotheses of (4.4.11), for \( \mathcal{L}'' \) to be ample relative to \( f'' \), it is necessary and sufficient for \( \mathcal{L} \) to be ample relative to \( f \) and \( \mathcal{L}' \) ample relative to \( f' \).

Proof. The necessity of the condition follows from (4.6.13, (i bis)). To see that the condition is sufficient, we can restrict to the case where \( Y \) is affine, and then the fact that \( \mathcal{L}'' \) is ample follows from criterion (4.5.2, (a)) applied to \( \mathcal{L} \), \( \mathcal{L}' \), and \( \mathcal{L}'' \), noting that a section of \( \mathcal{L} \) over \( X \) can be extended (by \( 0 \)) to a section of \( \mathcal{L}'' \) over \( X'' \). \( \square \)

Proposition (4.6.18). — Let \( Y \) be a quasi-compact prescheme, \( \mathcal{X} \) a quasi-coherent graded \( \mathcal{O}_Y \)-algebra of finite type, and \( X = \text{Proj}(\mathcal{X}) \), and let \( f : X \to Y \) the structure morphism. Then \( f \) is of finite type, and there exists an integer \( d > 0 \) such that \( \mathcal{O}_X(d) \) is invertible and \( f \)-ample.

Proof. By (3.1.10), there exists an integer \( d > 0 \) such that \( \mathcal{X}^{(d)} \) is generated by \( \mathcal{X}_d \). We know that, under the canonical isomorphism between \( X \) and \( X^{(d)} = \text{Proj}(\mathcal{X}^{(d)}) \), \( \mathcal{O}_X(d) \) is identified with \( \mathcal{O}_X^{(d)}(1) \) (3.2.9, (ii)). We thus see that we can restrict to the case where \( \mathcal{X} \) is generated by \( \mathcal{X}_d \); the proposition then follows from (4.4.3) and (4.6.2) (taking into account the fact that \( f \) is a morphism of finite type (3.4.1)). \( \square \)
§5. QUASI-AFFINE MORPHISMS; QUASI-PROJECTIVE MORPHISMS; PROPER MORPHISMS; PROJECTIVE MORPHISMS

5.1. Quasi-affine morphisms

Definition (5.1.1). — We define a quasi-affine scheme to be a scheme isomorphic to some subscheme induced on some quasi-compact open subset of an affine scheme. We say that a morphism \( f : X \to Y \) is quasi-affine, or that \( X \) (considered as a \( Y \)-prescheme via \( f \)) is a quasi-affine \( Y \)-scheme, if there exists a cover \( (U_n) \) of \( Y \) by affine open subsets such that the \( f^{-1}(U_n) \) are quasi-affine schemes.

It is clear that a quasi-affine morphism is separated (\((\text{I}, 5.5.5)\) and \((\text{I}, 5.5.8)\)) and quasi-compact (\((\text{I}, 6.6.1)\)); every affine morphisms is evidently quasi-affine.

Recall that, for any prescheme \( X \), setting \( A = \Gamma(X, \mathcal{O}_X) \), the identity homomorphism \( A \to A = \Gamma(X, \mathcal{O}_X) \) defines a morphism \( X \to \text{Spec}(A) \), said to be canonical (\((\text{I}, 2.2.4)\)); this is nothing but the canonical morphism defined in \((4.5.1)\) for the specific case where \( \mathcal{L} = \mathcal{O}_X \), if we remember that \( \text{Proj}(A[T]) \) is canonically identified with \( \text{Spec}(A) \) \((3.1.7)\).

Proposition (5.1.2). — Let \( X \) be a quasi-compact scheme or a prescheme whose underlying space is Noetherian, and \( A \) the ring \( \Gamma(X, \mathcal{O}_X) \). The following conditions are equivalent.

(a) \( X \) is a quasi-affine scheme.
(b) The canonical morphism \( u : X \to \text{Spec}(A) \) is an open immersion.
(b') The canonical morphism \( u : X \to \text{Spec}(A) \) is a homeomorphism from \( X \) to some subspace of the underlying space of \( \text{Spec}(A) \).
(c) The \( \mathcal{O}_X \)-module \( \mathcal{O}_X \) is very ample relative to \( u \) \((4.4.2)\).
(c') The \( \mathcal{O}_X \)-module \( \mathcal{O}_X \) is ample \((4.5.1)\).
(d) When \( f \) ranges over \( A \), the \( X_f \) form a basis for the topology of \( X \).
(d') When \( f \) varies over \( A \), the \( X_f \) that are affine form a cover of \( X \).
(e) Every quasi-coherent \( \mathcal{O}_X \)-module is generated by its sections over \( X \).
(e') Every quasi-coherent sheaf of ideals of \( \mathcal{O}_X \) of finite type is generated by its sections over \( X \).

Proof. It is clear that (b) implies (a), and (a) implies (c) by \((4.4.4, b)\) applied to the identity morphism (taking into account the remark preceding this proposition); Furthermore, (c) implies (c') \((4.5.10, i)\), and (c'), (b), and (b') are all equivalent by \((4.5.2, b)\) and \((4.5.2, b')\). Finally, (c') is the same as each of (d), (d'), (e), and (e') by \((4.5.2, a)\), \((4.5.2, a')\), \((4.5.2, c)\), and \((4.5.5, d'')\). □

We further observe that, with the previous notation, the \( X_f \) that are affine form a basis for the topology of \( X \), and that the canonical morphism \( u \) is dominant \((4.5.2)\).

Corollary (5.1.3). — Let \( X \) be a quasi-compact prescheme. If there exists a morphism \( v : X \to Y \) from \( X \) to some affine scheme \( Y \) (which would be a homeomorphism from \( X \) to some open subspace of \( Y \)), then \( X \) is quasi-affine.

Proof. There exists a family \((g_a)\) of sections of \( \mathcal{O}_Y \) over \( Y \) such that the \( D(g_a) \) form a basis for the topology of \( v(X) \); if \( v = (\psi, \theta) \) and we set \( f_a = \theta(g_a) \), then we have \( X_{f_a} = \psi^{-1}(D(g_a)) \) \((\text{I}, 2.2.4.1)\), so the \( X_{f_a} \) form a basis for the topology of \( X \), and the criterion \((5.1.2, d)\) is satisfied. □

Corollary (5.1.4). — If \( X \) is a quasi-affine scheme, then every invertible \( \mathcal{O}_X \)-module is very ample (relative to the canonical morphism), and a fortiori ample.

Proof. Such a module \( \mathcal{L} \) is generated by its sections over \( X \) \((5.1.2, e)\), so \( \mathcal{L} \otimes \mathcal{O}_X = \mathcal{L} \) is very ample \((4.4.8)\). □

Corollary (5.1.5). — Let \( X \) be a quasi-compact prescheme. If there exists an invertible \( \mathcal{O}_X \)-module \( \mathcal{L} \) such that \( \mathcal{L} \) and \( \mathcal{L}^{-1} \) are ample, then \( X \) is a quasi-affine scheme.

Proof. Indeed, \( \mathcal{O}_X = \mathcal{L} \otimes \mathcal{L}^{-1} \) is then ample \((4.5.7)\). □

Proposition (5.1.6). — Let \( f : X \to Y \) be a quasi-compact morphism. Then the following conditions are equivalent.

(a) The morphism \( f \) is quasi-affine.
(b) The $\mathcal{O}_Y$-algebra $f_*(\mathcal{O}_X) = \mathcal{A}(X)$ is quasi-coherent, and the canonical morphism $X \to \text{Spec}(\mathcal{A}(X))$ corresponding to the identity morphism $\mathcal{A}(X) \to \mathcal{A}(X)$ (1.2.7) is an open immersion.

(b') The $\mathcal{O}_Y$-algebra $\mathcal{A}(X)$ is quasi-coherent, and the canonical morphism $X \to \text{Spec}(\mathcal{A}(X))$ is a homeomorphism from $X$ to some subspace of $\text{Spec}(\mathcal{A}(X))$.

(c) The $\mathcal{O}_X$-module $\mathcal{O}_X$ is very ample for $f$.  

(c') The $\mathcal{O}_X$-module $\mathcal{O}_X$ is ample for $f$.

(d) The morphism $f$ is separated, and, for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the canonical homomorphism $\sigma : f^*(f_*(\mathcal{F})) \to \mathcal{F}$ (0.4.4.3) is surjective.

Furthermore, whenever $f$ is quasi-affine, every invertible $\mathcal{O}_X$-module $\mathcal{L}$ is very ample relative to $f$.

**Proof.** The equivalence between (a) and (c') follows from the local (on $Y$) character of the $f$-ampleness (4.6.4), Definition (5.1.1), and the criterion (5.1.2, c'). The other properties are local on $Y$ and thus follow immediately from (5.1.2) and (5.1.4), taking into account the fact that $f_*(\mathcal{F})$ is quasi-coherent whenever $f$ is separated (I, 9.2.2, a).

**Corollary (5.1.7).** — Let $f : X \to Y$ be a quasi-affine morphism. For every open subset $U$ of $Y$, the restriction $f^{-1}(U) \to U$ of $f$ is quasi-affine.

**Corollary (5.1.8).** — Let $Y$ be an affine scheme, and $f : X \to Y$ a quasi-compact morphism. For $f$ to be quasi-affine, it is necessary and sufficient for $X$ to be a quasi-affine scheme.

**Proof.** This is an immediate consequence of (5.1.6) and (4.6.6).

**Corollary (5.1.9).** — Let $Y$ be a quasi-compact scheme or a prescheme whose underlying space is Noetherian, and $f : X \to Y$ a morphism of finite type. If $f$ is quasi-affine, then there exists a quasi-coherent $\mathcal{O}_Y$-subalgebra $\mathcal{B}$ of $\mathcal{A}(X) = f_*(\mathcal{O}_X)$ of finite type (I, 9.6.2) such that the morphism $X \to \text{Spec}(\mathcal{B})$ corresponding to the canonical injection $\mathcal{B} \to \mathcal{A}(X)$ is an immersion. Further, every quasi-coherent $\mathcal{O}_Y$-subalgebra $\mathcal{B}$ of finite type over $\mathcal{A}(X)$ containing $\mathcal{B}$ has the same property.

**Proof.** Indeed, $\mathcal{A}(X)$ is the inductive limit of its quasi-coherent $\mathcal{O}_Y$-subalgebras of finite type (I, 9.6.5); the result is then a particular case of (3.8.4), taking into account the identification of $\text{Spec}(\mathcal{A}(X))$ with $\text{Proj}(\mathcal{A}(X)[T])$ (3.1.7).

**Proposition (5.1.10).**

(i) A quasi-compact morphism $X \to Y$ that is a homeomorphism from the underlying space of $X$ to some subspace of the underlying space of $Y$ (so, in particular, any closed immersion) is quasi-affine.

(ii) The composition of any two quasi-affine morphisms is quasi-affine.

(iii) If $f : X \to Y$ is a quasi-affine $S$-morphism, then $f(S') : X(S') \to Y(S')$ is a quasi-affine $S'$-morphism for any extension $S' \to S$ of the base prescheme.

(iv) If $f : X \to Y$ and $g : X' \to Y'$ are quasi-affine $S$-morphisms, then $f \times_S g$ is quasi-affine.

(v) If $f : X \to Y$ and $g : Y \to Z$ are morphisms such that $g \circ f$ is quasi-affine, and if $g$ is separated or the underlying space of $X$ is locally Noetherian, then $f$ is quasi-affine.

(vi) If $f$ is a quasi-affine morphism, then so is $f_{\text{red}}$.

**Proof.** Taking into account the criterion (5.1.6, c'), all of (i), (iii), (iv), (v), and (vi) follow immediately from (4.6.13, i bis), (4.6.13, iii), (4.6.13, iv), (4.6.13, v), and (4.6.13, vi) (respectively). To prove (ii), we can restrict to the case where $Z$ is affine, and then the claim follows directly from applying (4.6.13, ii) to $\mathcal{L} = \mathcal{O}_X$ and $\mathcal{H} = \mathcal{O}_Y$.

**Remark (5.1.11).** — Let $f : X \to Y$ and $g : Y \to Z$ be morphisms such that $X \times_Z Y$ is locally Noetherian. Then the graph immersion $\Gamma_f : X \to X \times_Z Y$ is quasi-affine, since it is quasi-compact (I, 6.3.5), and since (I, 5.5.12) shows that, in (v), the conclusion still holds true if we remove the hypothesis that $g$ is separated.

**Proposition (5.1.12).** — Let $f : X \to Y$ be a quasi-compact morphism, and $g : X' \to X$ a quasi-affine morphism. If $\mathcal{L}$ is an ample (for $f$) $\mathcal{O}_X$-module, then $g^*(\mathcal{L})$ is an ample (for $f \circ g$) $\mathcal{O}_{X'}$-module.
Theorem (5.2.1). — (Serre’s criterion). Let \( f : \mathcal{L} \to \mathcal{O}_X \) be quasi-ample for \( f \circ g \), and so \( g^*(\mathcal{L}) \) is ample for \( f \circ g \). Therefore, we necessarily have \( g^*(\mathcal{L}^\otimes n) = (g^*(\mathcal{L}))^\otimes n \) is ample for \( f \circ g \), and so \( g^*(\mathcal{L}) \) is ample for \( f \circ g \).

5.2. Serre’s criterion

Theorem (5.2.1). — (Serre’s criterion). Let \( X \) be a quasi-compact scheme or a prescheme whose underlying space is Noetherian. The following conditions are equivalent.

(a) \( X \) is an affine scheme.

(b) There exists a family of elements \( f_\alpha \in A = \Gamma(X, \mathcal{O}_X) \) such that the \( X_{f_\alpha} \) are affine, and such that the ideal generated by the \( f_\alpha \) in \( A \) is equal to \( A \) itself.

(c) The functor \( \Gamma(X, \mathcal{F}) \) is exact in \( \mathcal{F} \) on the category of quasi-coherent \( \mathcal{O}_X \)-modules, or, in other words, if

\[
(*) \quad 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0
\]

is an exact sequence of quasi-coherent \( \mathcal{O}_X \)-modules, then the sequence

\[
0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0
\]

is also exact.

(c’) Condition (c) holds for every exact sequence \( (*) \) of quasi-coherent \( \mathcal{O}_X \)-modules such that \( \mathcal{F} \) is isomorphic to a \( \mathcal{O}_X \)-submodule of \( \mathcal{O}_X^n \) for some finite \( n \).

(d) \( H^1(X, \mathcal{F}) = 0 \) for every quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \).

(d’) \( H^1(X, \mathcal{F}_n) = 0 \) for every quasi-coherent sheaf of ideals \( \mathcal{F} \) of \( \mathcal{O}_X \).

Proof. It is evident that (a) implies (b); furthermore, (b) implies that the \( X_{f_\alpha} \) cover \( X \), because, by hypothesis, the section 1 is a linear combination of the \( f_\alpha \), and the \( D(f_\alpha) \) thus cover \( \text{Spec}(A) \). The final claim of (4.5.2) thus implies that \( X \to \text{Spec}(A) \) is an isomorphism.

We know that (a) implies (c) (I, 1.3.11), and (c) trivially implies (c’). We now prove that (c’) implies (b). First of all, (c’) implies that, for every closed point \( x \in X \) and every open neighbourhood \( U \) of \( x \), there exists some \( f \in A \) such that \( x \in X_f \subset X \setminus U \). Let \( \mathcal{F} \) (resp. \( \mathcal{F}' \)) be the quasi-coherent sheaf of ideals of \( \mathcal{O}_X \) defining the reduced closed subscheme of \( X \) that \( X \setminus U \) (resp. \( X \setminus \{x\} \)) as its underlying space (I, 5.2.1); it is clear that we have \( \mathcal{F}' \subset \mathcal{F} \), and that \( \mathcal{F}'' = \mathcal{F}/\mathcal{F}' \) is a quasi-coherent \( \mathcal{O}_X \)-module that has support equal to \( \{x\} \), and such that \( \mathcal{F}'' \) is isomorphic to a \( \mathcal{O}_X \)-submodule of \( \mathcal{O}_X^n \) for some finite \( n \). Hypothesis (c’) applied to the exact sequence \( 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \) shows that \( \Gamma(X, \mathcal{F}', \mathcal{F}'') \) is surjective. The section of \( \mathcal{F}'' \) whose germ at \( x \) is \( 1_x \) is thus the image of some section \( f \in \Gamma(X, \mathcal{F}) \subset \Gamma(X, \mathcal{O}_X) \), and we have, by definition, that \( f(x) = 1_x \) and \( f(y) = 0 \) in \( X \setminus U \), which establishes our claim. Now, if \( U \) is affine, then so is \( X_f \) (I, 1.3.6), so the union of the \( X_f \) that are affine (\( f \in A \)) is an open set \( Z \) that contains all the closed points of \( X \); since \( X \) is a quasi-compact Kolmogorov space, we necessarily have \( Z = X \) (0, 2.1.3). Because \( X \) is quasi-compact, there are a finite number of elements \( f_i \in A \) (\( 1 \leq i \leq n \)) such that the \( X_{f_i} \) are affine and cover \( X \). So consider the homomorphism \( \mathcal{O}_X^n \to \mathcal{O}_X \) defined by the sections \( f_i \); since, for all \( x \in X \), at least one of the \( (f_i)_x \) is invertible, this homomorphism is surjective, and we thus have an exact sequence \( 0 \to \mathcal{F} \to \mathcal{O}_X^n \to \mathcal{O}_X \to 0 \), where \( \mathcal{F} \) is a quasi-coherent \( \mathcal{O}_X \)-submodule of \( \mathcal{O}_X \). It then follows from (c’) that the corresponding homomorphism \( \Gamma(X, \mathcal{O}_X^n) \to \Gamma(X, \mathcal{O}_X) \) is surjective, which proves (b).

Finally, (a) implies (d) (I, 5.1.9.2), and (d) trivially implies (d’). It remains to show that (d’) implies (c’). But if \( \mathcal{F}' \) is a quasi-coherent \( \mathcal{O}_X \)-submodule of \( \mathcal{O}_X^n \), then the filtration \( 0 \subset \mathcal{O}_X \subset \mathcal{O}_X^2 \subset \ldots \subset \mathcal{O}_X^n \) defines a filtration \( \mathcal{F}' \) given by the \( \mathcal{F}'_k = \mathcal{F} \cap \mathcal{O}_X^k \) (\( 0 \leq k \leq n \)), which are quasi-coherent \( \mathcal{O}_X \)-modules (I, 4.1.1), and \( \mathcal{F}'_k/\mathcal{F}'_{k+1} \) is isomorphic to a quasi-coherent \( \mathcal{O}_X \)-submodule of \( \mathcal{O}_X^{k+1}/\mathcal{O}_X^k = \mathcal{O}_X \), which is to say, a quasi-coherent sheaf of ideals of \( \mathcal{O}_X \). Hypothesis (d’) thus implies that \( H^1(X, \mathcal{F}'_k/\mathcal{F}'_{k+1}) = 0 \); the exact cohomology sequence \( H^1(X, \mathcal{F}'_k) \to H^1(X, \mathcal{F}'_{k+1}) \to H^1(X, \mathcal{F}'_{k+1}/\mathcal{F}'_k) = 0 \) then lets us prove by induction on \( k \) that \( H^1(X, \mathcal{F}'_k) = 0 \) for all \( k \). \( \square \)
Remark (5.2.1.1). — When $X$ is a Noetherian prescheme, we can replace “quasi-coherent” by “coherent” in the statements of (c’) and (d’). Indeed, in the proof of the fact that (c’) implies (b), $\mathcal{F}$ and $\mathcal{F}'$ are then coherent sheaves of ideals, and, furthermore, every quasi-coherent submodule of a coherent module is coherent (I, 6.1.1); whence the conclusion.

Corollary (5.2.2). — Let $f : X \to Y$ be a separated quasi-compact morphism. The following conditions are equivalent.

(a) The morphism $f$ is an affine morphism.
(b) The functor $f_*$ is exact on the category of quasi-coherent $\mathcal{O}_X$-modules.
(c) For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, we have $R^1f_*(\mathcal{F}) = 0$.
(c’) For every quasi-coherent sheaf of ideals $\mathcal{I}$ of $\mathcal{O}_X$, we have $R^1f_*(\mathcal{I}) = 0$.

Proof. All these conditions are local on $Y$, by definition of the functor $R^1f_*$ (T, 3.7.3), and so we can assume that $Y$ is affine. If $f$ is affine, then $X$ is affine, and property (b) is nothing more than (I, 1.6.4). Conversely, we now show that (b) implies (a): for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, we have that $f_*(\mathcal{F})$ is a quasi-coherent $\mathcal{O}_Y$-module (I, 9.2.2, a). By hypothesis, the functor $f_*(\mathcal{F})$ is exact in $\mathcal{F}$, and the functor $\Gamma(Y, \mathcal{I})$ is exact in $\mathcal{I}$ (in the category of quasi-coherent $\mathcal{O}_Y$-modules) because $Y$ is affine (I, 1.3.11); so $\Gamma(Y, f_*(\mathcal{F})) = \Gamma(X, \mathcal{F})$ is exact in $\mathcal{F}$, which proves our claim, by (5.2.1, c).

If $f$ is affine, then $f^{-1}(U)$ is affine for every affine open subset $U$ of $Y$ (1.3.2), and so $H^1(f^{-1}(U), \mathcal{F}) = 0$ (5.2.1, d), which, by definition, implies that $R^1f_*(\mathcal{F}) = 0$. Finally, suppose that condition (c’) is satisfied; the exact sequence of terms of low degree in the Leray spectral sequence (G, II, 4.17.1 and I, 4.5.1) give, in particular, the exact sequence

$$0 \to H^1(Y, f_*(\mathcal{I})) \to H^1(X, \mathcal{I}) \to H^0(Y, R^1f_*(\mathcal{I})).$$

Since $Y$ is affine, and $f_*(\mathcal{I})$ quasi-coherent (I, 9.2.2, a), we have that $H^1(Y, f_*(\mathcal{I})) = 0$ (5.2.1); hypothesis (c’) thus implies that $H^1(X, \mathcal{I}) = 0$, and we conclude, by (5.2.1), that $X$ is an affine scheme.

Corollary (5.2.3). — If $f : X \to Y$ is an affine morphism, then, for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the canonical homomorphism $H^1(Y, f_*(\mathcal{F})) \to H^1(X, \mathcal{F})$ is bijective.

Proof. We have the exact sequence

$$0 \to H^1(Y, f_*(\mathcal{F})) \to H^1(X, \mathcal{F}) \to H^0(Y, R^1f_*(\mathcal{F})).$$

of terms of low degree in the Leray spectral sequence, and the conclusion follows from (5.2.2).

Remark (5.2.4). — In Chapter III, §1, we prove that, if $X$ is affine, then we have $H^i(X, \mathcal{F}) = 0$ for all $i > 0$ and all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$.

5.3. Quasi-projective morphisms

Definition (5.3.1). — We say that a morphism $f : X \to Y$ is quasi-projective, or that $X$ (considered as a $Y$-prescheme via $f$) is quasi-projective over $Y$, or that $X$ is a quasi-projective $Y$-scheme, if $f$ is of finite type and there exists an invertible $f$-ample $\mathcal{O}_X$-module.

We note that this notion is not local on $Y$: the counterexamples of Nagata [Nag58b] and Hironaka show that, even if $X$ and $Y$ are non-singular algebraic schemes over an algebraically closed field, every point of $Y$ can have an affine neighbourhood $U$ such that $f^{-1}(U)$ is quasi-projective over $U$, without $f$ being quasi-projective.

We note that a quasi-projective morphism is necessarily separated (4.6.1). When $Y$ is quasi-compact, it is equivalent to say either that $f$ is quasi-projective, or that $f$ is of finite type and there exists a very ample (relative to $f$) $\mathcal{O}_X$-module (4.6.2 and 4.6.11). Further:

Proposition (5.3.2). — Let $Y$ be a quasi-compact scheme or a prescheme whose underlying space is Noetherian, and let $X$ be a $Y$-prescheme. The following conditions are equivalent.

(a) $X$ is a quasi-projective $Y$-scheme.
(b) $X$ is of finite type over $Y$, and there exists some quasi-coherent $O_Y$-module $\mathcal{E}$ of finite type such that $X$ is $Y$-isomorphic to a subscheme of $\mathbb{P}(\mathcal{E})$.

(c) $X$ is of finite type over $Y$, and there exists some quasi-coherent graded $O_Y$-algebra $\mathcal{S}$ such that $\mathcal{S}_1$ is of finite type and generates $\mathcal{S}$, and such that $X$ is $Y$-isomorphic to an induced subscheme on some everywhere-dense open subset of $\text{Proj}(\mathcal{S})$.

**Proof.** This follows immediately from the previous remark and from (4.4.3), (4.4.6), and (4.4.7).

We note that, whenever $Y$ is a Noetherian prescheme, we can, in conditions (b) and (c) of (5.3.2), remove the hypothesis that $X$ is of finite type over $Y$, since this automatically satisfied (I, 6.3.5).

**Corollary (5.3.3).** — Let $Y$ be a quasi-compact scheme such that there exists an ample $O_Y$-module $\mathcal{L}$ (4.5.3). For a $Y$-scheme $X$ to be quasi-projective, it is necessary and sufficient for it to be of finite type over $Y$ and also isomorphic to a $Y$-subscheme of a projective bundle of the form $\mathbb{P}_Y^{k-1}$.

**Proof.** If $\mathcal{E}$ is a quasi-coherent $O_Y$-module of finite type, then $\mathcal{E}$ is isomorphic to a quotient of an $O_Y$-module $\mathcal{L} \otimes_{O_Y} O_Y^{k-1}$ (4.5.5), and so $\mathbb{P}(\mathcal{E})$ is isomorphic to a closed subscheme of $\mathbb{P}_Y^{k-1}$ (4.1.2) and (4.1.4)).

**Proposition (5.3.4).** —

(i) A quasi-affine morphism of finite type (and, in particular, a quasi-compact immersion, or an affine morphism of finite type) is quasi-projective.

(ii) If $f : X \to Y$ and $g : Y \to Z$ are quasi-projective, and if $Z$ is quasi-compact, then $g \circ f$ is quasi-projective.

(iii) If $f : (S') \to Y$ is a quasi-projective $S'$-morphism, then $f(S') : X(S') \to Y(S')$ is quasi-projective for every extension $S' \to S$ of the base prescheme.

(iv) If $f : X \to Y$ and $g : X' \to Y'$ are quasi-projective $S'$-morphisms, then $f \times_S g$ is quasi-projective.

(v) If $f : X \to Y$ and $g : Y \to Z$ are morphisms such that $g \circ f$ is quasi-projective, and if $g$ is separated or $X$ locally Noetherian, then $f$ is quasi-projective.

(vi) If $f$ is a quasi-projective morphism, then so is $f_{\text{red}}$.

**Proof.** (i) follows from (5.1.6) and (5.1.10, i). The other claims are immediate consequences of Definition (5.3.1), of the properties of morphisms of finite type (I, 6.3.4), and of (4.6.13).

**Remark (5.3.5).** — We note that we can have $f_{\text{red}}$ being quasi-projective without $f$ being quasi-projective, even if we assume that $Y$ is the spectrum of an algebra of finite rank over $\mathbb{C}$ and that $f$ is proper.

**Corollary (5.3.6).** — If $X$ and $X'$ are quasi-projective $Y$-schemes, then $X \sqcup X'$ is a quasi-projective $Y$-scheme.

**Proof.** This follows from (4.6.18).

## 5.4. Proper morphisms and universally closed morphisms

**Definition (5.4.1).** — We say that a morphism of preschemes $f : X \to Y$ is **proper** if it satisfies the following two conditions:

(a) $f$ is separated and of finite type; and

(b) for every prescheme $Y'$ and every morphism $Y' \to Y$, the projection $f_{(Y')} : X \times_Y Y' \to Y'$ is a closed morphism (I, 2.2.6).

When this is the case, we also say that $X$ (considered as a $Y$-prescheme with structure morphism $f$) is proper over $Y$.

It is immediate that conditions (a) and (b) are local on $Y$. To show that the image of a closed subset $Z$ of $X \times_Y Y'$ under the projection $q : X \times_Y Y' \to Y'$ is closed in $Y'$, it suffices to see that $q(Z) \cap U'$ is closed in $U'$ for every affine open subset $U'$ of $Y'$; since $q(Z) \cap U' = q(Z \cap q^{-1}(U'))$, and since $q^{-1}(U')$ can be identified with $X \times_Y U'$ (I, 4.4.1), we see that to satisfy condition (b) of Definition (5.4.1), we can restrict to the case where $Y$ is an **affine** scheme. We further see (5.3.6) that, if $Y$ is locally Noetherian, then we can even restrict to proving (b) in the case where $Y'$ is of finite type over $Y$.

It is clear that every proper morphism is **closed**.
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Proposition (5.4.2). —

(i) A closed immersion is a proper morphism.
(ii) The composition of two proper morphisms is proper.
(iii) If $X$ and $Y$ are $S$-preschemes, and $f : X \to Y$ a proper $S$-morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is proper for every extension $S' \to S$ of the base prescheme.
(iv) If $f : X \to Y$ and $g : X' \to Y'$ are proper $S$-morphisms, then $f \times_S g : X \times_S Y \to X' \times_Y Y'$ is a proper $S$-morphism.

Proof. It suffices to prove (i), (ii), and (iii) (I, 3.5.1). In each of the three cases, verifying condition (a) of (5.4.1) follows from previous results ((I, 5.5.1) and (6.4.3)); it remains to verify condition (b). It is immediate in case (i), because if $X \to Y$ is a closed immersion, then so is $X \times_Y Y' \to Y \times_Y Y' = Y'$ ((I, 4.3.2) and (3.3.3)). To prove (ii), consider two proper morphisms $X \to Y$ and $Y \to Z$, and a morphism $Z' \to Z$.

If $X \times_Z Z' = X \times_Y (Y \times_Z Z')$ (I, 3.3.9.1), and so the projection $X \times_Z Z' \to Z'$ factors as $X \times_Y (Y \times_Z Z') \to Y \times_Z Z' \to Z'$. Taking the initial remark into account, (ii) follows from the fact that the composition of two closed morphisms is closed. Finally, for every morphism $S' \to S$, we can identify $X_{(S')}$ with $X \times_Y Y_{(S')}$ (I, 3.3.11); for every morphism $Z \to Y_{(S')}$, we can write

$$X_{(S')} \times_{Y_{(S')}} Z = (X \times_Y Y_{(S')}) \times_{Y_{(S')}} Z = X \times_Y Z;$$

since by hypothesis $X \times_Y Z \to Z$ is closed, this proves (iii). □

Corollary (5.4.3). — Let $f : X \to Y$ and $g : Y \to Z$ be morphisms such that $g \circ f$ is proper.

(i) If $g$ is separated, then $f$ is proper.
(ii) If $g$ is separated and of finite type, and if $f$ is surjective, then $g$ is proper.

Proof. (i) follows from (5.4.2) by the general procedure (I, 5.5.12). To prove (ii), we need only verify that condition (b) of Definition (5.4.1) is satisfied. For every morphism $Z' \to Z$, the diagram

$$\begin{array}{ccc}
X \times_Z Z' & \xrightarrow{f \times_Z Z'} & Y \times_Z Z' \\
\downarrow p & & \downarrow p' \\
Z' & & 
\end{array}$$

(where $p$ and $p'$ are the projections) commutes (I, 3.2.1); furthermore, $f \times 1_{Z'}$ is surjective because $f$ is surjective (I, 3.5.2), and $p$ is a closed morphism by hypothesis. Every closed subset $F$ of $Y \times_Z Z'$ is thus the image under $f \times 1_{Z'}$ of some closed subset $E$ of $X \times_Z Z'$, so $p'(F) = p(E)$ is closed in $Z'$ by hypothesis, whence the corollary. □

Corollary (5.4.4). — If $X$ is a proper prescheme over $Y$, and $\mathcal{O}$ a quasi-coherent $O_Y$-algebra, then every $Y$-morphism $f : X \to \text{Proj}(\mathcal{O})$ is proper (and a fortiori closed).

Proof. The structure morphism $p : \text{Proj}(\mathcal{O}) \to Y$ is separated, and $p \circ f$ is proper by hypothesis. □

Corollary (5.4.5). — Let $f : X \to Y$ be a separated morphism of finite type. Let $(X_i)_{1 \leq i \leq n}$ (resp. $(Y_i)_{1 \leq i \leq n}$) be a finite family of closed sub-preschemes of $X$ (resp. $Y$), and $j_i$ (resp. $h_i$) the canonical injection $X_i \to X$ (resp. $Y_i \to Y$). Suppose that the underlying space of $X$ is the union of the $X_i$, and that, for all $i$, there is a morphism $f_i : X_i \to Y_i$, such that the diagram

$$\begin{array}{ccc}
X_i & \xrightarrow{f_i} & Y_i \\
\downarrow j_i & & \downarrow h_i \\
X & \xrightarrow{f} & Y
\end{array}$$

commutes. Then, for $f$ to be proper, it is necessary and sufficient for all of the $f_i$ to be proper.
Proof. If \( f \) is proper, then so is \( f \circ j_i \), because \( j_i \) is a closed immersion \((5.4.2)\); since \( h_i \) is a closed immersion, and thus a separated morphism, \( f_i \) is proper, by \((5.4.3)\). Conversely, suppose that all of the \( f_i \) are proper, and consider the prescheme \( Z \) given by the sum of the \( X_i \); let \( u \) be the morphism \( Z \to X \) which reduces to \( j_i \) on each \( X_i \). The restriction of \( f \circ u \) to each \( X_i \) is equal to \( f \circ j_i = h_i \circ f_i \), and is thus proper, because both the \( h_i \) and the \( f_i \) are \((5.4.2)\); it then follows immediately from Definition \((5.4.1)\) that \( u \) is proper. But since by hypothesis \( u \) is surjective, we conclude that \( f \) is proper by \((5.4.3)\).

Corollary \((5.4.6)\). — Let \( f : X \to Y \) be a separated morphism of finite type; for \( f \) to be proper, it is necessary and sufficient for \( f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}} \) to be proper.

Proof. This is a particular case of \((5.4.5)\), with \( n = 1 \), \( X_1 = X_{\text{red}} \), and \( Y_1 = Y_{\text{red}} \) \((I, 5.1.5)\).

\((5.4.7)\). If \( X \) and \( Y \) are Noetherian preschemes, and \( f : X \to Y \) a separated morphism of finite type, then we can, to show that \( f \) is proper, restrict to the the case of dominant morphisms and integral preschemes. Indeed, let \( X_i \) \((1 \leq i \leq n)\) be the (finitely many) irreducible components of \( X \), and consider, for each \( i \), the unique reduced closed subscheme of \( X \) that has \( X_i \) as its underlying space, which we again denote by \( X_i \) \((I, 5.2.1)\). Let \( Y_i \) be the unique reduced closed subscheme of \( Y \) that has \( f(X_i) \) as its underlying space. If \( g_i \) (resp. \( h_i \)) is the injection morphism \( X_i \to X \) (resp. \( Y_i \to Y \)), then we conclude that \( f \circ g_i = h_i \circ f_i \), where \( f_i \) is a dominant morphism \( X_i \to Y_i \) \((I, 5.2.2)\); we are then under the right conditions to apply \((5.4.5)\), and for \( f \) to be proper, it is necessary and sufficient for all the \( f_i \) to be proper.

Corollary \((5.4.8)\). — Let \( X \) and \( Y \) be separated \( S \)-preschemes of finite type over \( S \), and \( f : X \to Y \) an \( S \)-morphism. For \( f \) to be proper, it is necessary and sufficient that, for every \( S \)-prescheme \( S' \), the morphism \( f \times_S 1_{S'} : X \times_S S' \to Y \times_S S' \) be closed.

Proof. First note that, if \( g : X \to S \) and \( h : Y \to S \) are the structure morphisms, then we have, by definition, \( g = h \circ f \), and so \( f \) is separated and of finite type \((I, 5.5.1)\) and \((6.3.4)\). If \( f \) is proper, then so is \( f \times_S 1_{S'} \) \((5.4.2)\); \( a \text{ fortiori, } f \times_S 1_{S'} \) is closed. Conversely, suppose that the conditions of the statement are satisfied, and let \( Y' \) be a \( Y \)-prescheme; \( Y' \) can also be considered as an \( S \)-prescheme, and since \( Y \to S \) is separated, \( X \times_Y Y' \) can be identified with a closed subscheme of \( X \times_S Y' \) \((I, 5.4.2)\). In the commutative diagram

\[
\begin{array}{ccc}
X \times_Y Y' & \xrightarrow{f \times_{1_{Y'}}} & Y' \times_Y Y' = Y' \\
\downarrow & & \downarrow \\
X \times_S Y' & \xrightarrow{f \times_{1_{Y'}}} & Y \times_S Y',
\end{array}
\]

the vertical arrows are closed immersions; it thus immediately follows that if \( f \times 1_{S'} \) is a closed morphism, then so is \( f \times 1_{Y'} \).

Remark \((5.4.9)\). — We say that a morphism \( f : X \to Y \) is universally closed if it satisfies condition (b) of Definition \((5.4.1)\). The reader will observe that, in \((5.4.2)\) to \((5.4.8)\), we can replace every occurrence of “proper” with “universally closed” without changing the validity of the results (and in the hypotheses of \((5.4.3)\), \((5.4.5)\), \((5.4.6)\), and \((5.4.8)\), we can omit the finiteness conditions).

\((5.4.10)\). Let \( f : X \to Y \) be a morphism of finite type. We say that a closed subset \( Z \) of \( X \) is proper on \( Y \) (or \( Y \)-proper, or proper for \( f \)) if the restriction of \( f \) to a closed subscheme of \( X \), with underlying space \( Z \) \((I, 5.2.1)\), is proper. Since this restriction is then separated, it follows from \((5.4.6)\) and \((I, 5.5.1, \text{vi})\) that the preceding property does not depend on the closed subscheme of \( X \) that has \( Z \) as its underlying space. If \( g : X' \to X \) is a proper morphism, then \( g^{-1}(Z) \) is a proper subset of \( X' \): if \( T \) is a subscheme of \( X \) that has \( Z \) as its underlying space, it suffices to note that the restriction of \( g \) to the closed subscheme \( g^{-1}(T) \) of \( X' \) is a proper morphism \( g^{-1}(T) \to T \), by \((5.4.2, \text{iii})\), and to then apply \((5.4.2, \text{ii})\). Further, if \( X'' \) is a \( Y \)-scheme of finite type, and \( u : X \to X'' \) a \( Y \)-morphism, then \( u(Z) \) is a proper subset of \( X'' \); indeed, let us take \( T \) to be the reduced closed subscheme of \( X \) having \( Z \) as its underlying space; then the restriction of \( f \) to \( T \) is proper, and thus so is the restriction of \( u \) to \( T \) \((5.4.3, \text{i})\), thus \( u(Z) \) is closed in \( X'' \); let \( T'' \) be a closed subscheme of \( X'' \)
having \( u(Z) \) as its underlying space (I, 5.2.1), such that \( u|T \) factors as \( T \stackrel{\pi}{\to} T'' \stackrel{j}{\to} X'' \), where \( j \) is the canonical injection (I, 5.2.2), and \( v \) is thus proper and surjective (5.4.5); if \( g \) is the restriction to \( T'' \) of the structure morphism \( X'' \to Y \), then \( g \) is separated and of finite type, and we have that \( f|T = g \circ v \); it thus follows from (5.4.3, ii) that \( g \) is proper, whence our assertion.

It follows, in particular, from these remarks that, if \( Z \) is a \( Y \)-proper subset of \( X \), then

1. for every closed subscheme \( X' \) of \( X \), \( Z \cap X' \) is a \( Y \)-proper subset of \( X' \); and
2. if \( X \) is a subscheme of a \( Y \)-scheme of finite type \( X'' \), then \( Z \) is also a \( Y \)-proper subset of \( X'' \) (and so, in particular, is closed in \( X'' \)).

### 5.5. Projective morphisms

#### Proposition (5.5.1).

Let \( X \) be a \( Y \)-prescheme. The following conditions are equivalent.

(a) \( X \) is \( Y \)-isomorphic to a closed subscheme of a projective bundle \( \mathbb{P}(\mathcal{E}) \), where \( \mathcal{E} \) is a quasi-coherent \( \mathcal{O}_Y \)-module of finite type.

(b) There exists a quasi-coherent graded \( \mathcal{O}_Y \)-algebra \( \mathcal{I} \) such that \( \mathcal{I}_1 \) is of finite type and generates \( \mathcal{I} \), and such that \( X \) is \( Y \)-isomorphic to \( \text{Proj}(\mathcal{I}) \).

**Proof.** Condition (a) implies (b), by (3.6.2, ii): if \( \mathcal{I} \) is a quasi-coherent graded sheaf of ideals of \( S(\mathcal{E}) \), then the quasi-coherent graded \( \mathcal{O}_Y \)-algebra \( \mathcal{I} = S(\mathcal{E})/\mathcal{J} \) is generated by \( \mathcal{I}_1 \), and \( \mathcal{I}_1 \), the canonical image of \( \mathcal{E} \), is an \( \mathcal{O}_Y \)-module of finite type. Condition (b) implies (a) by (3.6.2) applied to the case where \( \mathcal{M} \to \mathcal{I}_1 \) is the identity map. \( \square \)

#### Definition (5.5.2).

We say that a \( Y \)-prescheme \( X \) is projective on \( Y \), or is a projective \( Y \)-scheme, if it satisfies either of the (equivalent) conditions (a) and (b) of (5.5.1). We say that a morphism \( f : X \to Y \) is projective if it makes \( X \) a projective \( Y \)-scheme.

It is clear that if \( f : X \to Y \) is projective, then there exists a very ample (relative to \( f \)) \( \mathcal{O}_X \)-module (4.4.2).

#### Theorem (5.5.3).

(i) Every projective morphism is quasi-projective and proper.

(ii) Conversely, let \( Y \) be a quasi-compact scheme or a prescheme whose underlying space is Noetherian; then every morphism \( f : X \to Y \) that is quasi-projective and proper is projective.

**Proof.**

(i) It is clear that if \( f : X \to Y \) is projective, then it is of finite type and quasi-projective (thus, in particular, separated); furthermore, it follows immediately from (5.5.1, b) and (3.5.3) that if \( f \) is projective, then so is \( f \times Y : X \times Y Y' \to Y' \) for every morphism \( Y' \to Y \). To show that \( f \) is universally closed, it is thus enough to show that a projective morphism \( f \) is closed. Since the question is local on \( Y \), we can suppose that \( Y = \text{Spec}(A) \), thus (5.5.1) \( X = \text{Proj}(S) \), where \( S \) is a graded \( A \)-algebra generated by a finite number of elements of \( S_1 \). For all \( y \in Y \), the fibre \( f^{-1}(y) \) can be identified with \( \text{Proj}(S_y \times_Y \text{Spec}(k(y))) \) (I, 3.6.1), and so also with \( \text{Proj}(S_y \otimes_A k(y)) \) (2.8.10); so \( f^{-1}(y) \) is empty if and only if \( S_n \otimes_A k(y) \) satisfies condition (TN) (2.7.4), or, in other words, if \( S_n \otimes_A k(y) = 0 \) for sufficiently large \( n \). But since \( (S_n)_y \) is an \( \mathcal{O}_Y \)-module of finite type, the preceding condition implies that \( (S_n)_y = 0 \) for sufficiently large \( N \), by Nakayama’s lemma. If \( a_n \) is the annihilator in \( A \) of the \( A \)-module \( S_n \), then the preceding condition also implies that \( a_n \subseteq j_n \) for sufficiently large \( n \) (0, 1.7.4). But since \( S_n \otimes_A k(y) \) is an \( \mathcal{O}_Y \)-module of finite type, as well as a \( Y \)-immersion \( j : X \to \mathbb{P}(\mathcal{E}) \) (5.3.2). But since \( f \) is proper, \( j \) is closed, by (5.4.4), and so \( f \) is projective.

(ii) The hypothesis on \( Y \) and the fact that \( f \) is quasi-projective implies the existence of a quasi-coherent \( \mathcal{O}_Y \)-module \( \mathcal{E} \) of finite type, as well as a \( Y \)-immersion \( \mathcal{J} : X \to \mathbb{P}(\mathcal{E}) \).
Remark (5.5.4). —

(i) Let \( f : X \to Y \) be a morphism such that \( f \) is proper, such that there exists a very ample (relative to \( f \)) \( O_X \)-module \( \mathcal{L} \), and such that the quasi-coherent \( O_Y \)-module \( \mathcal{E} = f_* (\mathcal{L}) \) is of finite type. Then \( f \) is a projective morphism: indeed (4.4.4), there is then a \( Y \)-immersion \( r : X \to \mathbb{P}(\mathcal{E}) \), and, since \( f \) is proper, \( r \) is a closed immersion (5.4.4). We will see in Chapter III, §3, that when \( Y \) is locally Noetherian, the third condition above (\( \mathcal{E} \) being of finite type) is a consequence of the first two, and so the first two conditions characterise, in this case, the projective morphisms, and if \( Y \) is quasi-compact, then we can replace the second condition (the existence of a very ample (relative to \( f \)) \( O_X \)-module \( \mathcal{L} \)) by the hypothesis that there exists an ample (relative to \( f \)) \( O_X \)-module (4.6.11).

(ii) Let \( Y \) be a quasi-compact scheme such that there exists an ample \( O_Y \)-module. For a \( Y \)-scheme \( X \) to be projective, it is necessary and sufficient for it to be \( Y \)-isomorphic to a closed \( Y \)-subscheme of a projective bundle of the form \( \mathbb{P}_Y \). The condition is clearly sufficient. Conversely, if \( X \) is projective over \( Y \), then it is quasi-projective, and so there exists a \( Y \)-immersion \( j \) of \( X \) into some \( \mathbb{P}_Y \) (5.3.3) that is closed, by (5.4.4) and (5.5.3).

(iii) The argument of (5.5.3) shows that, for every prescheme \( Y \) and every integer \( r \geq 0 \), the structure morphism \( \mathbb{P}_Y \to Y \) is surjective, because if we set \( \mathcal{I} = S\mathcal{O}_Y (\mathcal{O}_Y (-1)^{r+1}) \), then we evidently have \( \mathcal{I}_Y = S_k(k)^{r+1} (1.7.3) \), and so \( (\mathcal{I}_Y)_y \neq 0 \) for any \( y \in Y \) or any \( n \geq 0 \).

(iv) It follows from the examples of Nagata [Nag58b] that there exist proper morphisms that are not quasi-projective.

Proposition (5.5.5). —

(i) A closed immersion is a projective morphism.

(ii) If \( f : X \to Y \) and \( g : Y \to Z \) are projective morphisms, and if \( Z \) is a quasi-compact scheme or a prescheme whose underlying space is Noetherian, then \( g \circ f \) is projective.

(iii) If \( f : X \to Y \) is a projective \( S \)-morphism, then \( f_{(S')} : X_{(S')} \to Y_{(S')} \) is projective for every extension \( S' \to S \) of the base prescheme.

(iv) If \( f : X \to Y \) and \( g : X' \to Y' \) are projective \( S \)-morphisms, then so is \( f \times g \).

(v) If \( g \circ f \) is a projective morphism, and if \( g \) is separated, then \( f \) is projective.

(vi) If \( f \) is projective, then so is \( f_{red} \).

Proof. (i) follows immediately from (3.1.7). We have to show (iii) and (iv) separately, because of the restriction introduced on \( Z \) in (ii) (cf. (I, 3.5.1)). To show (iii), we restrict to the case where \( S = Y \) (I, 3.3.11), and the claim then immediately follows from (5.5.1, b) and (3.5.3). To show (iv), we are immediately led to the case where \( X = \mathbb{P}(\mathcal{E}) \) and \( X = \mathbb{P}(\mathcal{E}') \), where \( \mathcal{E} \) (resp. \( \mathcal{E}' \)) is a quasi-coherent \( O_Y \)-module (resp. quasi-coherent \( O_{Y'} \)-module) of finite type. Let \( p \) and \( p' \) be the canonical projections of \( T = Y \times_S Y' \) to \( Y \) and \( Y' \) (respectively); by (4.1.3.1), we have \( \mathbb{P}(p^*(\mathcal{E})) = \mathbb{P}(\mathcal{E}) \times_Y T \) and \( \mathbb{P}(p'^*(\mathcal{E}')) = \mathbb{P}(\mathcal{E}') \times_{Y'} T \); whence
\[
\mathbb{P}(p^*(\mathcal{E})) \times_T \mathbb{P}(p'^*(\mathcal{E}')) = (\mathbb{P}(\mathcal{E}) \times_Y T) \times_T (T \times_{Y'} \mathbb{P}(\mathcal{E}')) = (\mathbb{P}(\mathcal{E}) \times_Y (T \times_{Y'} \mathbb{P}(\mathcal{E}'))) = \mathbb{P}(\mathcal{E}) \times_X \mathbb{P}(\mathcal{E}')
\]
by replacing \( T \) with \( Y \times_S Y' \), and using (I, 3.3.9.1). But \( p^*(\mathcal{E}) \) and \( p'^*(\mathcal{E}') \) are of finite type over \( T (0, 5.2.4) \), and thus so is \( p^*(\mathcal{E}) \otimes_{\mathcal{O}_Y} p'^*(\mathcal{E}') \); since \( \mathbb{P}(p^*(\mathcal{E})) \times_T \mathbb{P}(p'^*(\mathcal{E}')) \) can be identified with a closed subsheaf of \( p^*(\mathcal{E}) \otimes_{\mathcal{O}_Y} p'^*(\mathcal{E}') \) (4.3.3), this proves (iv). To show (v) and (vi), we can apply (I, 5.5.13), because every closed subsheaf of a projective \( Y \)-scheme is a projective \( Y \)-scheme, by (5.5.1, a).

It remains to prove (ii); by the hypothesis on \( Z \), this follows from (5.5.3), (5.3.4, ii), and (5.4.2, ii).

Proposition (5.5.6). — If \( X \) and \( X' \) are projective \( Y \)-schemes, then \( X \sqcup X' \) is a projective \( Y \)-scheme.

Proof. This is an evident consequence of (5.5.2) and (4.3.6).
Proposition (5.5.7). — Let $X$ be a projective $Y$-scheme, and $\mathcal{L}$ a $Y$-ample $\mathcal{O}_X$-module; then, for every section $f$ of $\mathcal{L}$ over $X$, $X_f$ is affine over $Y$.

Proof. Since the question is local on $Y$, we can assume that $Y = \text{Spec}(A)$; furthermore, $X_{f\circ x} = X_f$, so by replacing $\mathcal{L}$ with some suitable $\mathcal{L}\otimes P$, we can assume that $\mathcal{L}$ is very ample relative to the structure morphism $q : X \to Y$ (4.6.11). The canonical homomorphism $\sigma : q^*(q_*(\mathcal{L})) \to \mathcal{L}$ is thus surjective, and the corresponding morphism

$$r = r_{|_{\mathcal{L}:X}} : X \to \text{P} = \text{P}(q_*(\mathcal{L}))$$

is an immersion such that $\mathcal{L} = r^*(\mathcal{O}_\text{P}(1))$ (4.4.4); furthermore, since $X$ is proper over $Y$, the immersion $r$ is closed (5.4.4). But by definition, $f \in \Gamma(Y, q_*(\mathcal{L}))$, and $\sigma^r$ is the identity of $q_*(\mathcal{L})$; it then follows from Equation (3.7.3.1) that we have $X_f = r^{-1}(\text{D}_+(f))$; so $X_f$ is a closed subprescheme of the affine scheme $\text{D}_+(f)$, and is thus also an affine scheme. \(\square\)

In the particular case where $Y = X$, we obtain (taking (4.6.13, i) into account) the following corollary, whose direct proof is immediate anyway:

Corollary (5.5.8). — Let $X$ be a prescheme, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. For every section $f$ of $\mathcal{L}$ over $X$, $X_f$ is affine over $X$ (and thus also an affine scheme whenever $X$ is an affine scheme).

5.6. Chow’s lemma

Theorem (5.6.1). — (Chow’s lemma). Let $S$ be a prescheme, and $X$ an $S$-scheme of finite type. Suppose that the following conditions are satisfied:

- (a) $S$ is Noetherian;
- (b) $S$ is a quasi-compact scheme, and $X$ has a finite number of irreducible components.

Under these hypotheses,

- (i) there exists a quasi-projective $S$-scheme $X'$, and an $S$-morphism $f : X' \to X$ that is both projective and surjective;
- (ii) we can take $X'$ and $f$ to be such that there exists an open subset $U \subset X$ for which $U' = f^{-1}(U)$ is dense in $X'$, and for which the restriction of $f$ to $U'$ is an isomorphism $U' \simeq U$; and
- (iii) if $X$ is reduced (resp. irreducible, integral), then we can assume that $X'$ is reduced (resp. irreducible, integral).

Proof. The proof proceeds in multiple steps.

(A) We can first restrict to the case where $X$ is irreducible. Indeed, in hypothesis (a), $X$ is Noetherian, and so, in the two hypotheses, the irreducible components $X_i$ of $X$ are finite in number. If the theorem is shown to be true for each of the reduced closed preschemes of $X$ having the $X_i$ as their underlying spaces, and if $X'_i$ and $f_i : X'_i \to X_i$ are the prescheme and the morphism corresponding to $X_i$ (respectively), then the prescheme $X'$ given by the sum of the $X'_i$, and the morphism $f : X' \to X$ whose restriction to each $X'_i$ is $j_i \circ f_i$ (where $j_i$ is the canonical injection $X_i \to X$) satisfy the conclusion of the theorem. It is immediate that $X'$ is reduced if all of the $X'_i$ are; furthermore, we can satisfy (ii) by taking $U$ to be the union of the sets $U_i \cap \mathcal{C} \left( \bigcup_{j \neq i} X_j \right)$. Finally, since the $X'_i$ are quasi-projective over $S$, so is $X'$ (5.3.6); similarly, the morphisms $X'_i \to X$ are projective by (5.5.5, i) and (5.5.5, ii), and so $f$ is projective (5.5.6), and is clearly surjective, by definition.

(B) Now suppose that $X$ is irreducible. Since the structure morphism $r : X \to S$ is of finite type, there exists a finite cover $(S_i)$ of $S$ by affine open subsets, and for each $i$ there is a finite cover $(T_{ij})$ of $r^{-1}(S_i)$ by affine open subsets, and the morphisms $T_{ij} \to S_i$ are of finite type, and so quasi-projective (5.3.4, i); since in both hypotheses (a) and (b) the immersion $S_i \to X$ is quasi-compact, it is also quasi-projective (5.3.4, i), and so the restriction of $r$ to $T_{ij}$ is a quasi-projective morphism (5.3.4, ii). Denote the $T_{ij}$ by $U_k$ $(1 \leq k \leq n)$. There exists, for each index $k$, an open immersion $\phi_k : U_k \to P_k$, where $P_k$ is projective over $S$ (5.3.2) and...
(5.5.2)). Let $U = \bigcap_k U_k$; since $X$ is irreducible, and the $U_k$ nonempty, $U$ is nonempty, and thus dense in $X$; the restrictions of the $\phi_k$ to $U$ define a morphism

$$\phi : U \longrightarrow P = P_1 \times_S P_2 \times_S \cdots \times_S P_n$$

such that the diagrams

(5.6.1.1)

$$\begin{array}{ccc}
U & \xrightarrow{\phi} & P \\
\downarrow j_k & & \downarrow p_k \\
U_k & \xrightarrow{\phi_k} & P_k
\end{array}$$

commute, where $j_k$ is the canonical injection $U \rightarrow U_k$, and $p_k$ the canonical projection $P \rightarrow P_k$. If $j$ is the canonical injection $U \rightarrow X$, then the morphism $\psi = (j, \phi)_S : U \rightarrow X \times_S P$ is an immersion (I, 5.3.14). In hypothesis (a), $X \times_S P$ is locally Noetherian ((3.4.1), (I, 6.3.7), and (I, 6.3.8)); in hypothesis (b), $X \times_S P$ is a quasi-compact scheme ((I, 5.5.1) and (I, 6.6.4)); in both cases, the closure $X'$ in $X \times_S P$ of the subprescheme $Z$ associated to $\psi$ (and so with underlying space $\psi(U)$) exists, and $\psi$ factors as

(5.6.1.2)

$$\psi : U \xrightarrow{\psi'} X' \xrightarrow{h} X \times_S P$$

where $\psi'$ is an open immersion and $h$ a closed immersion (I, 9.5.10). Let $q_1 : X \times_S P \rightarrow X$ and $q_2 : X \times_S P \rightarrow P$ be the canonical projections; we set

(5.6.1.3)

$$f : X' \xrightarrow{h} X \times_S P \xrightarrow{q_1} X,$$

(5.6.1.4)

$$g : X' \xrightarrow{h} X \times_S P \xrightarrow{q_2} P.$$

We will see that $X'$ and $f$ satisfy the conclusion of the theorem.

(C) First we show that $f$ is projective and surjective, and that the restriction of $f$ to $U' = f^{-1}(U)$ is an isomorphism from $U'$ to $U$. Since the $P_i$ are projective over $S$, so is $P$ (5.5.5, iv), and so $X \times_S P$ is projective over $X$ (5.5.5, iii), and thus so is $X'$, which is a closed subscheme of $X \times_S P$. Furthermore, we have $f \circ \psi' = q_1 \circ (h \circ \psi') = q_1 \circ \psi = j$, so $f(X')$ contains the open everywhere-dense subset $U$ of $X$; but $f$ is a closed morphism (5.5.3), so $f(X') = X$. Now note that $q_1^{-1}(U) = U \times_S P$ is induced on an open subset of $X \times_S P$, and, by definition, the prescheme $U' = h^{-1}(U \times_S P)$ is induced by $X'$ on the open subset $U'$; it is thus the closure relative to $U \times_S P$ of the prescheme $Z$ (I, 9.5.8). But the immersion $\psi$ factors as

$$U \xrightarrow{\Gamma_{\psi}} U \times_S P \xrightarrow{i \times 1} X \times_S P,$$

and since $P$ is separated over $S$, the graph morphism $\Gamma_{\psi}$ is a closed immersion (I, 5.4.3), and so $Z$ is a closed subscheme of $U \times_S P$, whence $U'$ is $Z$. Since $\psi$ is an immersion, the restriction of $f$ to $U'$ is an isomorphism onto $U$, and the inverse of $\psi';$ finally, by the definition of $X'$, $U'$ is dense in $X'$.

(D) We now show that $g$ is an immersion, which will imply that $X'$ is quasi-projective over $S$, because $P$ is projective over $S$. Set

$$V_k = \phi_k(U_k) \quad (\text{open subset of } P_k)$$

$$W_k = p_k^{-1}(V_k) \quad (\text{open subset of } P)$$

$$U'_k = f^{-1}(U_k) \quad (\text{open subset of } X')$$

$$U''_k = g^{-1}(W_k) \quad (\text{open subset of } X').$$

It is clear that the $U'_k$ form an open cover of $X'$; we will first see that the $U''_k$ also form an open cover of $X'$, by showing that $U'_k \subset U''_k$. For this, it will suffice to show that the diagram

(5.6.1.5)

$$\begin{array}{ccc}
U'_k & \xrightarrow{g|U'_k} & P \\
\downarrow f|U'_k & & \downarrow p_k \\
U_k & \xrightarrow{\phi_k} & P_k
\end{array}$$
commutes. But the prescheme $U'_k = h^{-1}(U_k \times_S P)$ is induced by $X'$ on the open subset $U'_k$, and is thus the closure of $Z = U' \subset U'_k$ relative to $U'_k$ (I, 9.5.8). To show the commutativity of (5.6.1.5), it thus suffices (since $P_k$ is an $S$-scheme) to show that composing the diagram with the canonical injection $U' \to U'_k$ (or, equivalently, thanks to the isomorphism from $U'$ to $U$, with $\psi$) gives us a commutative diagram (I, 9.5.6). But, by definition, the diagram thus obtained is exactly (5.6.1.1), whence our claim.

The $W_k$ thus form an open cover of $g(X')$; to show that $g$ is an immersion, it suffices to show that each of the restrictions $g|U''_k$ is an immersion into $W_k$ (I, 4.2.4). For this, consider the morphism $u_k : W_k \to V_k$ such that $X$ is separated over $S$, the graph morphism $\Gamma_{u_k} : W_k \to X \times_S W_k$ is a closed immersion (I, 5.4.3), and the graph $\Gamma_k = \Gamma_{u_k}(W_k)$ is a closed subscheme of $X \times_S W_k$; if we show that $U' \to X \times_S W_k$ factors through this subscheme, then the map from the subscheme induced by $X'$ on the open subset $X'_k$ of $X'$ to $X \times_S W_k$ will also factor through this graph, by (I, 9.5.8). Since the restriction of $q_2$ to $T_k$ is an isomorphism onto $W_k$, the restriction of $g$ to $X'_k$ will be an immersion into $W_k$, and our claim will be proven. Let $v_k$ be the canonical injection $U' \to X \times_S W_k$; we have to show that there exists a morphism $w_k : U' \to W_k$ such that $v_k = w_k \circ u_k$. By the definition of the product, it suffices to prove that $q_1 \circ v_k = u_k \circ q_2 \circ w_k$ (I, 3.2.1), or, by composing on the right with the isomorphism $\psi' : U \to U'$, that $q_1 \circ \psi = u_k \circ q_2 \circ \psi$. But since $q_1 \circ \psi = j$ and $q_2 \circ \phi = \phi$, our claim follows from the commutativity of (5.6.1.1), taking into account the definition of $u_k$.

(E) It is clear that since $U$, and thus $U'$, is irreducible, so is the $X'$ from the preceding construction, and the morphism $f$ is thus birational (I, 2.2.9). If in addition $X$ is reduced, then so is $U'$, and hence $X'$ is also reduced (I, 9.5.9). This finishes the proof.

\[\square\]

**Corollary (5.6.2).** — Suppose that one of the hypotheses, (a) and (b), of (5.6.1) is satisfied. For $X$ to be proper over $S$, it is necessary and sufficient for there to exist a projective scheme $X'$ over $S$, and a surjective $S$–morphism $f : X' \to X$ (which is thus projective, by (5.5.5, v)). Whenever this is the case, we can further choose $f$ to be such that there exists a dense open subset $U$ of $X$ for which the restriction of $f$ to $f^{-1}(U)$ is an isomorphism $f^{-1}(U) \cong U$, and for which $f^{-1}(U)$ is dense in $X'$. If in addition $X$ is irreducible (resp. reduced), then we can assume that $X'$ is also irreducible (resp. reduced); when $X$ and $X'$ are irreducible, $f$ is a birational morphism.

**Proof.** The condition is sufficient, by (5.5.3) and (5.4.3, ii). It is necessary because, with the notation of (5.6.1), if $X$ is proper over $S$, then $X'$ is proper over $S$, because it is projective over $X$, and thus proper over $X$ (5.5.3), and our claim follows from (5.4.2, ii); furthermore, since $X'$ is quasi-projective over $S$, it is projective over $S$ by (5.5.3).

**Corollary (5.6.3).** — Let $S$ be a locally Noetherian prescheme, and $X$ an $S$-scheme of finite type over $S$, with structure morphism $f_0 : X \to S$. For $X$ to be proper over $S$, it is necessary and sufficient that, for every morphism of finite type $S' \to S$, $(f_0)_{S'} : X(S') \to S'$ be a closed morphism. It even suffices for this condition to be verified only for every $S$-prescheme of the form $S' = S \otimes_S Z[T_1, \ldots, T_n]$ (where the $T_i$ are indeterminates).

**Proof.** The condition being clearly necessary, we now show that it is sufficient. Since the question is local on $S$ and $S'$ (5.4.1), we can suppose that $S$ and $S'$ are affine and Noetherian. By Chow’s lemma, there exists a projective $S$-scheme $P$, an immersion $j : X' \to P$, and a surjective projective morphism $f : X' \to X$, such that the diagram

$$
\begin{array}{ccc}
X & \xleftarrow{f} & X' \\
\downarrow{f_0} & & \downarrow{j} \\
S & \xleftarrow{r} & P
\end{array}
$$

commutes. Since $P$ is of finite type over $S$, the first hypothesis implies that the projection $q_2 : X \times_S P \to P$ is a closed morphism. But the immersion $j$ is the composition of $q_2$ and the morphism
7. VALUATIVE CRITERIA

7.1. Reminder on valuation rings

(7.1.1). Amongst the many diverse equivalent properties that characterise valuation rings, we will use the following: a ring \( A \) is said to be a valuation ring if it is an integral ring which is not a field, and \( A \) is maximal in the set of local rings strictly contained in the field of fractions \( K \) of \( A \) under the domination relation (I, 8.1.1). Recall that a valuation ring is integrally closed. If \( A \) is a valuation ring, then so too is \( A_p \) for any prime ideal \( p \neq 0 \) of \( A \).

(7.1.2). Let \( K \) be a field, and \( A \) a local subring of \( K \) that is not a field; then there exists a valuation ring that both dominates \( A \) and has \( K \) as its field of fractions ([CC, p. 1-07, lemma 2]).

Now let \( B \) be a valuation ring, \( k \) its residue field, \( K \) its field of fractions, and \( L \) an extension of \( k \). Then there exists a complete valuation ring \( C \) that dominates \( B \) and whose residue field is \( L \). Indeed, \( L \) is the algebraic extension of a pure transcendental extension \( k' = (T_{\mu})_{\mu \in M} \); we know that we can extend the valuation of \( B \) corresponding to \( k' \) to a valuation of \( K' = K(T_{\mu})_{\mu \in M} \) in such a way that \( L' \) is the residue field of this valuation ([Jaf60, p. 98]); replacing \( B \) by the completion of the ring of this extended valuation, we see that we can restrict to the case where \( B \) is complete and \( L \) is an algebraic closure of \( k \). If \( \overline{K} \) is an algebraic closure of \( K \), we can then extend the valuation that defines \( B \) to \( \overline{K} \), and the corresponding residue field is an algebraic closure of \( k \), as we can see by lifting to \( \overline{K} \) the coefficients of a unitary polynomial of \( k[T] \). We are thus finally led to the case where \( L = k \), and it then suffices to take \( C \) to be the completion of \( B \) in order to satisfy our claim.

(7.1.3). Let \( K \) be a field, and \( A \) a subring of \( K \); the integral closure \( A' \) of \( A \) in \( K \) is the intersection of the valuation rings that contain \( A \) and have \( K \) as their field of fractions ([Sam53b, p. 51, th. 2]). Proposition (7.1.2) can then be expressed geometrically in an equivalent form:

**Proposition (7.1.4).** — Let \( Y \) be a prescheme, \( p : X \to Y \) a morphism, \( x \) a point of \( X \), \( y = p(x) \), and \( y' \neq y \) a specialisation (0, 2.1.2) of \( y \). Then there exists a local scheme \( Y' \) which is the spectrum of some valuation ring, and a separated morphism \( f : Y' \to Y \) such that, denoting the unique closed point of \( Y' \) by a
and the generic point of \( Y' \) by \( b \), we have \( f(a) = y' \) and \( f(b) = y \). We can furthermore suppose that one of the two additional following properties are satisfied:

(i) \( Y' \) is the spectrum of a complete valuation ring whose residue field is algebraically closed, and there exists a \( k(y) \)-homomorphism \( k(x) \to k(b) \).

(ii) There exists a \( k(y) \)-isomorphism \( k(x) \xrightarrow{\sim} k(b) \).

**Proof.** Let \( Y_1 \) be the reduced closed subscheme of \( Y \) that has \( \{y\} \) as its underlying space (I, 5.2.1), and let \( X_1 \) be the closed subscheme given by the inverse image \( p^{-1}(Y_1) \); since \( y' \in \{y\} \) by hypothesis, and since \( k(x) \) is the same in \( X \) and \( X_1 \), we can assume that \( Y \) is integral, with generic point \( y \); \( \mathcal{O}_{Y'} = k(y) \) is then an integral local ring that is not a field, and whose field of fractions is \( \mathcal{O}_y = k(y) \), and \( k(x) \) is then an extension of \( k(y) \). To satisfy the conditions \( f(a) = y' \) and \( f(b) = y \) as well as the additional condition (i) (resp. (ii)), we take \( Y' = \text{Spec}(A') \), where \( A' \) is a valuation ring that dominates \( \mathcal{O}_{Y'} \) (resp. a valuation ring that dominates \( \mathcal{O}_y \) and whose field of fractions is \( k(x) \)); the existence such an of \( A' \) is guaranteed by (7.1.2). \( \square \)

(7.1.5). Recall that a local ring \( A \) is said to be of dimension 1 if there exists a prime ideal distinct from the maximal ideal \( m \), and if every prime ideal of \( A \) distinct from \( m \) is a minimal prime ideal; when \( A \) is integral, it is equivalent to ask that \( m \) and \( (0) \) be the only prime ideals, with \( m \neq (0) \); in other words, \( Y = \text{Spec}(A) \) consists of two points \( a \) and \( b : a \) is the unique closed point, we have \( I_a = m \), and \( k(a) = k \) is the residue field \( k = A/m; b \) is the generic point of \( Y \), \( I_b = (0) \), with the set \( \{b\} \) being the unique open subset of \( Y \) distinct from both \( \emptyset \) and \( Y \) (an open subset which is thus everywhere dense), and \( k(b) = K \) is the field of fractions of \( A \).

(7.1.6). For a local ring \( A \), Noetherian and of dimension 1, we know ([CC, pp. 2-08 and 17-01]) that the following conditions are equivalent:

(a) \( A \) is normal;

(b) \( A \) is regular;

(c) \( A \) is a valuation ring;

furthermore, \( A \) is then a discrete valuation ring. Propositions (7.1.2) and (7.1.3) then have the following analogues for discrete valuation rings:

**Proposition (7.1.7).** — Let \( A \) be an integral local Noetherian ring that is not a field, \( K \) its field of fractions, and \( L \) an extension of finite type of \( K \); then there exists a discrete valuation ring that dominates \( A \) and \( L \) as its field of fractions.

**Proof.** Suppose first of all that \( L = K \). Let \( m \) be the maximal ideal of \( A \), \( (x_1, \ldots, x_n) \) a system of non-null generators of \( m \), and \( B \) the subring \( A[x_2/x_1, \ldots, x_n/x_1] \) of \( K \), which is Noetherian. It is immediate that the ideal \( mB \) of \( B \) is identical to the principal ideal \( x_1B \); if \( p \) is a minimal prime ideal of \( x_1B \), then \( p \) is of rank 1 ([SZ60, t. I, p. 277]); in other words, \( B_p \) is a local Noetherian ring of dimension 1; it is clear that \( pB_p \cap A \) is an ideal of \( A \) that contains \( m \) and that does not contain 1, and is thus equal to \( m \), and so \( B_p \) dominates \( A \) (I, 8.1.1). It follows from the Krull-Akizuki Theorem ([Nag55, p. 293]) that the integral closure \( C \) of \( B_p \) is a Noetherian ring (even though \( C \) is not necessarily a \( B_p \)-module of finite type); if \( n \) is a maximal ideal of \( C \), then \( C_n \) is a normal local Noetherian ring of dimension 1 ([Nag55, p. 295]), and thus a discrete valuation ring that dominates \( B_p \) and is a fortiori \( A \).

Now, if \( L \) is an extension of finite type of \( K \), we can, by the above, restrict to the case where \( A \) is already a discrete valuation ring. Let \( w \) be a valuation of \( K \) associated to \( A \); there exists a discrete valuation \( w' \) of \( L \) that extends \( w \); we can restrict, by induction on the number of generators of \( L \), to the case where \( L = K(a) \), and then the proposition is classical ([Jaf60, p. 106]). \( \square \)

**Corollary (7.1.8).** — Let \( A \) be a Noetherian integral ring, \( K \) its field of fractions, and \( L \) an extension of finite type of \( K \). Then the integral closure of \( A \) in \( L \) is the intersection of the discrete valuation rings that have \( L \) as their field of fractions and that contain \( A \).

**Proof.** Indeed, such a discrete valuation ring, being normal, contains a fortiori every element of \( L \) that is integral over \( A \). It thus suffices to prove that, if \( x \in L \) is not integral over \( A \), then there exists a discrete valuation ring \( C \) that has \( L \) as its field of fractions, contains \( A \), and does not contain \( x \). The hypothesis on \( x \) implies that \( x \notin B = A[1/x] \), or, in other words, that \( 1/x \) is not invertible in
the Noetherian ring \( B \). There is thus a prime ideal \( p \) of \( B \) that contains \( 1/x \). The integral local ring \( B_p \) is Noetherian and contained in \( L \), which is an extension of finite type of the field of fractions of \( B_p \) (with the latter containing \( K \)). By (7.1.7), there thus exists a discrete valuation ring \( C \) that dominates \( B_p \) and has \( L \) as its field of fractions; since \( 1/x \in pB_p \) belongs to the maximal ideal of \( C \), we have that \( x \not\in C \), which concludes the proof. \( \square \)

The geometric form of (7.1.7) is the following:

\[
\begin{array}{c}
\text{X} \\
\downarrow \\
\text{Y}
\end{array}
\]

\[
\begin{array}{c}
k(x) \\
\downarrow \\
k(b)
\end{array}
\]

\[
\begin{array}{c}
k(y)
\end{array}
\]

(7.1.9.1)

\( \gamma \)

\( \pi \)

\( \varphi \)

(Proposition 7.1.9). — Let \( Y \) be a locally Noetherian prescheme, \( p : X \to Y \) a morphism locally of finite type, \( x \) a point of \( X \), \( y = p(x) \), and \( y' \neq y \) a specialisation of \( y \). Then there exists a local scheme \( Y' \), spectrum of a discrete valuation ring, a separated morphism \( f : Y' \to Y \), and a rational \( Y \)-map \( g \) from \( Y' \) to \( Y \), such that, denoting the closed point of \( Y' \) by \( a \), and the generic point of \( Y' \) by \( b \), we have \( f(a) = y' \), \( f(b) = y \), \( g(b) = x \), and such that, in the commutative diagram

\[
\begin{array}{c}
k(x) \\
\downarrow \\
k(b)
\end{array}
\]

\[
\begin{array}{c}
k(y)
\end{array}
\]

(where \( \pi \), \( \varphi \), and \( \gamma \) are the homomorphisms corresponding to \( p \), \( f \), and \( g \), respectively) the morphism \( \gamma \) is a bijection.

**Proof.** As in (7.1.4), we can restrict to the case where \( Y \) is integral with generic point \( y \) (taking (I, 6.4.3, iv) into account), and, since the question is local on \( X \) and \( Y \), we can assume that \( p \) is of finite type; we are then in the situation of (7.1.4), with the additional property that \( k(x) \) is an extension of finite type of \( k(y) \) (I, 6.4.11) and that \( \mathcal{O}_{y'} \) is Noetherian; this lets us apply (7.1.7) and take \( Y' = \text{Spec}(A') \), where \( A' \) is a discrete valuation ring that dominates \( \mathcal{O}_{y'} \) and whose field of fractions is \( k(x) \). We have thus defined a commutative diagram (7.1.9.1) where \( \gamma \) is a bijection, with \( \pi \) and \( \varphi \) corresponding to the morphisms \( p \) and \( f \). Furthermore, since \( X \) and \( Y \) are locally Noetherian (I, 6.6.2) and since \( Y' \) is integral, there exists exactly one rational \( Y \)-map \( g \) from \( Y' \) to \( X \) to which corresponds the isomorphism \( \gamma \) (I, 7.1.15), which finishes the proof. \( \square \)

### 7.2. Valuative criterion for separatedness

**Proposition (7.2.1).** — Let \( X \) and \( Y \) be preschemes, and \( f : X \to Y \) a quasi-compact morphism. The following two conditions are equivalent:

(a) The morphism \( f \) is closed.

(b) For all \( x \in X \), and every specialisation \( y' \neq y = f(x) \), there exists a specialisation \( x' \) of \( x \) such that \( f(x') = x \).

**Proof.** Condition (b) implies that \( f(\{x\}) = \{y\} \), and is thus a consequence of (a). To show that (b) implies (a), consider a closed subset \( X' \) of the underlying space \( X \); let \( Y' = \overline{f(X')} \), and show that \( Y' = f(X') \) as follows. Consider the closed reduced subschemes of \( X \) and \( Y \) whose underlying spaces are \( X' \) and \( Y' \) (respectively) (I, 5.2.1); there then exists a morphism \( f' : X' \to Y' \) such that the diagram

\[
\begin{array}{c}
X' \\
\downarrow \\
X
\end{array}
\]

\[
\begin{array}{c}
f
\end{array}
\]

\[
\begin{array}{c}
Y \\
\downarrow \\
Y'
\end{array}
\]

commutes (I, 5.2.2), and, since \( f \) is quasi-compact, so too is \( f' \). We are thus led to proving that, if \( f \) is a quasi-compact and dominant morphism, then condition (b) implies that \( f(X) = Y \). But let \( y' \) be a point of \( Y \), and let \( y \) be the generic point of an irreducible component of \( Y \) that contains \( y' \); by (b), it suffices to show that \( f^{-1}(y) \) is not empty. But we know that this property is a consequence of the fact that \( f \) is quasi-compact and dominant (I, 6.6.5). \( \square \)
Corollary (7.2.2). — Let \( f : X \to Y \) be a quasi-compact immersion. For the underlying space \( X \) to be closed in \( Y \), it is necessary and sufficient for it to contain every specialisation (in \( Y \)) of all of its points.

Proposition (7.2.3). — Let \( Y \) be a prescheme (resp. a locally Noetherian prescheme), and \( f : X \to Y \) a morphism (resp. a morphism locally of finite type). The following conditions are equivalent:

(a) \( f \) is separated.
(b) The diagonal morphism \( X \to X \times_Y X \) is quasi-compact, and, for every \( Y \)-prescheme of the form \( Y' = \text{Spec}(A) \), with \( A \) a valuation ring (resp. some discrete valuation ring), any two \( Y \)-morphisms from \( Y' \) to \( X \) that agree on the generic point of \( Y' \) are equal.
(c) The diagonal morphism \( X \to X \times_Y X \) is quasi-compact, and, for every \( Y \)-prescheme of the form \( Y' = \text{Spec}(A) \), with \( A \) a valuation ring (resp. some discrete valuation ring), any two \( Y' \)-sections of \( X' = X_{(Y')} \) that agree on the generic point of \( Y' \) are equal.

Proof. The equivalence of (b) and (c) follows from the bijective correspondence between \( Y \)-morphisms from \( Y' \) to \( X \) and \( Y' \)-sections of \( X' \) (I, 3.3.14). If \( X \) is separated over \( Y \), condition (b) is satisfied, by (I, 7.2.2.1), since \( Y' \) is integral. It remains to show that (b) implies that the diagonal morphism \( \Delta : X \to X \times_Y X \) is closed, and it is equivalent to show that it satisfies the criteria of (7.2.2). But let \( z \) be a point of the diagonal \( \Delta(X) \), and \( z' \neq z \) a specialisation of \( z \) in \( X \times_Y X \). There then exists (7.1.4) a valuation ring \( A \) and a morphism \( f \) from \( Y' = \text{Spec}(A) \) to \( X \times_Y X \) such that \( f \) sends the closed point \( a \) of \( Y' \) to \( z' \), and the generic point \( b \) of \( Y' \) to \( z \); this morphism makes \( Y' \) an \( (X \times_Y X) \)-prescheme, and \( a \) fortiori a \( Y \)-prescheme. If we compose the two projections of \( X \times_Y X \) with \( f \), then we obtain two \( Y \)-morphisms, \( g_1 \) and \( g_2 \), from \( Y' \) to \( X \), which, by hypothesis, agree on the point \( b \); they are thus equal to one single morphism \( g \), which implies (I, 5.3.1) that \( f \) factors as \( f = \Delta \circ g \), and thus \( z' \in \Delta(X) \). If we suppose that \( Y \) is locally Noetherian and \( f \) is locally of finite type, then \( X \times_Y X \) is locally Noetherian (I, 6.6.7); we can thus follow the same argument as before by supposing that \( A \) is a discrete valuation ring, by (7.1.9).

Remark (7.2.4). — (i) The hypothesis that the morphism \( \Delta \) is quasi-compact is always satisfied whenever \( Y \) is locally Noetherian and \( f \) is locally of finite type, because \( X \times_Y X \) is then locally Noetherian (I, 6.6.4, i). In the general case, this also implies that, for every cover \( (U_a) \) of \( X \) by affine opens, the sets \( U_a \cap U_b \) are quasi-compact.

(ii) For \( f \) to be separated, it is sufficient for condition (b) or (c) to be satisfied for some valuation ring \( A \) that is complete and whose residue field is algebraically closed; this follows from the proofs of (7.2.3) and (7.2.4).

7.3. Valuative criterion for properness

Proposition (7.3.1). — Let \( A \) be a valuation ring, \( Y = \text{Spec}(A) \), \( b \) the generic point of \( Y \), \( X \) an integral scheme, and \( f : X \to Y \) a closed morphism such that \( f^{-1}(b) \) consists of a single point \( x \) and such that the corresponding homomorphism \( k(b) \to k(x) \) is bijective. Then \( f \) is an isomorphism.

Proof. Since \( f \) is closed and dominant, we have that \( f(X) = Y \); it suffices (I, 4.2.2) to prove that, for all \( y' \neq b \) in \( Y \), there exists exactly one point \( x' \) such that \( f(x') = y' \), and that the corresponding homomorphism \( \mathcal{O}_{y'} \to \mathcal{O}_{x'} \) is bijective, since then \( f \) will be a homeomorphism. But if \( f(x') = y' \) then \( \mathcal{O}_{x'} \) is a local ring contained in \( K = k(x) = k(b) \) and dominating \( \mathcal{O}_{y'} \); the latter is the local ring \( A_{y'} \), and is thus a valuation ring (7.1.1) that has \( K \) as its field of fractions. Also, \( \mathcal{O}_{y'} \neq K \), since \( x' \) is not the generic point of \( X \) (0, 2.1.3); we thus conclude that \( \mathcal{O}_{x'} = \mathcal{O}_{y'} \). Since \( X \) is an integral scheme, the fact that \( \mathcal{O}_{x'} = \mathcal{O}_{x''} \) implies that \( x' = x'' \) (I, 8.2.2), which finishes the proof.

(7.3.2). Let \( A \) be a valuation ring, \( K \) its field of fractions, \( Y = \text{Spec}(A) \), and \( b \) the generic point of \( Y \), such that \( \mathcal{O}_b = k(b) \) is equal to \( K \); let \( f : X \to Y \) be a morphism. We know (I, 7.1.4) that the rational \( Y \)-sections of \( X \) are in bijective correspondence with the germs of \( Y \)-sections (defined in a neighbourhood of \( b \)) at the point \( b \), whence we have a canonical map

\[
\Gamma_{\text{rat}}(X/Y) \to \Gamma(f^{-1}(b)/\text{Spec}(K))
\]
with the elements of $\Gamma(f^{-1}(b) / \text{Spec}(K))$ being identified, by definition (I, 3.4.5), with the points of $f^{-1}(b) = X \otimes_A K$ that are rational over $K$. When $f$ is separated, it follows from (I, 5.4.7) that the map (7.3.2.1) is injective, since $Y$ is an integral scheme.

Composing (7.3.2.1) with the canonical map $\Gamma(X/Y) \to \Gamma_{\text{rat}}(X/Y)$ (I, 7.1.2), we obtain a canonical map

$$\Gamma(X/Y) \to \Gamma(f^{-1}(b) / \text{Spec}(K)).$$

When $f$ is separated, this map is again injective (I, 5.4.7).

**Proposition (7.3.3).** — Let $A$ be a valuation ring with field of fractions $K$, $Y = \text{Spec}(A)$, $b$ the generic point of $Y$, and $f : X \to Y$ a separated and closed morphism. Then the canonical map (7.3.2.2) is bijective (which is equivalent to saying that it is surjective, and implies that the rational $Y$-sections of $X$ are everywhere defined).

**Proof.** So let $x$ be a point of $f^{-1}(b)$ that is rational over $K$. Since $f$ is separated, so too is the morphism $f^{-1}(b) \to \text{Spec}(K)$ corresponding to $f$ (I, 5.5.1, iv), and, since every section of $f^{-1}(b)$ is a closed immersion (I, 5.4.6), $\{x\}$ is closed in $f^{-1}(b)$. Consider the reduced closed subscheme $X'$ of $X$ that has the closure $\{x\}$ of $\{x\}$ in $X$ as its underlying space. It is clear that the restriction of $f$ to $X'$ satisfies the hypotheses of (7.3.1), and is thus an isomorphism from $X'$ to $Y$, whose inverse isomorphism is the desired $Y$-section of $X$. □

**Corollary (7.3.5).** — Let $Y$ be a locally Noetherian reduced prescheme, and $N$ the set of points $y \in Y$ where $Y$ is not regular (0, 4.1.4); suppose that codim$_Y N \geq 2$. Let $f : X \to Y$ be a morphism of finite type, both separated and closed, and let $g$ be a rational $Y$-section of $X$; if $Y'$ is the set of points of $Y$ where $g$ is not defined (a set which is closed (I, 7.2.1)), then codim$_Y Y' \geq 2$.

**Proof.** It suffices to prove that $g$ is defined at every point $z \in Y$ such that dim $\mathcal{O}_z \leq 1$. If dim $\mathcal{O}_z = 0$, then $z$ is the generic point of an irreducible component of $Y$ (I, 1.1.14), and so belongs to every everywhere-dense open subset of $Y$, and, in particular, to the domain of definition of $g$. So suppose that dim $\mathcal{O}_z = 1$; by hypothesis, $\mathcal{O}_z$ is then a regular Noetherian local ring, and thus (7.1.6) a discrete valuation ring. Let $Z = \text{Spec}(\mathcal{O}_z)$; since $U = Y - Y'$ is everywhere dense, $U \cap Z$ is nonempty (I, 2.4.2); let $g'$ be the rational map from $Z$ to $X$ induced by $g$ (I, 7.2.8); it suffices to show that $g'$ is a morphism (I, 7.2.9). But $g'$ can be thought of as a rational $Z$-section of the $Z$-prescheme $f^{-1}(Z) = X \times_Y Z$; it is clear that the morphism $f^{-1}(Z) \to Z$ corresponding to $f$ is closed, and it follows from (I, 5.5.1, i) that it is separated; we thus conclude from (7.3.3) that $g'$ is everywhere defined; since $Z$ is reduced, and $X$ is separated over $Y$, $g'$ is a morphism (I, 7.2.2). □

**Corollary (7.3.6).** — Let $S$ be a locally Noetherian prescheme, and $X$ and $Y$ both $S$-preschemes; suppose that $Y$ is reduced, and further that the set $N$ of points $y \in Y$ where $Y$ is not regular is such that codim$_Y N \geq 2$; suppose finally that the structure morphism $X \to S$ is proper. Let $f$ be a rational $S$-map from $Y$ to $X$, and let $Y'$ be the points of $Y$ where $f$ is not defined; then codim$_Y Y' \geq 2$.

**Proof.** We know (I, 7.1.2) that we can identify the rational $S$-maps from $Y$ to $X$ with the rational $Y$-sections of $X \times_S Y$; since the structure morphism $X \times_S Y \to Y$ is closed (5.4.1), we can apply (7.3.5), whence the corollary. □

**Remark (7.3.7).** — The hypotheses on $Y$ in (7.3.5) and (7.3.6) will be satisfied in particular when $Y$ is normal (0, 4.1.4), by (7.1.6).

We can characterise the universally closed morphisms (resp. proper morphisms) by a converse of (7.3.3):

**Theorem (7.3.8).** — Let $Y$ be a prescheme (resp. a locally Noetherian prescheme), and $f : X \to Y$ a quasi-compact separated morphism (resp. a morphism of finite type). The following conditions are equivalent:

(a) $f$ is universally closed (resp. proper).
(b) For every $Y$-scheme of the form $Y' = \text{Spec}(A)$, where $A$ is a valuation ring (resp. a discrete valuation ring) with field of fractions $K$, the canonical map

$$\text{Hom}_Y(Y', X) \longrightarrow \text{Hom}_Y(\text{Spec}(K), X)$$

corresponding to the canonical injection $A \rightarrow K$ is surjective (resp. bijective).

(c) For every $Y$-scheme of the form $Y' = \text{Spec}(A)$, where $A$ is a valuation ring (resp. a discrete valuation ring), the canonical map (7.3.2.2) relative to the $Y'$-prescheme $X_{(Y)}$ is surjective (resp. bijective).

**Proof.** The equivalence of (b) and (c) follows immediately from (I, 3.3.14); (a) implies (b), since (a) implies, in either case, that $f(Y')$ is separated (I, 5.5.1, iv) and closed, and it suffices to apply (7.3.3). It remains to prove that (b) implies (a). We first consider the case where $Y$ is arbitrary, and $f$ is separated and quasi-compact. If condition (b) is satisfied by $f$, then it is also satisfied by $f(Y'_{(y)}) : X_{(Y')} \rightarrow Y''$, where $Y''$ is an arbitrary $Y$-prescheme, thanks to the equivalence between (b) and (c), and the fact that $X_{(Y')} \times_{Y''} Y' = X \times_Y Y'$ for every morphism $Y' \rightarrow Y''$ (I, 3.3.9.1); since, further, $f(Y')$ is separated and quasi-compact whenever $f$ is ((I, 5.5.1, iv) and (I, 6.6.4, iii)), we are led to proving that (b) implies that $f$ is closed. For this, it suffices to verify condition (b) of (7.2.1). So let $x \in X$, and $y'$ be a specialisation of $y = f(x)$, distinct from $y$; by (7.1.4), there exists a scheme $Y'$, the spectrum of some valuation ring, and a separated morphism $g : Y' \rightarrow Y$ such that, letting $a$ denote the closed point and $b$ the generic point of $Y'$, we have that $g(a) = y'$, $g(b) = y$, and that there exists a $k(y')$-homomorphism $k(x) \rightarrow k(b)$. The latter corresponds canonically to a $Y$-morphism $\text{Spec}(k(b)) \rightarrow X$ (I, 2.4.6), and it thus follows from (b) that there exists a $Y$-morphism $h : Y' \rightarrow X$ to which the previous morphism corresponds. We then have that $h(b) = x$; if we set $h(a) = x'$, then $x'$ is a specialisation of $x$, and we have that $f(x') = f(h(a)) = g(a) = y'$.

If now $Y$ is locally Noetherian and $f$ of finite type, then hypothesis (b) implies, first of all, that $f$ is separated, by (7.2.3), with the diagonal morphism $X \rightarrow X \times_Y X$ being quasi-compact (7.2.4). Further, to show that $f$ is proper, it suffices to show that $f(Y'_{(y)}) : X_{(Y')} \rightarrow Y''$ is closed for every $Y$-prescheme $Y''$ of finite type, taking (5.6.3) into account. Since $Y''$ is then locally Noetherian, we can follow the same reasoning as in the first case by taking $Y'$ to be the spectrum of a discrete valuation ring, and applying (7.1.9) instead of (7.1.4).

**Remarks (7.3.9).**

(i) Whenever $Y$ is an arbitrary prescheme and $f$ a separated morphism, for $f$ to be universally closed, it suffices that condition (b) or (c) be satisfied for the complete valuation rings $A$ whose residue field is algebraically closed; this follows from the above proof and from (7.1.4).

(ii) From criterion (c) of (7.3.8) we obtain a new proof of the fact that a projective morphism $X \rightarrow Y$ is closed (5.5.3), and it is closer to the classical approach. We can indeed assume that $Y$ is affine, and thus that $X$ can be identified with a closed subscheme of a projective bundle $P^n_Y$ (5.3.3); to prove that $X \rightarrow Y$ is closed, it suffices to verify that the structure morphism $P^n_Y \rightarrow Y$ is closed, and criteria (c) of (7.3.8), combined with (4.1.3.1), tells us that we can reduce to proving the following fact: if $Y$ is the spectrum of a valuation ring $A$, with field of fractions $K$, then every point of $P^n_Y$ with values in $K$ comes from (by restriction to the generic point of $Y$) a point of $P^n_A$ with values in $A$. But every invertible $O_Y$-module is trivial (I, 2.4.8); so it follows from (4.2.6) that a point of $P^n_A$ with values in $K$ can be identified with a class of elements $(\zeta c_0, \zeta c_1, \ldots, \zeta c_n)$ of $K$, where $\zeta \neq 0$ and the $c$ are elements of $K$ that are not all zero. However, by multiplying the $c_i$ by an element of $A$ of suitable valuation, we can suppose that the $c_i$ all belong to $A$, and that at least one of them is invertible. But then (4.2.6) the system $(c_0, \ldots, c_n)$ also defines a point of $P^n_A$ with values in $A$, which proves our claim.

(iii) Criteria (7.2.3) and (7.3.8) are particularly simple when we consider the data of a $Y$-prescheme $X$ as being equivalent to the data of the functor

$$X(Y') = \text{Hom}_Y(Y', X)$$

for $Y$-preschemes $Y'$; these criteria allow us, for example, to prove that, under certain conditions, the “Picard schemes” are proper.
Corollary (7.3.10). — Let $Y$ be an integral scheme (resp. a locally Noetherian integral scheme), $X$ an integral scheme, and $f : X \to Y$ a dominant morphism.

(i) If $f$ is quasi-compact and universally closed, then every valuation ring whose field of fractions is the field $R(X)$ of rational functions on $X$, and which is dominated by a local ring $Y$, also dominates by a local ring of $X$.

(ii) Conversely, suppose that $f$ is of finite type, and that the property described in (i) is verified by every valuation ring (resp. every discrete valuation ring) that has $R(X)$ as its field of fractions. Then $f$ is proper.

Proof. Note first that all the hypotheses imply, in any case, that $f$ is separated (I, 5.5.9).

(i) Let $K = R(Y)$, $L = R(X)$, $y$ a point of $Y$, and $A$ a valuation ring that dominates $O_y$ and has $L$ as its field of fractions; the injection $O_y \to A$ then defines a morphism $h$ from $Y' = \text{Spec}(A)$ to $Y$ (I, 2.4.4) such that $h(a) = y$, where we write $a$ to denote the closed point of $Y'$; furthermore, if $\eta$ is the generic point of $Y$, which is also the generic point of $\text{Spec}(O_y)$, then we have $h(b) = \eta$, writing $b$ to denote the generic point of $Y'$ (since $K \subseteq L$ by hypothesis). If $\xi$ is the generic point of $X$, then $k(\xi) = k(b) = L$ by hypothesis, whence we have a $Y$-morphism $g : \text{Spec}(L) \to X$ such that $g(b) = \xi$; by (7.3.8), $g$ comes from a $Y$-morphism $g' : Y' \to X$. If $x = g'(a)$, it is clear that $A$ dominates $O_X$.

(ii) Since the questions is local on $Y$, we can always suppose that $Y$ is affine (resp. affine and Noetherian). Since $f$ is of finite type, we can apply, in either case, Chow’s lemma (5.6.1). There is thus a projective morphism $p : P \to Y$, an immersion morphism $j : X' \to P$, and a projective morphism $g : X' \to X$ that is both surjective and birational, with $X$ integral, such that the diagram

\[
P \leftarrow X' \rightarrow Y \leftarrow X
\]

commutes. It suffices to prove that $j$ is a closed immersion, since then $f \circ g = p \circ j$ will be a projective morphism, and thus closed, and, since $g$ is surjective, $f$ will also be proper (5.4.3). Let $Z$ be the reduced closed subscheme of $P$ that has $\overline{j(X')}$ as its underlying space (I, 5.2.1); since $X'$ is integral, $j$ factors as $i \circ h$, where $i : Z \to P$ is the canonical injection, $h : X' \to Z$ a dominant open immersion (I, 5.2.3), and $Z$ is integral; furthermore, $Z$ is projective over $Y$, and we see that we can restrict to the case where $P$ is integral and $j$ is dominant and birational, and everything then reduces to showing that $j$ is surjective. But let $z \in P$; then $O_z$ is an integral (resp. integral and Noetherian) local ring whose field of fractions is

$L = R(P) = R(X') = R(X)$.

We can restrict to the case where $z$ is not the generic point of $P$. There then exists ((7.1.2) and (7.1.7)) a valuation ring (resp. a discrete valuation ring) $A$ which dominates $O_z$ and has $L$ as its field of fractions. A fortiori, $A$ dominates $O_y$, where $y = p(z)$, and, by hypothesis, there thus exists some $x \in X$ such that $A$ dominates $O_x$. Since $g$ is proper, the first part of the proof shows that $A$ also dominates $O_{x'}$ for some $x' \in X'$; it then follows that $O_{x'}$ and $O_{j(x')} = O_{x'}$ are allied (I, 8.1.4), and, since $P$ is a scheme, this implies that $z = j(x')$ (I, 8.2.2) and finishes the proof.

Corollary (7.3.11). — Let $X$ and $Y$ be integral schemes, and $f : X \to Y$ a dominant, quasi-compact, and universally closed morphism. Suppose further that $Y$ is affine of (integral) ring $B$. Then $\Gamma(X, O_X)$ is canonically isomorphic to a subring of the integral closure of $B$ in $R(X)$.

Proof. Indeed (I, 8.2.1.1), $\Gamma(X, O_X)$ can be identified with the intersection of the $O_x$ over $x \in X$; by (7.3.10), (7.1.2), and (7.1.3), $\Gamma(X, O_X)$ is then contained in the intersection of the valuation rings that contain $B$ and that have $R(X)$ as their field of fractions; the conclusion then follows from (7.1.3).
Remarks (7.3.12). — Under the hypotheses of (7.3.11), and when we suppose that \( R(X) \) is an extension of finite type of \( R(Y) \), we can, in many cases, conclude that \( \Gamma(X, \mathcal{O}_X) \) is a module of finite type over the ring \( B = \Gamma(Y, \mathcal{O}_Y) \). For example, this will be the case whenever \( B \) is an algebra of finite type over a field, since then we already know that the integral closure of \( B \) in an extension of finite type of its field of fractions is a \( B \)-module of finite type ([SZ60, t. I, p. 267, th. 9]); the conclusion then follows from (7.3.11) and the fact that \( B \) is Noetherian.

In particular, a proper affine scheme \( X \) over a field \( K \) is finite. Indeed, by (1.6.4), (5.4.6), and (I, 6.4.4, (c)), we can restrict to the case where \( X \) is reduced. Furthermore, it suffices to prove that each of the closed subschemes of \( X \) that have an irreducible component of \( X \) as their underlying space (of which there are finitely many) is finite over \( K \), which means (taking (5.4.5) into account) that we are finally reduced to the case where \( X \) is integral. But then the result follows from the above remarks.

In chapter III, we will again prove this above proposition by other methods, and as a consequence of more general results, by showing that, if \( f : X \to Y \) is proper and \( Y \) is locally Noetherian, \( f_*(\mathcal{F}) \) is coherent for any coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \) (III, 4.4.2).

Finally, note that criterion (7.3.10) is taken as the definition of proper morphisms in classical algebraic geometry. We only mention this here as a remark, since criterion (7.3.8) seems more manageable in all the applications with which we are familiar.

7.4. Algebraic curves and function fields of dimension 1

The aim of this section is to show how to formulate the classical notion of algebraic curves (as introduced, by example, in the book of C. Chevalley [Che51]) in the language of schemes. All throughout this section, we write \( k \) to mean a field, all the schemes in question are \( k \)-schemes of finite type, and all the morphisms are \( k \)-morphisms.

Proposition (7.4.1). — Let \( X \) be a prescheme of finite type over \( k \) (and thus Noetherian); let \( x_i \) (\( 1 \leq i \leq n \)) be the generic points of the irreducible components \( X_i \) of \( X \), and let \( K_i = k(x_i) \) (\( 1 \leq i \leq n \)). Then the following conditions are equivalent:

(a) Each of the \( K_i \) is an extension of \( k \) with transcendence degree equal to 1.

(b) For every closed point \( x \) of \( X \), the local ring \( \mathcal{O}_x \) is of dimension 1 (7.1.5).

(c) The closed irreducible subsets of \( X \) that are distinct from the \( X_i \) are exactly the closed points of \( X \).

Proof. Since \( X \) is quasi-compact, every closed irreducible subset \( F \) of \( X \) contains a closed point (0, 2.1.3). By (1, 2.4.2), there is a bijective correspondence between the prime ideals of \( \mathcal{O}_x \) and the closed irreducible subsets of \( X \) that contain \( x \) (I, 1.1.14); the equivalence between (b) and (c) follows immediately from this. Now, if \( p_x \) (\( 1 \leq x \leq r \)) are the minimal prime ideals of the local Noetherian ring \( \mathcal{O}_x \), then the local rings \( \mathcal{O}_x/p_x \) are integral, and have the \( K_i \) such that \( x \in X_i \) as their fields of fractions. Furthermore, we know ([CC, p. 4-06, th. 2]) that the dimension of a local integral \( k \)-algebra of finite type is equal to the transcendence degree over \( k \) of its field of fractions. Finally, the dimension of \( \mathcal{O}_x \) is bounded above by the dimensions of the \( \mathcal{O}_x/p_x \); but condition (a) implies that these dimensions are equal to 1, and so (a) implies (b); conversely, if \( \mathcal{O}_x \) is of dimension 1, then none of the \( p_x \) can be equal to the maximal ideal of \( \mathcal{O}_x \), otherwise \( \mathcal{O}_x \) would be of dimension 0; thus each of the \( \mathcal{O}_x/p_x \) are of dimension 1, which shows that (b) implies (a).

We note that, under the conditions of (7.4.1), the set \( X \) is either empty or infinite, as an immediate result of (I, 6.4.4).

Definition (7.4.2). — We define an algebraic curve over \( k \) to be a non-empty algebraic scheme over \( k \) that satisfies the conditions of (7.4.1).

In the language of dimensions, which will be introduced in Chapter IV, this can be expressed by saying that an algebraic curve over \( k \) is a non-empty algebraic \( k \)-scheme whose irreducible components are all of dimension 1.

We note that, if \( X \) is an algebraic curve over \( k \), then the closed reduced subschemes \( X_i \) (\( 1 \leq i \leq n \)) of \( X \) that have the irreducible components of \( X \) as their underlying space are also algebraic curves over \( k \).
Corollary (7.4.3). — Let \( X \) be an irreducible algebraic curve. The only non-closed point of \( X \) is its generic point. The closed subsets of \( X \) that are distinct from \( X \) are the finite sets of closed points; these are also the only subsets of \( X \) that are not everywhere dense.

Proof. If a point \( x \in X \) is not closed, then its closure in \( X \) is an irreducible closed subset of \( X \), and thus necessarily the whole of \( X \), by (7.4.1), and thus \( x \) is the generic point of \( X \). A closed subset \( F \) of \( X \) that is distinct from \( X \) cannot contain the generic point of \( X \), and so all its points are closed (in \( X \), and \textit{a fortiori} in \( F \)); by considering the closed reduced subpreschemes of \( X \) that have \( F \) as their underlying space (I.5.2.1), it thus follows from (I.6.2.2) that \( F \) is finite and discrete. The closure in \( X \) of any infinite subset of \( X \) is thus necessarily equal to \( X \) itself.

If \( X \) is an arbitrary algebraic curve, by applying (7.4.3) to the irreducible components of \( X \), we see that the only non-closed points of \( X \) are the generic points of these components.

Corollary (7.4.4). — Let \( X \) and \( Y \) be irreducible algebraic curves over \( k \), and \( f : X \to Y \) a \( k \)-map. For \( f \) to be dominant, it is necessary and sufficient for \( f^{-1}(y) \) to be finite for all \( y \in Y \).

Proof. Indeed, if \( f \) is not dominant, then \( f(X) \) is necessarily a finite subset of \( Y \), by (7.4.3), and so it is not possible for \( f^{-1}(y) \) to be finite for every point of \( Y \), since otherwise \( X \) would be finite, which is a contradiction (7.4.1). Conversely, if \( f \) is dominant, then for any \( y \in Y \) distinct from the generic point \( \eta \) of \( Y \), we have that \( f^{-1}(y) \) is closed in \( X \), since \( \{ y \} \) is closed in \( Y \) (7.4.3); also, by hypothesis, \( f^{-1}(y) \) does not contain the generic point \( \xi \) of \( X \), and is thus finite, by (7.4.3). Finally, to see that, when \( f \) is dominant, \( f^{-1}(\eta) \) is finite, we note that the fibre \( f^{-1}(\eta) \) is an irreducible scheme of finite type over \( k(\eta) \), and with generic point \( \xi \) ((I.6.3.9) and (I.4.11)). Since \( k(\xi) \) and \( k(\eta) \) are extensions of finite type of \( k \), both of transcendence degree 1, we have that \( k(\xi) \) is necessarily an extension of finite degree of \( k(\eta) \), and so \( \xi \) is closed in \( f^{-1}(\eta) \) (I.6.4.2), and \( f^{-1}(\eta) \) thus consists of a single point \( \xi \).

We will see, in Chapter III, that, if \( f : X \to Y \) is a proper morphism of Noetherian preschemes such that \( f^{-1}(y) \) is finite for all \( y \in Y \), then \( f \) is necessarily finite; it will thus follow from (7.4.4) that a proper dominant morphism from an irreducible algebraic curve to an algebraic curve is finite.

Corollary (7.4.5). — Let \( X \) be an algebraic curve over \( k \). For \( X \) to be regular, it is necessary and sufficient for \( X \) to be normal, or for the local rings of its closed points to be discrete valuation rings.

Proof. This follows immediately from (7.4.1, (b)) and (7.1.6).

Corollary (7.4.6). — Let \( X \) be a reduced algebraic curve, and \( \mathcal{A} \) a reduced coherent \( \mathcal{A}(X) \)-algebra; then the integral closure \( X' \) of \( X \) relative to \( \mathcal{A} \) (6.3.4) is a normal algebraic curve, and the canonical morphism \( X' \to X \) is finite.

Proof. The fact that \( X' \to X \) is finite follows from (6.3.10); \( X' \) is thus an algebraic \( k \)-scheme; furthermore, if \( x_i \) (1 \( \leq \) \( i \) \( \leq \) \( n \)) are the generic points of the irreducible components of \( X \), and \( x_i' \) (1 \( \leq \) \( j \) \( \leq \) \( m \)) the generic points of the irreducible components of \( X' \), then each of the \( k(x_i') \) is a finite algebraic extension of one of the \( k(x_i) \) (6.3.6), and thus of transcendence degree 1 over \( k \). So \( X' \) is indeed an algebraic curve over \( k \), and, furthermore, we know that \( X' \) is a finite sum of normal integral schemes ((6.3.6) and (6.3.7)).

(7.4.7). We say that an algebraic curve \( X \) over \( k \) is complete if it is proper over \( k \).

Corollary (7.4.8). — For a reduced algebraic curve \( X \) over \( k \) to be complete, it is necessary and sufficient for its normalisation \( X' \) to be complete.

Proof. The canonical morphism \( f : X' \to X \) is finite (7.4.6), and thus proper (6.1.11) and surjective (6.3.8); if \( g : X \to \text{Spec}(k) \) is the structure morphism, then \( g \) and \( \circ f \) are both proper, by (5.4.2, (ii)) and (5.4.3, (ii)), since \( g \) is separated by hypothesis.

Proposition (7.4.9). — Let \( X \) be a normal algebraic curve over \( k \), and \( Y \) a proper algebraic \( k \)-scheme over \( k \). Then every rational \( k \)-map from \( X \) to \( Y \) is everywhere defined, or, in other words, is a morphism.
Proof. It follows from (7.3.7) that, at the points \( x \in X \) where such a map is not defined, the dimension of \( \partial x \) must be \( \geq 2 \), and so the set of such points is empty; the final claim follows from (I, 7.2.3). □

Corollary (7.4.10). — A normal algebraic curve over \( k \) is quasi-projective over \( k \).

Proof. Since \( X \) is a finite sum of normal integral algebraic curves (6.3.8), we can restrict to the case where \( X \) is integral (5.3.6). Since \( X \) is quasi-compact, it is covered by a finite number of affine open subsets \( U_i \) \((1 \leq i \leq n)\), and, since each of these \( U_i \) is of finite type over \( k \), for each \( i \) there exists some integer \( n_i \) along with a \( k \)-immersion \( f_i : U_i \rightarrow P^m_k \) (5.3.3) and (5.3.4, (ii)). Since \( U_i \) is dense in \( X \), it follows from (7.4.9) that \( f_i \) can be extended to a \( k \)-morphism \( g_i : X \rightarrow P^m_k \), whence we obtain a \( k \)-morphism \( g = (g_1, \ldots, g_n)_k \) from \( X \) to the product \( P \) of the \( P^m_k \) over \( k \). Furthermore, for each \( i \), since the restriction of \( g_i \) to \( U_i \) is an immersion, so too is the restriction of \( g \) to \( U_i \) (I, 5.3.14). Since the \( U_i \) cover \( X \), and since \( g \) is separated (I, 5.5.1, (v)), \( g \) is an immersion from \( X \) into \( P \) (I, 8.2.8). Since the Segre morphism (4.3.3) gives an immersion of \( P \) into \( P^N_k \), this proves that \( X \) is quasi-projective. □

Corollary (7.4.11). — Any normal algebraic curve \( X \) is isomorphic to the scheme induced by some complete normal algebraic curve \( \hat{X} \) on some everywhere dense open subset, and this \( \hat{X} \) is unique up to unique isomorphism.

Proof. If \( X_1 \) and \( X_2 \) are complete normal curves, then it follows from (7.4.9) that every isomorphism from any dense open \( U_1 \) in \( X_1 \) to any dense open \( U_2 \) in \( X_2 \) can be uniquely extended to an isomorphism from \( X_1 \) to \( X_2 \); whence the uniqueness claim. To prove the existence of \( \hat{X} \), it suffices to note that we can consider \( X \) as a subscheme of a projective bundle \( P^n_k \) (7.4.10). Let \( \overline{X} \) be the closure of \( X \) in \( P^n_k \) (I, 9.5.11); since \( X \) is induced by \( \overline{X} \) on a dense open subset of \( \overline{X} \) (I, 9.5.10), the generic points \( x_i \) of the irreducible components of \( X \) are also the generic points of the irreducible components of \( \overline{X} \), and the \( k(x_i) \) are the same for both of these schemes, and so (7.4.1) \( \overline{X} \) is an algebraic curve over \( k \) that is reduced (I, 9.5.9) and projective over \( k \) (5.5.1), whence complete (5.5.3). So we take for \( \hat{X} \) the normalisation of \( X \), which is again complete (7.4.8); furthermore, if \( h : \hat{X} \rightarrow \overline{X} \) is the canonical morphism, then the restriction of \( h \) to \( h^{-1}(X) \) is an isomorphism to \( X \), since \( X \) is normal (6.3.4), and since \( h^{-1}(X) \) contains the generic points of the irreducible components of \( \hat{X} \) (6.3.8), it is dense in \( \hat{X} \), which finishes the proof. □

Remark (7.4.12). — We will show, in Chapter V, that the conclusion of (7.4.10) still holds true without the assumption that the curve is normal (or even reduced); we will also show that, for an algebraic curve (reduced or not) to be affine, it is necessary and sufficient for its (reduced) irreducible components to not be complete.

Corollary (7.4.13). — Let \( X \) be a normal irreducible curve over the field \( K = R(X) \), and \( Y \) a complete integral curve over the field \( L = R(Y) \). Then there is a canonical bijective correspondence between dominant \( k \)-morphisms \( X \rightarrow Y \) and \( k \)-monomorphisms \( L \rightarrow K \).

Proof. By (7.4.9), rational \( k \)-map from \( X \) to \( Y \) can be identified with \( k \)-morphisms \( u : X \rightarrow Y \). Since the dominant morphisms \( u : X \rightarrow Y \) are characterised by being those such that \( u(x) = y \) (writing \( x \) and \( y \) to denote the generic points of \( X \) and \( Y \), respectively), the corollary follows from these remarks and from (I, 7.1.13). □

Corollary (7.4.14). We can refine the result of (7.4.13) in the case where \( Y \) is the projective line \( P^1_k = \text{Proj}(k[T_0, T]), \) where \( T_0 \) and \( T \) are indeterminates. Then \( Y \) is an integral scheme (2.4.4), and the scheme induced on the open subset \( D_+ (T_0) \) of \( Y \) is isomorphic to \( \text{Spec}(k[T]) \) (2.3.6), and so the generic point of \( Y \) is the ideal \( (0) \) of \( k[T] \), and the field of rational functions of \( Y \) is \( k(T) \), which proves that \( Y \) is a complete algebraic curve over \( k \). Furthermore, the only graded prime ideal of \( S = k[T_0, T] \) that contains \( T_0 \) and is distinct from \( S_+ \) is the principal ideal \( (T_0) \), and so the complement of \( D_+ (T_0) \) in \( Y = P^1_k \) consists of one closed point, called the “point at infinity”, which we denote by \( \infty \) (for a general study of the links between vector bundles and projective bundles, see (8.4)). With these notations:
Corollary (7.4.15). — Let \( X \) be a normal irreducible curve over the field \( K = \mathbb{R}(X) \). Then there exists a canonical bijective correspondence between the set \( K \) and the set of morphisms \( u \) from \( X \) to \( \mathbb{P}^1_k \) that are distinct from the constant morphism with value \( \infty \). For such a rational map to be dominant, it is necessary and sufficient for the corresponding element of \( K \) to be transcendental over \( k \).

Proof. This claim follows immediately from (7.4.9) and the following:

Lemma. — Let \( X \) be an integral prescheme over \( k \), and let \( K = \mathbb{R}(X) \) be its field of rational functions. Then there exists a canonical bijective correspondence between the set \( K \) and the set of rational maps \( u \) from \( X \) to \( \mathbb{P}^1_k \) that are distinct from the constant morphism with value \( \infty \). For such a rational map to be dominant, it is necessary and sufficient for the corresponding element of \( K \) to be transcendental over \( k \).

First of all, rational maps from \( X \) to \( \mathbb{P}^1_k \) correspond bijectively to points of \( \mathbb{P}^1_k \) with values in the extension \( K \) of \( k \) (I, 7.1.12). If such a point is located (I, 3.4.5) at the generic point of \( \mathbb{P}^1_k \), then the corresponding rational map is clearly dominant. In the converse case, since every point of \( \mathbb{P}^1_k \) that is distinct from the generic point is closed (7.4.3), the image of the domain of definition \( U \) of \( u \) by the unique morphism \( U \to \mathbb{P}^1_k \) of the class \( u \) (I, 7.2.2) consists of one closed point \( y \) of \( \mathbb{P}^1_k \), and this morphism (which is not necessarily everywhere defined on \( X \)) is thus not dominant; as an abuse of language, we thus say that the rational map \( u \) is “constant, of value \( y \)”. It remains to place in bijective correspondence the points of \( \mathbb{P}^1_k \) with value in \( K \) that are located (I, 3.4.5) not at \( \infty \), and the elements of \( K \), and then to verify that the location of such a point is the generic point of \( \mathbb{P}^1_k \) if and only if it corresponds to an element that is transcendental over \( k \). But this is immediate (4.2.6, example 1).

Corollary (7.4.16). — Let \( X \) and \( Y \) be algebraic curves over \( k \) that are normal, complete, and irreducible; let \( K = \mathbb{R}(X) \) and \( L = \mathbb{R}(Y) \) be their fields. Then there exists a canonical bijective correspondence between the set of \( k \)-isomorphisms \( X \sim \to Y \) and the set of \( k \)-isomorphisms \( L \sim \to K \).

Proof. This is an evident consequence of (7.4.13).

Corollary (7.4.17). This corollary (7.4.16) shows that an algebraic curve over \( k \) that is normal, complete, and irreducible, is determined by its field of rational functions \( K \) up to unique isomorphism; by definition, \( K \) is an extension of finite type of \( k \), of transcendence degree 1 (we classically call this a field of algebraic functions of one variable). Furthermore:

Proposition (7.4.18). — For every extension \( K \) of \( k \) of finite type and of transcendence degree 1, there exists an algebraic curve \( X \) (determined up to unique isomorphism) that is normal, complete, and irreducible, and such that \( \mathbb{R}(X) = K \). The set of local rings of \( X \) can be identified (I, 8.2.1) with the set consisting of the elements of \( K \) and the elements of the valuation rings that contain \( k \) and have \( K \) as their field of fractions.

Proof. Indeed, \( K \) is an extension of finite degree of a pure transcendental extension \( k(T) \) of \( k \), which can be identified, as we have seen, with the field of rational functions of the projective line \( Y = \mathbb{P}^1_k \). Let \( X \) be the integral closure of \( Y \) relative to \( K \) (6.3.4); then \( X \) is a normal algebraic curve over the field \( K \) (6.3.7), and it is complete, since the morphism \( X \to Y \) is finite (7.4.6). The local rings \( \mathcal{O}_x \) of \( X \) are either the field \( k \), when \( x \) is the generic point, or discrete valuation rings that contain \( k \) and have \( K \) as their field of fractions, when \( x \) is distinct from the generic point (7.4.5). Conversely, let \( A \) be such a ring; since the morphism \( X \to \text{Spec}(k) \) is proper, the fact that \( A \) dominates \( k \) implies that \( A \) also dominates a local ring \( \mathcal{O}_x \) of \( X \) (7.3.10); since the latter is a valuation ring that has \( K \) as a field of fractions, it is necessarily equal to \( A \).

Remarks (7.4.19). — It follows from (7.4.16) and (7.4.18) that the data of an algebraic curve over \( k \) that is normal, complete, and irreducible, is essentially equivalent to the data of an extension \( K \) of \( k \) that is of finite type and of transcendence degree 1. We note that, if \( k' \) is an extension of the base field \( k \), then \( X \otimes_k k' \) will again be a complete algebraic curve over \( k' \) (5.4.2, (iii)), but, in general, it will be neither reduced nor irreducible. It will, however, be both reduced and irreducible if \( K \) is a separable extension of \( k \), and \( k \) is algebraically closed in \( K \) (this can be expressed, in classical terminology, which we will not use, by saying that \( K \) is a “regular extension” of \( k \)). But even in this case, it is possible for \( X \otimes_k k' \) to not be normal. The reader will find details on these questions in Chapter IV.
§8. Blowup schemes; based cones; projective closure

8.1. Blowup preschemes

(8.1.1). Let \( Y \) be a prescheme, and, for every integer \( n \geq 0 \), let \( \mathcal{I}_n \) be a quasi-coherent sheaf of ideals of \( \mathcal{O}_Y \); suppose that the following conditions are satisfied:

\[
\mathcal{I}_0 = \mathcal{O}_Y, \quad \mathcal{I}_n \subset \mathcal{I}_m \text{ for } m \leq n, \quad \mathcal{I}_m \mathcal{I}_n \subset \mathcal{I}_{m+n} \text{ for any } m, n.
\]

We note that these hypotheses imply

\[
\mathcal{I}_1 \subset \mathcal{I}_n.
\]

Set

\[
\mathcal{I} = \bigoplus_{n \geq 0} \mathcal{I}_n.
\]

It follows from (8.1.1.1) and (8.1.1.2) that \( \mathcal{I} \) is a quasi-coherent graded \( \mathcal{O}_Y \)-algebra, and thus defines a \( Y \)-scheme \( X = \text{Proj}(\mathcal{I}) \). If \( \mathcal{I} \) is an invertible sheaf of ideals of \( \mathcal{O}_Y \), then \( \mathcal{I}_n \otimes \mathcal{O}_Y \mathcal{I}^n \) is canonically identified with \( \mathcal{I}_n \mathcal{I}^n \). If we then replace the \( \mathcal{I}_n \) by the \( \mathcal{I}_n \mathcal{I}^n \) and, in doing so, replace \( \mathcal{I} \) by a quasi-coherent \( \mathcal{O}_Y \)-algebra \( (\mathcal{I}) \), then \( X(\mathcal{I}) = \text{Proj}(\mathcal{I}(\mathcal{I})) \) is canonically isomorphic to \( X \).

(8.1.2). Suppose that \( Y \) is locally integral, so that the sheaf \( \mathcal{R}(Y) \) of rational functions is a quasi-coherent \( \mathcal{O}_Y \)-algebra (1, 7.3.7). We say that a \( \mathcal{O}_Y \)-submodule \( \mathcal{I} \) of \( \mathcal{R}(Y) \) is a fractional ideal of \( \mathcal{R}(Y) \) if it is of finite type (0, 5.2.1). Suppose we have, for all \( n \geq 0 \), a quasi-coherent fractional ideal \( \mathcal{I}_n \) of \( \mathcal{R}(Y) \), such that \( \mathcal{I}_0 = \mathcal{O}_Y \), and such that condition (8.1.1.2) (but not necessarily the second condition (8.1.1.1)) is satisfied; we can then again define a quasi-coherent graded \( \mathcal{O}_Y \)-algebra by Equation (8.1.1.4), and the corresponding \( Y \)-scheme \( X = \text{Proj}(\mathcal{I}) \); we will again have a canonical isomorphism from \( X \) to \( X(\mathcal{I}) \) for every invertible fractional ideal \( \mathcal{I} \) of \( \mathcal{R}(Y) \).

Definition (8.1.3). — Let \( Y \) be a prescheme (resp. a locally integral prescheme), and \( \mathcal{I} \) a quasi-coherent ideal of \( \mathcal{O}_Y \) (resp. a quasi-coherent fractional ideal of \( \mathcal{R}(Y) \)). We say that the \( Y \)-scheme \( X = \text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}^n) \) is obtained by blowing up the ideal \( \mathcal{I} \), or is the blow-up prescheme of \( Y \) relative to \( \mathcal{I} \). When \( \mathcal{I} \) is a quasi-coherent ideal of \( \mathcal{O}_Y \), and \( Y' \) is the closed subscheme of \( Y \) defined by \( \mathcal{I} \), we also say that \( X \) is the \( Y \)-scheme obtained by blowing up \( Y' \).

By definition, \( \mathcal{I} = \bigoplus_{n \geq 0} \mathcal{I}^n \) is then generated by \( \mathcal{I}_1 = \mathcal{I} \); if \( \mathcal{I} \) is an \( \mathcal{O}_Y \)-module of finite type, then \( X \) is projective over \( Y \) (5.5.2). Without any hypotheses on \( \mathcal{I} \), the \( \mathcal{O}_X(1) \) is invertible (3.2.5) and very ample, by (4.4.3) applied to the structure morphism \( X \rightarrow Y \).

We note that, if \( j : X \rightarrow Y \) is the structure morphism, then the restriction of \( f \) to \( f^{-1}(Y - Y') \) is an isomorphism to \( Y - Y' \) whenever \( \mathcal{I} \) is an ideal of \( \mathcal{O}_Y \) and \( Y' \) is the closed subscheme that it defines: indeed, since the questions is local on \( Y \), it suffices to assume that \( \mathcal{I} = \mathcal{O}_Y \), and our claim then follows from (3.1.7).

If we replace \( \mathcal{I} \) by \( \mathcal{I}^d (d > 0) \), then the blow-up \( Y \)-scheme \( X \) is replaced by a canonically isomorphic \( Y \)-scheme \( X' \) (8.1.1); similarly, for every invertible ideal (resp. invertible fractional ideal) \( \mathcal{I} \), the blow-up prescheme \( X(\mathcal{I}) \) relative to the ideal \( \mathcal{I} \) is canonically isomorphic to \( X \).

In particular, whenever \( \mathcal{I} \) is an invertible ideal (resp. invertible fractional ideal), the \( Y \)-scheme obtained by blowing up \( \mathcal{I} \) is isomorphic to \( Y \).

Proposition (8.1.3). — Let \( Y \) be an integral prescheme.

(i) For every sequence \( (\mathcal{I}_n) \) of quasi-coherent fractional ideals of \( \mathcal{R}(Y) \) that satisfies (8.1.1.2) and such that \( \mathcal{I}_0 = \mathcal{O}_Y \), the \( Y \)-scheme \( X = \text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}^n) \) is integral, and the structure morphism \( f : X \rightarrow Y \) is dominant.

(ii) Let \( \mathcal{I} \) be a quasi-coherent fractional ideal of \( \mathcal{R}(Y) \), and let \( X \) be the \( Y \)-scheme given by the blow up of \( Y \) relative to \( \mathcal{I} \). If \( \mathcal{I} \neq 0 \), then the structure morphism \( f : X \rightarrow Y \) is then birational and surjective.

Proof.
(i) This follows from the fact that \( \mathcal{S} = \bigoplus_{n \geq 0} \mathcal{I}_n \) is an integral \( \mathcal{O}_Y \)-algebra ((3.1.12) and (3.1.14)), since, for all \( y \in Y \), \( \mathcal{O}_y \) is an integral ring (I, 5.1.4).

(ii) By (i), \( X \) is integral; if, furthermore, \( x \) and \( y \) are the generic points of \( X \) and \( Y \) (respectively), then we have \( f(x) = y \), and it remains to show that \( k(x) \) is of rank 1 over \( k(y) \). But \( x \) is also the generic point of the fibre \( f^{-1}(y) \); if \( \psi \) is the canonical morphism \( Z \to Y \), where \( Z = \text{Spec}(k(y)) \), then the prescheme \( f^{-1}(y) \) can be identified with \( \text{Proj}(\mathcal{S}') \), where \( \mathcal{S}' = \psi^*(\mathcal{S}) \) (3.5.3). But it is clear that \( \mathcal{S}' = \bigoplus_{n \geq 0} (\mathcal{I}_y)^n \), and, since \( \mathcal{S} \) is a quasi-coherent fractional ideal of \( \mathcal{O}(Y) \) that is not zero, \( \mathcal{I}_y \neq 0 \) (I, 7.3.6), whence \( \mathcal{I}_y = k(y) \); then \( \text{Proj}(\mathcal{S}') \) can be identified with \( \text{Spec}(k(y)) \) (3.1.7), whence the conclusion.

We show a converse of (8.1.4) in (III, 2.3.8).

**8.15.** We return to the setting and notation of (8.1.1). By definition, the injection homomorphisms \( \mathcal{I}_{n+1} \to \mathcal{I}_n \) (8.1.1.1) define, for every \( k \in \mathbb{Z} \), an injective homomorphism of degree zero of graded \( \mathcal{S} \)-modules

\[
8.15.1 \quad u_k : \mathcal{S}_+(k+1) \rightarrow \mathcal{S}(k);
\]

since \( \mathcal{S}_+(k+1) \) and \( \mathcal{S}(k) \) are canonically (TN)-isomorphic, they give a canonical correspondence between \( u_k \) and an injective homomorphism of \( \mathcal{O}_X \)-modules (3.4.2):

\[
8.15.2 \quad \tilde{u}_k : \mathcal{O}_X(k+1) \rightarrow \mathcal{O}_X(k).
\]

Recall as well (3.2.6) that we have defined canonical homomorphisms

\[
8.15.3 \quad \lambda : \mathcal{O}_X(h) \otimes \mathcal{O}_X \mathcal{O}_X(k) \rightarrow \mathcal{O}_X(h+k)
\]

and, since the diagram

\[
\begin{array}{ccc}
\mathcal{S}(h) \otimes \mathcal{S} \mathcal{S}(k) \otimes \mathcal{S}(l) & \rightarrow & \mathcal{S}(h+k) \otimes \mathcal{S}(l) \\
\downarrow & & \downarrow \\
\mathcal{S}(h) \otimes \mathcal{S} \mathcal{S}(k+l) & \rightarrow & \mathcal{S}(h+k+l)
\end{array}
\]

commutes, it follows from the functoriality of the \( \lambda \) (3.2.6) that the homomorphisms (8.1.5.3) define the structure of a quasi-coherent graded \( \mathcal{O}_X \)-algebra on

\[
8.15.4 \quad \mathcal{S}_X = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n).
\]

Furthermore, the diagram

\[
\begin{array}{ccc}
\mathcal{S}(h) \otimes \mathcal{S} \mathcal{S}(k+1) & \rightarrow & \mathcal{S}(h+k+1) \\
\downarrow 1 \otimes u_k & & \downarrow u_{k+1} \\
\mathcal{S}(h) \otimes \mathcal{S} \mathcal{S}(k) & \rightarrow & \mathcal{S}(h+k)
\end{array}
\]

commutes; the functoriality of the \( \lambda \) then implies that we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_X(h) \otimes \mathcal{O}_X \mathcal{O}_X(k+1) & \rightarrow & \mathcal{O}_X(h+k+1) \\
\downarrow 1 \otimes \tilde{u}_k & & \downarrow \tilde{u}_{k+1} \\
\mathcal{O}_X(h) \otimes \mathcal{O}_X \mathcal{O}_X(k) & \rightarrow & \mathcal{O}_X(h+k)
\end{array}
\]

where the horizontal arrows are the canonical homomorphisms. We can thus say that the \( \tilde{u}_k \) define an injective homomorphism (of degree zero) of graded \( \mathcal{S}_X \)-modules

\[
8.15.6 \quad \tilde{u} : \mathcal{S}_X(1) \rightarrow \mathcal{S}_X.
\]
(8.1.6). Keeping the notation from (8.1.5), we now note that, for \( n \geq 0 \), the composite homomorphism
\[
\tilde{v}_n = \tilde{u}_{n-1} \circ \tilde{u}_{n-2} \circ \ldots \circ \tilde{u}_0
\]
is an injective homomorphism \( \mathcal{O}_X(n) \to \mathcal{O}_X \); we denote by \( \mathcal{I}_{n,X} \) its image, which is thus a quasi-coherent ideal of \( \mathcal{O}_X \), isomorphic to \( \mathcal{O}_X(n) \). Furthermore, the diagram
\[
\begin{array}{ccc}
\mathcal{O}_X(m) \otimes \mathcal{O}_X & \xrightarrow{\lambda} & \mathcal{O}_X(m+n) \\
\tilde{v}_m \otimes \tilde{v}_n & \downarrow & \tilde{v}_{m+n} \\
\mathcal{O}_X & \xrightarrow{id} & \mathcal{O}_X
\end{array}
\]
commutes for \( m \geq 0, n \geq 0 \). We thus deduce the following inclusions:

(8.1.6.1) \[ \mathcal{I}_{0,X} = \mathcal{O}_X, \quad \mathcal{I}_{n,X} \subset \mathcal{I}_{m,X} \quad \text{for} \quad 0 \leq m \leq n; \]

(8.1.6.2) \[ \mathcal{I}_{m,X} \mathcal{I}_{n,X} \subset \mathcal{I}_{m+n,X} \quad \text{for} \quad m \geq 0, n \geq 0. \]

Proposition (8.1.7). — Let \( Y \) be a prescheme, \( \mathcal{I} \) a quasi-coherent ideal of \( \mathcal{O}_Y \), and \( X = \text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}^n) \) the \( Y \)-scheme given by blowing up \( \mathcal{I} \). We then have, for all \( n > 0 \), a canonical isomorphism

(8.1.7.1) \[ \mathcal{O}_X(n) \xrightarrow{\sim} \mathcal{I}^n \mathcal{O}_X = \mathcal{I}_{n,X} \]

(cf. (0, 4.3.5)), and thus that \( \mathcal{I}^n \mathcal{O}_X \) is a very-ample invertible \( \mathcal{O}_X \)-module if \( n > 0 \).

Proof. The last claim is immediate, since \( \mathcal{O}_X(1) \) is invertible (3.2.5) and very ample for \( Y \) by definition ((4.4.3) and (4.4.9)). Also by definition, the image of \( v_n \) is exactly \( \mathcal{I}^n \mathcal{I} \), and (8.1.7.1) then follows from the exactness of the functor \( M(3.2.4) \) and from Equation (3.2.4.1).

Corollary (8.1.8). — Under the hypotheses of (8.1.7), if \( f : X \to Y \) is the structure morphism, and \( Y' \) the closed subscheme of \( Y \) defined by \( \mathcal{I} \), then the closed subscheme \( X' = f^{-1}(Y') \) of \( X \) is defined by \( \mathcal{I} \mathcal{O}_X \) (which is canonically isomorphic to \( \mathcal{O}_X(1) \)), from which we obtain a canonical short exact sequence

(8.1.8.1) \[ 0 \to \mathcal{O}_X(1) \to \mathcal{O}_X \to \mathcal{O}_{X'} \to 0. \]

Proof. This follows from (8.1.7.1) and from (I, 4.4.5).

(8.1.9). Under the hypotheses of (8.1.7), we can be more precise about the structure of the \( \mathcal{I}_{n,X} \). Note that the homomorphism
\[
\tilde{u}_0 : \mathcal{O}_X \to \mathcal{O}_X(1)
\]
canonically corresponds to a section \( s \) of \( \mathcal{O}_X(-1) \) over \( X \), which we call the canonical section (relative to \( \mathcal{I} \)) (0, 5.1.1). In the diagram in (8.1.5.5), the horizontal arrows are isomorphisms (3.2.7); by replacing \( h \) with \( k \), and \( k \) with \( -1 \) in this diagram, we obtain that \( \tilde{u}_k = 1_k \otimes \tilde{u}_0 \) (where \( 1_k \) denotes the identity on \( \mathcal{O}_X(k) \)), or, equivalently, that the homomorphism \( \tilde{u}_k \) is given exactly by tensoring with the canonical section \( s \) (for all \( k \in \mathbb{Z} \)). The homomorphism \( \tilde{u} \) (8.1.5.6) can then be understood in the same way.

Thus, for all \( n \geq 0 \), the homomorphism \( \tilde{v}_n : \mathcal{O}_X(n) \to \mathcal{O}_X \) is given exactly by tensoring with \( s \otimes n \); we thus deduce:

Corollary (8.1.10). — With the notation of (8.1.8), the underlying space of \( X' \) is the set of \( x \in X \) such that \( s(x) = 0 \), where \( s \) denotes the canonical section of \( \mathcal{O}_X(-1) \).

Proof. Indeed, if \( c_x \) is a generator of the fibre \( (\mathcal{O}_X(1))_x \) at a point \( x \), then \( s_x \otimes c_x \) is canonically identified with a generator of the fibre of \( \mathcal{I}_1 \) at the point \( x \), and is thus invertible if and only if \( s_x \notin m_x(\mathcal{O}_X(-1))_x \), or, equivalently, if and only if \( s(x) \neq 0 \).

Proposition. — Let \( Y \) be an integral prescheme, \( \mathcal{I} \) a quasi-coherent fractional ideal of \( \mathcal{O}(Y) \), and \( X \) the \( Y \)-scheme given by blowing up \( \mathcal{I} \). Then \( \mathcal{I} \mathcal{O}_X \) is an invertible \( \mathcal{O}_X \)-module that is very ample for \( Y \).

Proof. Since the questions is local on \( Y \) (4.4.5), we can reduce to the case where \( Y = \text{Spec}(A) \), with \( A \) some integral ring of ring of fractions \( K \), and \( \mathcal{I} = \mathfrak{I} \), with \( \mathcal{I} \) some fractional ideal of \( K \); there then exists an element \( a \neq 0 \) of \( A \) such that \( a \mathfrak{I} \subset A \). Let \( S = \bigoplus_{n \geq 0} \mathfrak{I}^n \); the map \( x \mapsto ax \)
is an $A$-isomorphism from $\mathcal{O}^{n+1} = (S(1))_n$ to $a\mathcal{O}^{n+1} = aS_n \subset \mathcal{O}^n = S_n$, and thus defines a (TN)-isomorphism of degree zero of graded $S$-modules $S_+(1) \to a\mathcal{O}S$. On the other hand, $x \mapsto a^{-1}x$ is an isomorphism of degree zero of graded $S$-modules $a\mathcal{O}S \xrightarrow{\sim} \mathcal{O}S$. We thus obtain, by composition (3.2.4), an isomorphism of $\mathcal{O}_X$-modules $\mathcal{O}_X(1) \xrightarrow{\sim} \mathcal{O}_X$, and, since $S$ is generated by $S_1 = \mathcal{O}_X$, $\mathcal{O}_X(1)$ is invertible (3.2.5) and very ample ((4.4.3) and (4.4.9)), whence our claim. \hfill \square

8. Preliminary results on the localisation of graded rings

(8.2.1). Let $S$ be a graded ring, but not assumed (for the moment) to be only in positive degree. We define

\[ S^\geq = \bigoplus_{n \geq 0} S_n, \quad S^\leq = \bigoplus_{n \leq 0} S_n \]

which are both graded subrings of $S$, in only positive and negative degrees (respectively). If $f$ is a homogeneous elements of degree $d$ (positive or negative) of $S$, then the ring of fraction $S_f = S'$ is again endowed with the structure of a graded ring, by taking $S'_n (n \in \mathbb{Z})$ to be the set of the $x/f^k$ for $x \in S_{n+kd}$ ($k \geq 0$); we define $S(f) = S'_0$, and will write $S^\geq_f$ and $S^\leq_f$ for $S'^{\geq}_f$ and $S'^{\leq}_f$ (respectively). If $d > 0$, then

\[ (S^{\geq}_f)_f = S_f \]

since, if $x \in S_{n+kd}$ with $n + kd < 0$, then we can write $x/f^k = x f^h/f^{h+k}$, and we also have that $n + (h + k)d > 0$ for $h$ sufficiently large and $> 0$. We thus conclude, by definition, that

\[ (S^{\geq}_f)_f = (S^\geq)_0 = S(f). \]

If $M$ is a graded $S$-module, then we similarly define

\[ M^{\geq} = \bigoplus_{n \geq 0} M_n, \quad M^{\leq} = \bigoplus_{n \leq 0} M_n \]

which are (respectively) a graded $S^{\geq}$-module and a graded $S^{\leq}$-module, and their intersection is the $S_0$ module $M_0$. If $f \in S_d$, then we define $M_f$ to be the graded $S_f$-module whose elements of degree $n$ are the $z/f^k$ for $z \in M_{n+kd}$ ($k \geq 0$); we denote by $M_f$ the set of elements of degree zero of $M_f$, and this is an $S_f^{\geq}$-module, and we will write $M^{\geq}_f$ and $M^{\leq}_f$ to mean $(M_f)^{\geq}$ and $(M_f)^{\leq}$ (respectively). If $d > 0$, then we see, as above, that

\[ (M^{\geq})_f = M_f \]

and

\[ (M^{\geq})_f = (M^\geq)_0 = M(f). \]

(8.2.2). Let $z$ be an indeterminate, we we will call the **homogenisation variable**. If $S$ is a graded ring (in positive or negative degrees), then the polynomial algebra

\[ \hat{S} = S[z] \]

is a graded $S$-algebra, where we define the degree of $fz^n$ ($n \geq 0$), with $f$ homogeneous, as

\[ \deg(fz^n) = n + \deg f. \]

**Lemma (8.2.3).** — (i) There are canonical isomorphisms of (non-graded) rings

\[ \hat{S}(z) \xrightarrow{\sim} \hat{S}/(z-1)\hat{S} \xrightarrow{\sim} S. \]

(ii) There is a canonical isomorphism of (non-graded) rings

\[ \hat{S}(f) \xrightarrow{\sim} S^\leq_f \]

for all $f \in S_d$ with $d > 0$.

---

1This should not be confused with the use of the notation $\hat{S}$ to denote the completed separation of a ring.
Proof. The first of the isomorphisms in (8.2.3.1) was defined in (2.2.5), and the second is trivial; the isomorphism $\hat{S}(z) \sim S$ thus defined thus gives a correspondence between $xz^n/z^{n+k}$ (where $\deg(x) = k$ for $k \geq -n$) and the element $x$. The homomorphism (8.2.3.2) gives a correspondence between $xz^n/f^k$ (where $\deg(x) = kd - n$) and the element $x/f^k$ of degree $-n$ in $S_f$, and it is again clear that this does indeed give an isomorphism.

(8.2.5). Let $M$ be a graded $S$-module. It is clear that the $S$-module
\[(8.2.4.1) \hat{M} = M \otimes_S \hat{S} = M \otimes_S S[z]\]
is the direct sum of the $S$-modules $M \otimes Sz^n$, and thus of the abelian groups $M_k \otimes S^z$ ($k \in \mathbb{Z}$, $n \geq 0$); we define on $\hat{M}$ the structure of a graded $\hat{S}$-module by setting
\[(8.2.4.2) \deg(x \otimes z^n) = n + \deg x\]
for all homogeneous $x \in M$. We leave it to the reader to prove the analogue of (8.2.3):

**Lemma (8.2.5).** —

(i) There is a canonical di-isomorphism of (non-graded) modules
\[(8.2.5.1) \hat{M}(z) \sim M.\]

(ii) For all $f \in S_d$ ($d > 0$), there is a di-isomorphism of (non-graded) modules
\[(8.2.5.2) \hat{M}(f) \sim M_f^\leq.\]

(8.2.6). Let $S$ be a positively-graded ring, and consider the decreasing sequence of graded ideals of $S$
\[(8.2.6.1) S[n] = \bigoplus_{m \geq n} S_m \quad (n \geq 0)\]
(so, in particular, we have $S[0] = S$ and $S[1] = S_+$. Since it is evident that $S[n]S[n] \subset S[n+n]$, we can define a graded ring $S^\leq$ by setting
\[(8.2.6.2) S^\leq = \bigoplus_{n \geq 0} S^\leq_n \quad \text{with} \quad S^\leq_0 = S[n].\]

$S^\leq_0$ is then the ring $S$ considered as a non-graded ring, and $S^\leq$ is thus an $S^\leq_0$-algebra. For every homogeneous element $f \in S_d$ ($d > 0$), we denote by $f^\leq$ the element $f$ considered as belonging to $S[\leq]$; with this notation:

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**Lemma (8.2.7).** — Let $S$ be a positively-graded ring, and $f$ a homogeneous element of $S_d$ ($d > 0$). There are canonical ring isomorphisms

\[(8.2.7.1) S_f \sim \bigoplus_{n \in \mathbb{Z}} S(n)(f)\]
\[(8.2.7.2) (S^\leq_f)_{f/1} \sim S_f\]
\[(8.2.7.3) S^\leq_{f/(f^\leq)} \sim S^\leq_f\]

where the first two are isomorphisms of graded rings.

Proof. It is immediate, by definition, that we have $(S_f)_n = (S(n)(f))_0$, whence the isomorphism in (8.2.7.1), which is exactly the identity. Next, since $f/1$ is invertible in $S_f$, there is a canonical isomorphism $S_f \sim (S^\leq_f)(f/1) = (S_f)_{f/1}$ by (8.2.1.2) applied to $S_f$; the inverse isomorphism is, by definition, the isomorphism in (8.2.7.2). Finally, if $x = \sum_{m \geq n} y_m$ is an element of $S[n]$ with $n = kd$, then the element $x/(f^\leq)^k$ corresponds to the element $\sum_{m \geq n} y_m/f^k$ of $S^\leq_f$, and we can quickly verify that this defines an isomorphism (8.2.7.3).
If $M$ is a graded $S$-module, then we similarly define, for all $n \in \mathbb{Z}$,

(8.2.8.1) \[ M_{[n]} = \bigoplus_{m \geq n} M_m \]

and, since $S_{[m]}M_{[n]} \subset M_{[m+n]}$ ($m \geq 0$), we can define a graded $S^\natural$-module $M^\natural$ by setting

(8.2.8.2) \[ M^\natural = \bigoplus_{n \in \mathbb{Z}} M^\natural_n \quad \text{with} \quad M^\natural_n = M_{[n]} . \]

We leave to the reader the proof of:

**Lemma (8.2.9).** — With the notation of (8.2.7) and (8.2.8), there are canonical di-isomorphisms of modules

(8.2.9.1) \[ M_f \sim \bigoplus_{n \in \mathbb{Z}} M(n)(f) \]

(8.2.9.2) \[ (M^\natural_f)_{f/1} \sim M_f \]

(8.2.9.3) \[ M^\natural_{(f^2)} \sim M^\natural_f \]

where the first two are di-isomorphisms of graded modules.

**Lemma (8.2.10).** — Let $S$ be a positively-graded ring.

(i) For $S^\natural$ to be an $S^\natural_0$-algebra of finite type (resp. a Noetherian $S^\natural_0$-algebra), it is necessary and sufficient for $S$ to be an $S^\natural_0$-algebra of finite type (resp. a Noetherian $S^\natural_0$-algebra).

(ii) For $S^\natural_{i+1} = S^\natural_i S^\natural_0$ ($n \geq n_0$), it is necessary and sufficient for $S_{n+1} = S_1 S_n$ ($n \geq n_0$).

(iii) For $S^\natural_n = S^\natural_i$ ($n \geq n_0$), it is necessary and sufficient for $S_n = S^\natural_i$ ($n \geq n_0$).

(iv) If $(f_a)$ is a set of homogeneous elements of $S_+$ such that $S_+$ is the radical in $S_+$ of the ideal of $S_+$ generated by the $f_a$, then $S^\natural_i$ is the radical in $S^\natural_i$ of the ideal of $S^\natural_i$ generated by the $f_a$.

**Proof.**

(i) If $S^\natural$ is an $S^\natural_0$-algebra of finite type, then $S_+ = S^\natural_0$ is a module of finite type over $S = S^\natural_0$, by (2.1.6, i), and so $S$ is an $S_0$-algebra of finite type (2.1.4); if $S^\natural$ is a Noetherian ring, then so too is $S^\natural_0 = S$ (2.1.5). Conversely, if $S$ is an $S_0$-algebra of finite type, then we know (2.1.6, ii) that there exist $h > 0$ and $m_0 > 0$ such that $S_{n+h} = S_h S_n$ for $n \geq m_0$; we can clearly assume that $m_0 \geq h$. Furthermore, the $S_m$ are $S_0$-modules of finite type (2.1.6, i). So, if $n \geq m_0 + h$, then $S_{n} = S_0 S_{n-h} = S_h^\natural S_{n-h}$; and if $m < m_0 + h$ then, letting $E = S_{m_0} + \ldots + S_{m_0+h-1}$, we have that

\[ S^\natural_m = S_m + \ldots + S_{m_0+h-1} + S_h E + S_h^2 E + \ldots . \]

For $1 \leq m \leq m_0$, let $G_m$ be the union of the finite systems of generators of the $S_0$-modules $S_i$ for $m \leq i \leq m_0 + h - 1$, thought of as a subset of $S_{[m]}$. For $m_0 + 1 \leq m \leq m_0 + h - 1$, let $G_m$ be the union of the finite system of generators of the $S_0$-modules $S_i$ for $m \leq i \leq m_0 + h - 1$ and of $S_h E$, thought of as a subset of $S_{[m]}$. It is clear that $S^\natural_m = S^\natural_0 G_m$ for $1 \leq m \leq m_0 + h - 1$, and thus the union $G$ of the $G_m$ for $1 \leq m \leq m_0 + h - 1$ is a system of generators of the $S^\natural_0$-algebra $S^\natural$. We thus conclude that, if $S = S^\natural_0$ is a Noetherian ring, then so too is $S^\natural$.

(ii) It is clear that, if $S_{n+1} = S_1 S_n$ for $n \geq n_0$, then $S^\natural_{n+1} = S_1 S^\natural_n$, and a fortiori $S^\natural_{n+1} = S_1 S^\natural_n$ for $n \geq n_0$. Conversely, this last equality can be written as

\[ S_{n+1} + S_{n+2} + \ldots = (S_1 + S_2 + \ldots)(S_n + S_{n+1} + \ldots) \]

and comparing terms of degree $n+1$ (in $S$) on both sides gives that $S_{n+1} = S_1 S_n$.

(iii) If $S_n = S^\natural_n$ for $n \geq n_0$, then $S^\natural_n = S^\natural_1 + S^\natural_1 + \ldots$ since $S^\natural_1$ contains $S_1 + S^\natural_2 + \ldots$, we have that $S^\natural_n \subset S^\natural_1$, and thus $S^\natural_n = S^\natural_1$ for $n \geq n_0$. Conversely, the only terms of $S^\natural_1 = (S_1 + S_2 + \ldots)^n$ that are of degree $n$ in $S$ are those of $S^\natural_1$; the equality $S^\natural_n = S^\natural_1$ thus implies that $S_n = S^\natural_1$. \

(iv) It suffices to show that, if an element $g \in S_{k+h}$ is considered as an element of $S_k^2$ ($k > 0$, $h \geq 0$), then there exists an integer $n > 0$ such that $g^n$ is a linear combination (in $S_k^2$) of the $f^2_i$ with coefficients in $S^2$. By hypothesis, there exists an integer $m_0$ such that, for $m \geq m_0$, we have, in $S$, that $g^m = \sum a_c f^c$, where the indices $a$ here are independent of $m$; furthermore, we can clearly assume that the $c_{am}$ are homogeneous, with
\[ \deg(c_{am}) = m(k + h) - \deg f^c \]
in $S$. So take $m_0$ sufficiently large enough to ensure that $km_0 > \deg f^c$ for all the $f^c$ that appear in $g^{m_0}$; for all $a$, let $c'_a = a$ be the element $c_{am}$ considered as having degree $km - \deg f^c$ in $S^2$; we then have, in $S^2$, that $g^m = \sum c'_a f^c$, which finishes the proof.

(8.2.11). Consider the graded $S_0$-algebra
\[ S^2 \otimes_S S_0 = S^2 / S \otimes_S S_0 = \bigoplus_{n \geq 0} S_{[n]} / S \otimes_S S_{[n]} \cdot \]
Since $S_0$ is a quotient $S_0$-module of $S_{[n]} / S \otimes_S S_{[n]}$, there is a canonical homomorphism of graded $S_0$-algebras
\[ S^2 \otimes_S S_0 \longrightarrow S \]
which is clearly surjective, and thus corresponds (2.9.2) to a canonical closed immersion
\[ \text{Proj}(S) \longrightarrow \text{Proj}(S^2 \otimes_S S_0) \cdot \]

Proposition (8.2.12). — The canonical morphism (8.2.11.3) is bijective. For the homomorphism (8.2.11.2) to be (TN)-bijective, it is necessary and sufficient for there to exist some $n_0$ such that $S_{n+1} = S_1 S_n$ for $n \geq n_0$. If this latter condition is satisfied, then (8.2.11.3) is an isomorphism; the converse is true whenever $S$ is Noetherian.

Proof. To prove the first claim, it suffices (2.8.3) to show that the kernel $\mathcal{I}$ of the homomorphism (8.2.11.2) consists of nilpotent elements. But if $f \in S_{[n]}$ is an element whose class modulo $S_1 S_{[n]}$ belongs to this kernel, then this implies that $f \in S_{[n+1]}$; then $f^{n+1}$, considered as an element of $S_1 S_{(n+1)}$, is also an element of $S_1 S_{(n+1)}$, since it can be written as $f \cdot f^n$; so the class of $f^{n+1}$ modulo $S_1 S_{(n+1)}$ is zero, which proves our claim. Since the hypothesis that $S_{n+1} = S_1 S_n$ for $n \geq n_0$ is equivalent to $S_{n+1}^2 = S_1 S_{n+1}^0$ for $n \geq n_0$ (8.2.10, ii), this hypothesis is equivalent, by definition, to the fact that (8.2.11.2) is (TN)-injective, and thus (TN)-bijective, and so (8.2.11.3) is an isomorphism, by (2.9.1). Conversely, if (8.2.11.3) is an isomorphism, then the sheaf $\mathcal{I}$ on $\text{Proj}(S^2 \otimes_S S_0)$ is zero (2.9.2, i); since $S^2 \otimes_S S_0$ is Noetherian, as a quotient of $S^2$ (8.2.10, i), we conclude from (2.7.3) that $\mathcal{I}$ satisfies condition (TN), and so $S_{n+1}^2 = S_1 S_{n+1}^0$ for $n \geq n_0$, and this finishes the proof, by (8.2.10, ii).

(8.2.13). Consider now the canonical injections $(S_+)^n \to S_{[n]}$, which define an injective homomorphism of degree zero of graded rings
\[ \bigoplus_{n \geq 0} (S_+)^n \longrightarrow S^2 \cdot \]

Proposition (8.2.14). — For the homomorphism (8.2.13.1) to be a (TN)-isomorphism, it is necessary and sufficient for there to exist some $n_0$ such that $S_n = S^n_1$ for all $n \geq n_0$. Whenever this is the case, the morphisms corresponding to (8.2.13.1) is everywhere defined and also an isomorphism
\[ \text{Proj}(S^2) \cong \text{Proj}(\bigoplus_{n \geq 0} (S_+)^n) \cdot \]
the converse is true whenever $S$ is Noetherian.

Proof. The first two claims are evident, given (8.2.10, iii) and (2.9.1). The third will follow from (8.2.10, i and iii) and the following lemma:
Lemma (8.2.14.1). — Let $T$ be a positively-graded ring that is also a $T_0$-algebra of finite type. If the morphism corresponding to the injective homomorphism $\bigoplus_{n \geq 0} T^n_1 \to T$ is everywhere defined and also an isomorphism $\text{Proj}(T) \to \text{Proj}(\bigoplus_{n \geq 0} T^n_1)$, then there exists some $n_0$ such that $T^n_1 = T^n_0$ for $n \geq n_0$.

Let $g_i (1 \leq i \leq r)$ be generators of the $T_0$-module $T_1$. The hypothesis implies first of all that all the $D_+(g_i)$ cover $\text{Proj}(T)$ (2.8.1). Let $(h_j)_{1 \leq j \leq s}$ be a system of homogeneous elements of $T_+$, with $\deg(h_j) = n_j$, that form, with the $g_i$, a system of generators of the ideal $T_+$, or, equivalently (2.1.3), a system of generators of $T$ as a $T_0$-algebra; if we set $T' = \bigoplus_{n \geq 0} T^n_1$, then the element $h_j / T^n_1 h_j$ of the ring $T(n)$ must, by hypothesis, belong to the subring $T'(n)$, and so there exists some integer $k$ such that $T_1^n h_j \subseteq T_1^{k+n}$. We thus conclude, by induction on $r$, that $T_1^n h_j' \subset T'$ for all $r \geq 1$, and, by definition of the $h_j$, we thus have that $T_1^n T \subset T'$. Also, there exists, for all $j$, an integer $m_j$ such that $h_j^{m_j}$ belongs to the ideal of $T$ generated by the $g_i$ (2.3.14), so $h_j^{m_j} \subseteq T_1 T$, and $h_j^{m_j} \subseteq T_1^n T \subset T'$.

There is thus an integer $m_0 \geq k$ such that $h_j^{m_j} \subseteq T_1^{m_j}$ for $m \geq m_0$. So, if $q$ is the largest of the integers $n_j$, then $n_0 = qsm_0 + k$ is the required number. Indeed, an element of $S_n$, for $n \geq n_0$, is the sum of monomials belonging to $T_1^n u$, where $u$ is a product of powers of the $h_j$; if $\alpha \geq k$, then it follows from the above that $T_1^n u \subset T_1^n$; in the other case, one of the exponents of the $h_j$ is $\geq m_0$, so $u \subset T_1^\beta v$, where $\beta \geq k$ and $v$ is again a product of powers of the $h_j$; we can then reduce to the previous case, and so we conclude that $T_1^n u \subset T_1^n$ in all cases. \hfill \Box

Remark (8.2.15). — The condition $S_n = S^n_1$ for $n \geq n_0$ clearly implies that $S_{n+1} = S_1 S_n$ for $n \geq n_0$, but the converse is not necessarily true, even if we assume that $S$ is Noetherian. For example, let $K$ be a field, $A = K[x]$, and $B = K[y] / y^2 K[y]$, where $x$ and $y$ are indeterminates, with $x$ taken to have degree 1 and $y$ to have degree 2, and let $S = A \otimes_K B$, so that $S$ is a graded algebra over $K$ that has a basis given by the elements $1, x^n (n \geq 1), \text{and } x^n y (n \geq 0)$. It is immediate that $S_{n+1} = S_1 S_n$ for $n \geq 2$, but $S^n_1 = Kx^n$ while $S^n_1 = Kx^n + Kx^ny$ for $n \geq 2$.

8.3. Based cones

(8.3.1). Let $Y$ be a prescheme; in all of this section, we will consider only $Y$-preschemes and $Y$-morphisms. Let $\mathcal{S}$ be a quasi-coherent positively-graded $O_Y$-algebra; we further assume that $\mathcal{S}_0 = O_Y$.

Following the notation introduced in (8.2.2), we let

$$\mathcal{S} = \mathcal{S}[z] = \mathcal{S} \otimes_{O_Y} O_Y[z]$$

which we consider as a positively-graded $O_Y$-algebra by defining the degrees as in (8.2.2.2), so that, for every affine open subset $U$ of $Y$, we have

$$\Gamma(U, \mathcal{S}) = (\Gamma(U, \mathcal{S}))[z].$$

In what follows, we write

$$(8.3.1.2) \quad X = \text{Proj}(\mathcal{S}), \quad C = \text{Spec}(\mathcal{S}), \quad \tilde{C} = \text{Proj}(\mathcal{S})$$

(8.3.1.2) (where, in the definition of $C$, we consider $\mathcal{S}$ as a non-graded $O_Y$-algebra), and we say that $C$ (resp. $\tilde{C}$) is the affine cone (resp. projective cone) defined by $\mathcal{S}$; we will sometimes say “cone” instead of “affine cone”. By an abuse of language, we also say that $C$ (resp. $\tilde{C}$) is the affine cone based at $X$ (7) (resp. the projective cone based at $X$ (7)2), with the implicit understanding that the prescheme $X$ is given in the form $\text{Proj}(\mathcal{S})$; finally, we say that $\tilde{C}$ is the projective closure of $C$ (with the data of $\mathcal{S}$ being implicit in the structure of $C$).

Proposition (8.3.2). — There exist canonical $Y$-morphisms

$$\begin{align*}
(8.3.2.1) & \quad Y \xrightarrow{i} C \xrightarrow{j} \tilde{C} \\
(8.3.2.2) & \quad X \xrightarrow{k} \tilde{C}
\end{align*}$$

2[Trans.] A more literal translation of the French (cône projetant (affine/projectif)) would be the projecting (affine/projective) cone, but it seems that this terminology already exists to mean something else.
such that \( \epsilon \) and \( j \) are closed immersions, and \( i \) is an affine morphism, which is a dominant open immersion, for which
\[
(8.3.2.3) \quad i(C) = \hat{C} - j(X);
\]
furthermore, \( \hat{C} \) is the smallest closed subscheme of \( \hat{C} \) containing \( i(C) \).

**Proof.** To define \( i \), consider the open subset of \( \hat{C} \) given by
\[
(8.3.2.4) \quad \hat{C}_z = \text{Spec}(\hat{S}/(z - 1)).
\]
(3.1.4), where \( z \) is canonically identified with a section of \( \mathcal{S} \) over \( Y \). The isomorphism \( i : C \rightarrow \hat{C}_z \) then corresponds to the canonical isomorphism \( (8.2.3.1) \)
\[
\mathcal{S}/(z - 1) \rightarrow \mathcal{S}.
\]

The morphism \( \epsilon \) corresponds to the augmentation homomorphism \( \mathcal{S} \rightarrow \mathcal{S}_0 = \mathcal{O}_Y \), which has kernel \( \mathcal{S}_+ \) (1.2.7), and, since the latter is surjective, \( \epsilon \) is a closed immersion (1.4.10). Finally, \( j \) corresponds (3.5.1) to the surjective homomorphism of degree zero \( \hat{S} \rightarrow \mathcal{S} \), which restricts to the identity on \( \mathcal{S}_+ \) and is zero on \( z \hat{S} \), which is its kernel; \( j \) is everywhere defined, and is a closed immersion, by (3.6.2).

To prove the other claims of (8.3.2), we can clearly restrict to the case where \( Y = \text{Spec}(A) \) is affine, and \( \mathcal{S} = \hat{S} \), with \( S \) a graded \( A \)-algebra, whence \( \hat{S} = (\hat{S}) \); the homogeneous elements \( f \) of \( S_+ \) can then be identified with sections of \( \mathcal{S} \) over \( Y \), and the open subset of \( \hat{C} \), denoted \( D_+(f) \) in (2.3.3), can then be written as \( \hat{C}_f \) (3.1.4); similarly, the open subset of \( C \) denoted \( D(f) \) in (I, 1.1.1) can be written as \( C_f \) (0, 5.5.2). With this in mind, it follows from (2.3.14) and from the definition of \( \hat{S} \) that, in this case, the open subsets \( \hat{C}_z = i(C) \) and \( \hat{C}_f \) (with \( f \) homogeneous in \( S_+ \)) form a cover of \( \hat{C} \).

Furthermore, with this notation,
\[
(8.3.2.5) \quad i^{-1}(\hat{C}_f) = C_f;
\]
indeed, \( \hat{C}_f \cap i(C) = \hat{C}_f \cap \hat{C}_z = \hat{C}_f \) is canonically isomorphic to \( (\hat{S}(z))_{f/d} \) (2.2.2), and it follows from the definition of the isomorphism in (8.2.3.1) that the image of \( (\hat{S}(z))_{f/d} \) under the corresponding isomorphism of rings of fractions is exactly \( \hat{S}_f \).

Since \( C_f = \text{Spec}(S_f) \), this proves (8.3.2.5) and shows, at the same time, that the morphism \( i \) is affine; furthermore, the restriction of \( i \) to \( C_f \), thought of as a morphism to \( \hat{C}_f \), corresponds (I, 1.7.3) to the canonical homomorphism \( \hat{S}_f \rightarrow \hat{S}(z) \), and, by the above and (8.2.3.2), we can claim the following result:

\[
(8.3.2.6) \quad \text{If} \ Y = \text{Spec}(A) \text{ is affine, and} \ \mathcal{S} = \hat{S}, \text{then, for every homogeneous} \ f \text{ in} \ S_+, \hat{C}_f \text{is canonically identified with} \text{Spec}(S_f^{S_f}), \text{and the morphism} \ C_f \rightarrow \hat{C}_f \text{given by restricting} \ i \text{then corresponds to the canonical injection} \ S_f^{S_f} \rightarrow S_f.
\]

Now note that (for \( Y \) affine) the complement of \( \hat{C}_z \) in \( \hat{C} = \text{Proj}(\hat{S}) \) is, by definition, the set of graded prime ideals of \( \hat{S} \) containing \( z \), which is exactly \( j(X) \), by definition of \( j \), which proves (8.3.2.3).

Finally, to prove the last claim of (8.3.2), we can assume that \( Y \) is affine. With the above notation, note that, in the ring \( \hat{S} \), \( z \) is not a zero divisor; since \( i(C) = \hat{C} \), it suffices to prove the following lemma:

**Lemma (8.3.2.7).** — Let \( T \) be a positively-graded ring, \( Z = \text{Proj}(T) \), and \( g \) a homogeneous element of \( T \) of degree \( d > 0 \). If \( g \) is not a zero divisor in \( T \), then \( Z \) is the smallest closed subscheme of \( Z \) that contains \( Z_h = D_+(g) \).

By (I, 4.1.9), the question is local on \( Z \); for every homogeneous element \( h \in T_e \) \((e > 0)\), it thus suffices to prove that \( Z_h \) is the smallest closed subscheme of \( Z_h \) that contains \( Z_{gh} \); it follows from the definitions and from (I, 4.3.2) that this condition is equivalent to asking for the canonical homomorphism \( T_{(h)} \rightarrow T_{(gh)} \) to be injective. But this homomorphism can be identified with the canonical homomorphism \( T_{(h)} \rightarrow (T_{(h)})^{g^e/h^d} \) (2.2.3). But since \( g^e \) is not a zero divisor in \( T \), \( g^e/h^d \) is not a zero divisor in \( T_h \) (nor a fortiori in \( T_{(h)} \)), since the fact that \( (g^e/h^d)(t/h^m) = 0 \) for \( t \in T \) and
\( m > 0 \) implies the existence of some \( n > 0 \) such that \( h^n g^t = 0 \), whence \( h^n t = 0 \), and thus \( t / h^m = 0 \) in \( T_h \). This thus finishes the proof (0, 1.2.2).

**8.3.3.** We will often identify the affine cone \( C \) with the subscheme induced by the projective cone \( \hat{C} \) on the open subset \( i(C) \) by means of the open immersion \( i \). The closed subscheme of \( C \) associated to the closed immersion \( \epsilon \) is called the **vertex prescheme** (?) of \( C \); we also say that \( \epsilon \), which is a \( Y \)-section of \( C \), is the **vertex section** (?) of \( C \), or the **null section**, or \( C \); we can identify \( Y \) with the vertex prescheme (?) of \( C \) by means of \( \epsilon \). Also, \( i \circ \epsilon \) is a \( Y \)-section of \( \hat{C} \), and thus also a closed immersion (I, 5.4.6), corresponding to the canonical surjective homomorphism of degree zero \( \hat{\mathcal{T}} = \mathcal{T}[z] \to \mathcal{O}_Y[z] \) (3.1.7), whose kernel is \( \mathcal{T}_+[z] = \mathcal{T}_z \); the subscheme of \( \hat{C} \) associated to this closed immersion is also called the vertex prescheme (?) of \( \hat{C} \), and \( i \circ \epsilon \) the vertex section (?) of \( \hat{C} \); it can be identified with \( Y \) by means of \( i \circ \epsilon \). Finally, the closed subscheme of \( \hat{C} \) associated to \( j \) is called the **part at infinity** of \( \hat{C} \), and can be identified with \( X \) by means of \( j \).

**8.3.4.** The subschemes of \( C \) (resp. \( \hat{C} \)) induced on the open subsets

\[
E = C - \epsilon(Y), \quad \hat{E} = \hat{C} - i(\epsilon(Y))
\]

are called (by an abuse of language) the **pointed affine cone** and the **pointed projective cone** (respectively) defined by \( \mathcal{T} \); we note that, despite this nomenclature, \( E \) is not necessarily affine over \( Y \), nor \( \hat{E} \) projective over \( Y \) (8.4.3). When we identify \( C \) with \( i(C) \), we thus have the underlying spaces

\[
C \cup \hat{E} = \hat{C}, \quad C \cap \hat{E} = E
\]

so that \( \hat{C} \) can be considered as being obtained by **gluing** the open subschemes \( C \) and \( \hat{E} \); furthermore, by (8.3.2.3),

\[
E = \hat{E} - j(X).
\]

If \( Y = \text{Spec}(A) \) is affine, then, with the notation of (8.3.2),

\[
E = \bigcup C_f, \quad \hat{E} = \bigcup \hat{C}_f, \quad C_f = C \cap \hat{C}_f
\]

where \( f \) runs over the set of homogeneous elements of \( S_+ \) (or only a subset \( M \) of this set, with \( M \) generating an ideal of \( S_+ \) whose radical in \( S_+ \) is \( S_+ \) itself, or, equivalently, such that the \( X_f \) for \( f \in M \) cover \( X \) (2.3.14)). The gluing of \( C \) and \( \hat{C}_f \) along \( C_f \) is thus determined by the injection morphisms \( C_f \to C \) and \( C_f \to \hat{C}_f \), which, as we have seen (8.3.2.6), correspond (respectively) to the canonical homomorphisms \( S \to S_f \) and \( S_f^{\leq} \to S_f \).

**Proposition (8.3.5).** With the notation of (8.3.1) and (8.3.4), the morphism associated (3.5.1) to the canonical injection \( \varphi : \mathcal{T} \to \hat{\mathcal{T}} = \mathcal{T}[z] \) is a surjective affine morphism (called the canonical retraction)

\[
(8.3.5.1) \quad p : \hat{E} \to X
\]

such that

\[
(8.3.5.2) \quad p \circ j = 1_X.
\]

**Proof.** To prove the proposition, we can restrict to the case where \( Y \) is affine. Taking into account the expression in (8.3.4.4) for \( \hat{E} \), the fact that the domain of definition \( G(\varphi) \) of \( p \) is equal to \( \hat{E} \) will follow from the first of the following claims:

**8.3.5.3.** If \( Y = \text{Spec}(A) \) is affine, and \( \mathcal{T} = \hat{S} \), then, for all homogeneous \( f \in S_+ \),

\[
(8.3.5.4) \quad p^{-1}(X_f) = \hat{C}_f
\]

and the restriction of \( p \) to \( \hat{C}_f = \text{Spec}(S_f^{\leq}) \), thought of as a morphism from \( \hat{C}_f \) to \( X_f \), corresponds to the canonical injection \( S_{(f)} \to S_f^{\leq} \). If, further, \( f \in S_1 \), then \( \hat{C}_f \) is isomorphic to \( X_f \otimes_{\mathbb{Z}} \mathbb{Z}[T] \) (where \( T \) is an indeterminate).
Indeed, Equation (8.3.5.4) is exactly a particular case of (2.8.1.1), and the second claim is exactly the definition of \( \text{Proj}(\varphi) \) whenever \( Y \) is affine (2.8.1). Then Equation (8.3.5.2) and the fact that \( p \) is surjective show that the composition \( \mathcal{S} \rightarrow \mathcal{S} \rightarrow \mathcal{S} \) of the canonical homomorphisms is the identity on \( \mathcal{S} \). Finally, the last claim of (8.3.5.3) follows from the fact that \( S_f^E \) is isomorphic to \( S(f)[T] \) whenever \( f \in S_1 \) (2.2.1).

**Corollary (8.3.6).** — The restriction

\[
(8.3.6.1) \quad \pi : E \rightarrow X
\]
of \( p \) to \( E \) is a surjective affine morphism. If \( Y \) is affine and \( f \) homogeneous in \( S_+ \), then

\[
(8.3.6.2) \quad \pi^{-1}(X_f) = C_f
\]
and the restriction of \( \pi \) to \( C_f \) corresponds to the canonical injection \( S(f) \rightarrow S_f \). If, further, \( f \in S_1 \), then \( C_f \) is isomorphic to \( X_f \otimes_{\mathbb{Z}} \mathbb{Z}[T, T^{-1}] \) (where \( T \) is an indeterminate).

**Proof.** Equation (8.3.6.2) follows immediately from (8.3.5.3) and (8.3.2.5), and shows the surjectivity of \( \pi \); we have already seen that the immersion \( i \), restricted to \( C_f \), corresponds to the injection \( S_f^E \rightarrow S_f \) (8.3.2). Finally, the last claim is a consequence of the fact that, for \( f \in S_1 \), \( S_f \) is isomorphic to \( S(f)[T, T^{-1}] \) (2.2.1).

**Remark (8.3.7).** — Whenever \( Y \) is affine, the elements of the underlying space of \( E \) are the (not-necessarily-graded) prime ideals \( p \) of \( S \) not containing \( S_+ \), by definition of the immersion \( \varepsilon \) (8.3.2). For such an ideal \( p \), the \( p \cap S_n \) clearly satisfy the conditions of (2.1.9), and so there exists exactly one graded prime ideal \( q \) of \( S \) such that \( q \cap S_n = p \cap S_n \) for all \( n \); the map \( \pi : E \rightarrow X \) of underlying spaces can then be understood via the equation

\[
(8.3.7.1) \quad \pi(p) = q.
\]

Indeed, to prove this equation, it suffices to consider some homogeneous \( f \) in \( S_+ \) such that \( p \in D(f) \), and to note that \( q(f) \) is the inverse image of \( p_f \) under the injection \( S(f) \rightarrow S_f \).

**Corollary (8.3.8).** — If \( \mathcal{S} \) is generated by \( \mathcal{A}_t \), then the morphisms \( p \) and \( \pi \) are of finite type; for all \( x \in X \), the fibre \( p^{-1}(x) \) is isomorphic to \( \text{Spec}(k(x)[T]) \), and the fibre \( \pi^{-1}(x) \) isomorphic to \( \text{Spec}(k(x)[T, T^{-1}]) \).

**Proof.** This follows immediately from (8.3.5) and (8.3.6) by noting that, whenever \( Y \) is affine and \( S \) is generated by \( S_1 \), the \( X_f \), for \( f \in S_1 \), form a cover of \( X \) (2.3.14).

**Remark (8.3.9).** — The pointed affine cone corresponding to the graded \( \mathcal{O}_Y \)-algebra \( \mathcal{O}_Y[T] \) (where \( T \) is an indeterminate) can be identified with \( G_m = \text{Spec}(\mathcal{O}_Y[T, T^{-1}]) \), since it is exactly \( C_T \), as we have seen in (8.3.2) (see (8.4.4) for a more general result). This prescheme is canonical endowed with the structure of a “\( Y \)-scheme in commutative groups”. This idea will be explained in detail later on, but, for now, can be quickly summarised as follows. A \( Y \)-scheme in groups is a \( Y \)-scheme \( G \) endowed with two \( Y \)-morphisms, \( p : G \times_Y G \rightarrow G \) and \( s : G \rightarrow G \), that satisfy conditions formally analogous to the axioms of the composition law and the symmetry law of a group: the diagram

\[
\begin{array}{ccc}
G \times G & \xrightarrow{p \times 1} & G \times G \\
1 \times p & \downarrow & \downarrow \quad p \\
G \times G & \xrightarrow{p} & G
\end{array}
\]

should commute (“associativity”), and there should be a condition which corresponds to the fact that, for groups, the maps

\[
(x, y) \mapsto (x, x^{-1}, y) \mapsto (x, x^{-1}y) \mapsto x(x^{-1}y)
\]

and

\[
(x, y) \mapsto (x, x^{-1}, y) \mapsto (x, yx^{-1}) \mapsto (yx^{-1})x
\]
should both reduce to \((x, y) \mapsto y\); the sequence of morphisms corresponding, for example, to the first composite map is

\[
G \times G \xrightarrow{(1, s) \times 1} G \times G \xrightarrow{1 \times p} G \times G \xrightarrow{p} G
\]

and the reader should write down the second sequence.

It is immediate (I, 3.4.3) that the data of a structure of a \(Y\)-scheme in groups on a \(Y\)-scheme \(G\) is equivalent to the data, for every \(Y\)-prescheme \(Z\), of a \(\Gamma\) structure on the set \(\text{Hom}_Y(Z, G)\), where these structures should be such that, for every \(Y\)-morphism \(Z \to Z'\), the corresponding map \(\text{Hom}_Y(Z', G) \to \text{Hom}_Y(Z, G)\) is a group homomorphism. In the particular case of \(G_m\) that we consider here, \(\text{Hom}_Y(Z, G)\) can be identified with the set of \(Z\)-sections of \(Z \times Y G_m\) (I, 3.3.14), and thus with the set of \(Z\)-sections of \(\text{Spec}(\mathcal{O}_Z[T, T^{-1}])\). Finally, the same reasoning as in (I, 3.3.15) shows that this set is canonically identified with the set of invertible elements of the ring \(\Gamma(Z, \mathcal{O}_Z)\), and the group structure on this set is the structure coming from the multiplication in the ring \(\Gamma(Z, \mathcal{O}_Z)\). The reader can verify that the morphisms \(p\) and \(s\) from above are obtained in the following way: they correspond, by (1.2.7) and (1.4.6), to the homomorphisms of \(\mathcal{O}_Y\)-algebras

\[
\pi : \mathcal{O}_Y[T, T^{-1}] \longrightarrow \mathcal{O}_Y[T, T^{-1}, T', T'^{-1}],
\]

\[
\sigma : \mathcal{O}_Y[T, T^{-1}] \longrightarrow \mathcal{O}_Y[T, T^{-1}],
\]

and are entirely defined by the data of \(\pi(T) = TT'\) and \(\sigma(T) = T^{-1}\).

With this in mind, \(G_m\) can be considered as a “universal domain of operators” for every affine cone \(C = \text{Spec}(\mathcal{S})\), where \(\mathcal{S}\) is a quasi-coherent positively-graded \(\mathcal{O}_Y\)-algebra. This means that we can canonically define a \(Y\)-morphism \(G_m \times_Y C \to C\) which has the formal properties of an external law of a set endowed with a group of operators; or, again, as above for schemes in groups, we can give, for every \(Y\)-prescheme \(Z\), an external law on \(\text{Hom}_Y(Z, C)\), having the group \(\text{Hom}_Y(Z, G_m)\) as its set of operators, with the usual axioms of sets endowed with a group of operators, and a compatibility condition with respect to the \(Y\)-morphisms \(Z \to Z'\). In the current case, the morphism \(G_m \times_Y C \to C\) is defined by the data of a homomorphism of \(\mathcal{O}_Y\)-algebras \(\mathcal{S} \to \mathcal{S} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y[T, T^{-1}] = \mathcal{S}[T, T^{-1}]\), which associates, to each section \(s_n \in \Gamma(U, \mathcal{S}_n)\) (where \(U\) is an open subset of \(Y\)), the section \(s_n T^n \in \Gamma(U, \mathcal{S}_n \otimes_{\mathcal{O}_Y} \mathcal{O}_Y[T, T^{-1}])\).

Conversely, suppose that we are given a quasi-coherent, \(a\ priori\ non-graded, \mathcal{O}_Y\)-algebra, and, on \(C = \text{Spec}(\mathcal{S})\), a structure of a “\(Y\)-scheme in sets endowed with a group of operators” that has the \(Y\)-scheme in groups \(G_m\) as its domain of operators; then we canonically obtain a grading of \(\mathcal{O}_Y\)-algebras on \(\mathcal{S}\). Indeed, the data of a \(Y\)-morphism \(G_m \times_Y C \to C\) is equivalent to that of a homomorphism of \(\mathcal{O}_Y\)-algebras \(\psi : \mathcal{S} \to \mathcal{S}[T, T^{-1}]\), which can be written as \(\psi = \sum_n \psi_n T^n\), where the \(\psi_n : \mathcal{S} \to \mathcal{S}\) are homomorphisms of \(\mathcal{O}_Y\)-modules (with \(\psi_n(s) = 0\) except for finitely many \(n\) for every section \(s \in \Gamma(U, \mathcal{S})\), for any open subset \(U\) of \(Y\). We can then prove that the axioms of sets endowed with a group of operators imply that the \(\psi_n(\mathcal{S}) = \mathcal{S}_n\) define a grading (in positive or negative degree) of \(\mathcal{O}_Y\)-algebras on \(\mathcal{S}\), with the \(\psi_n\) being the corresponding projectors. We also have the notation of a structure of an “affine cone” on every affine \(Y\)-scheme, defined in a “geometric” way without any reference to any prior grading. We will not further develop this point of view here, and we leave the work of precisely formulating the definitions and results corresponding to the information given above to the reader.

### 8.4. Projective closure of a vector bundle

(8.4.1). Let \(Y\) be a prescheme, and \(\mathcal{E}\) a quasi-coherent \(\mathcal{O}_Y\)-module. If we take \(\mathcal{S}\) to be the graded \(\mathcal{O}_Y\)-algebra \(S_{\psi}(\mathcal{E})\), then Definition (8.3.1.1) shows that \(\mathcal{S}\) can be identified with \(S_{\psi}(\mathcal{S} \oplus \mathcal{O}_Y)\). With the affine cone \(\text{Spec}(\mathcal{S})\) defined by \(\mathcal{S}\) being, by definition, \(\mathcal{V}(\mathcal{E})\), and \(\text{Proj}(\mathcal{S})\) being, by definition, \(\text{Proj}(\mathcal{E})\), we see that:

**Proposition (8.4.2).** — The projective closure of a vector bundle \(\mathcal{V}(\mathcal{E})\) on \(Y\) is canonically isomorphic to \(\text{Proj}(\mathcal{S} \oplus \mathcal{O}_Y)\), and the part at infinity of the latter is canonically isomorphic to \(\text{Proj}(\mathcal{E})\).

**Remark (8.4.3).** — Take, for example, \(\mathcal{E} = \mathcal{O}_Y^r\) with \(r \geq 2\); then the pointed cones \(E\) and \(\mathcal{E}\) defined by \(\mathcal{S}\) are neither affine nor projective on \(Y\) if \(Y \neq \emptyset\). The second claim is immediate, because \(\mathcal{C} = \text{Proj}(\mathcal{O}_Y^{r+1})\) is projective on \(Y\), and the underlying spaces of \(E\) and \(\mathcal{E}\) are non-closed open subsets
of \( \hat{\mathcal{C}} \), and so the canonical immersions \( E \to \hat{\mathcal{C}} \) and \( \hat{\mathcal{E}} \to \hat{\mathcal{C}} \) are not projective (5.5.3), and we conclude by appealing to (5.5.5, v). Now, supposing, for example, that \( Y = \text{Spec}(A) \) is affine, and \( r = 2 \), then \( C = \text{Spec}(A[T_1, T_2]) \), and \( E \) is then the prescheme induced by \( C \) on the open subset \( D(T_1) \cup D(T_2) \); but we have already seen that the latter is not affine (I, 5.5.11); \( a \) fortiori \( \hat{\mathcal{E}} \) cannot be affine, since \( E \) is the open subset where the section \( z \) over \( \hat{\mathcal{E}} \) does not vanish (8.3.2).

However:

**Proposition (8.4.4).** — If \( \mathcal{L} \) is an invertible \( \mathcal{O}_Y \)-module, then there are canonical isomorphisms for both the pointed cones \( E \) and \( \hat{\mathcal{E}} \) corresponding to \( C = \text{V}(\mathcal{L}) \):

\[
(8.4.4.1) \quad \text{Spec}\left( \bigoplus_{n \in \mathbb{Z}} \mathcal{L}^\otimes n \right) \xrightarrow{\sim} E
\]

\[
(8.4.4.2) \quad \text{V}(\mathcal{L}^{-1}) \xrightarrow{\sim} \hat{\mathcal{E}}.
\]

Furthermore, there exists a canonical isomorphism from the projective closure of \( \text{V}(\mathcal{L}) \) to the projective closure of \( \text{V}(\mathcal{L}^{-1}) \) that sends the null section (resp. the part at infinity) of the former to the part at infinity (resp. the null section) of the second.

**Proof.** We have here that \( \mathcal{J} = \bigoplus_{n \geq 0} \mathcal{L}^\otimes n \); the canonical injection

\[
\mathcal{J} \to \bigoplus_{n \in \mathbb{Z}} \mathcal{L}^\otimes n
\]

defines a canonical dominant morphism

\[
(8.4.4.3) \quad \text{Spec}\left( \bigoplus_{n \in \mathbb{Z}} \mathcal{L}^\otimes n \right) \to \text{V}(\mathcal{L}) = \text{Spec}\left( \bigoplus_{n \geq 0} \mathcal{L}^\otimes n \right)
\]

and it suffices to prove that this morphism is an isomorphism from the scheme \( \text{Spec}(\bigoplus_{n \in \mathbb{Z}} \mathcal{L}^\otimes n) \) to \( E \). Since the questions is local on \( Y \), we can assume that \( Y = \text{Spec}(A) \) is affine and that \( \mathcal{L} = \mathcal{O}_Y \), and so \( \mathcal{J} = (A[T])^{-} \) and \( \bigoplus_{n \in \mathbb{Z}} \mathcal{L}^\otimes n = (A[T, T^{-1}])^{-} \). But \( A[T, T^{-1}] \) is the ring of fractions \( A[T]^{-} \) of \( A[T] \), and thus (8.4.4.3) identifies \( \bigoplus_{n \in \mathbb{Z}} \mathcal{L}^\otimes n \) with the prescheme induced by \( C = \text{V}(\mathcal{L}) \) on the open subset \( D(T) \); the complement \( V(T) \) of this open subset in \( C \) is the underlying space of the closed subscheme of \( C \) defined by the ideal \( TA[T] \), which is exactly the null section of \( C \), and so \( E = D(T) \).

The isomorphism in (8.4.4.2) will be a consequence of the last claim, since \( \text{V}(\mathcal{L}^{-1}) \) is the complement of the part at infinity in the projective closure of \( \mathcal{E} \) and \( \hat{\mathcal{E}} \) is the complement of the null section of the projective closure \( \hat{\mathcal{C}} = \text{V}(\mathcal{L}) \). But these projective closures are \( \text{P}(\mathcal{L}^{-1} \oplus \mathcal{O}_Y) \) and \( \text{P}(\mathcal{L} \oplus \mathcal{O}_Y) \) (respectively); but we can write \( \mathcal{L} \oplus \mathcal{O}_Y = \mathcal{L} \oplus (\mathcal{L}^{-1} \oplus \mathcal{O}_Y) \). The existence of the desired canonical isomorphism then follows from (4.1.4), and everything reduces to showing that this isomorphism swaps the null sections and the parts at infinity. For this, we can reduce to the case where \( Y = \text{Spec}(A) \) is affine, \( L = Ac \), and \( L^{-1} = Ac' \), with the canonical isomorphism \( L \otimes L^{-1} \to A \) sending \( c \otimes c' \) to the element 1 of \( A \). Then \( S(L \oplus A) \) is the tensor product of \( A[z] \) with \( \bigoplus_{n \geq 0} Ac^\otimes h \), and \( S(L^{-1} \oplus A) \) is the tensor product of \( A[z] \) with \( \bigoplus_{n \geq 0} Ac'^\otimes h \), and the isomorphism defined in (4.1.4) sends \( z^h \otimes c'^\otimes(n-h) \) to the element \( z^{n-h} \otimes c'^h \). But, in \( \text{P}(\mathcal{L}^{-1} \oplus \mathcal{O}_Y) \), the part at infinity is the set of points where the section \( z \) vanishes, and the null section is the set of points where the section \( c' \) vanishes; since we have analogous definitions for \( \text{P}(\mathcal{L} \oplus \mathcal{O}_Y) \), the conclusion follows immediately from the above explanation. \( \square \)
8.5. Functorial behaviour

Let \( Y \) and \( Y' \) be prescheme, \( q : Y' \to Y \) a morphism, and \( \mathcal{O} \) (resp. \( \mathcal{O}' \)) a positively-graded quasi-coherent \( \mathcal{O}_Y \)-algebra (resp. positively-graded quasi-coherent \( \mathcal{O}_{Y'} \)-algebra). Consider a \( q \)-morphism of graded algebras

\[
\varphi : \mathcal{O} \to \mathcal{O}'.
\]

We know \((1.5.6)\) that this corresponds, canonically, to a morphism

\[
\Phi = \text{Spec}(\varphi) : \text{Spec}(\mathcal{O}') \to \text{Spec}(\mathcal{O})
\]

such that the diagram

\[
\begin{array}{ccc}
C' & \xrightarrow{\Phi} & C \\
\downarrow & & \downarrow \\
Y' & \xrightarrow{q} & Y
\end{array}
\]

commutes, where we write \( C = \text{Spec}(\mathcal{O}) \) and \( C' = \text{Spec}(\mathcal{O}') \). Suppose, further, that \( \mathcal{O}_0 = \mathcal{O}_Y \) and \( \mathcal{O}'_0 = \mathcal{O}_{Y'} \); let \( \varepsilon : Y \to C \) and \( \varepsilon' : Y' \to C' \) be the canonical immersions \((8.3.2)\); we then have a commutative diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{q} & Y \\
\downarrow & & \downarrow \\
C' & \xrightarrow{\Phi} & C
\end{array}
\]

which corresponds to the diagram

\[
\begin{array}{ccc}
\mathcal{O}_Y & \xrightarrow{q} & \mathcal{O}_Y' \\
\downarrow & & \downarrow \\
\mathcal{O}_C & \xrightarrow{\Phi} & \mathcal{O}_C'
\end{array}
\]

where the vertical arrows are the augmentation homomorphisms, and so the commutativity follows from the hypothesis that \( \varphi \) is assumed to be a homomorphism of graded algebras.

**Proposition (8.5.2).** — If \( E \) (resp. \( E' \)) is the pointed affine cone defined by \( \mathcal{O} \) (resp. \( \mathcal{O}' \)), then \( \Phi^{-1}(E) \subset E' \); if, further, \( \text{Proj}(\varphi) : \text{Proj}(\mathcal{O}) \to \text{Proj}(\mathcal{O}') \) is everywhere defined (or, equivalently, if \( \text{Proj}(\varphi) = \text{Proj}(\mathcal{O}') \)), then \( \Phi^{-1}(E) = E' \), and conversely.

**Proof.** The first claim follows from the commutativity of \((8.5.1.3)\). To prove the second, we can restrict to the case where \( Y = \text{Spec}(A) \) and \( Y' = \text{Spec}(A') \) are affine, and \( \mathcal{O} = \mathcal{S} \) and \( \mathcal{O}' = \mathcal{S}' \). For every homogeneous \( f \) in \( S_+ \), writing \( f' = \varphi(f) \), we have that \( \Phi^{-1}(C_f) = C'_{f'} \) \((1.2.2.4.1)\); saying that \( G(\varphi) = \text{Proj}(S') \) implies that the radical (in \( S'_+ \)) of the ideal generated by the \( f' = \varphi(f) \) is \( S'_+ \) itself \((2.8.1)\) and \((2.3.14)\)), and this is equivalent to saying that the \( C'_{f'} \) cover \( E' \) \((8.3.4.4)\). \( \square \)

**Proposition (8.5.3).** The \( q \)-morphism \( \varphi \) canonically extends to a \( q \)-morphism of graded algebras

\[
\hat{\varphi} : \hat{\mathcal{O}} \to \hat{\mathcal{O}}'
\]

by letting \( \hat{\varphi}(z) = z \). This induces a morphism

\[
\hat{\Phi} = \text{Proj}(\hat{\varphi}) : \text{Proj}(\hat{\mathcal{O}}) \to \hat{\mathcal{C}} = \text{Proj}(\hat{\mathcal{O}})
\]

such that the diagram

\[
\begin{array}{ccc}
G(\hat{\varphi}) & \xrightarrow{\Phi} & \hat{\mathcal{C}} \\
\downarrow & & \downarrow \\
Y' & \xrightarrow{q} & Y
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{O}_Y & \xrightarrow{q} & \mathcal{O}_Y' \\
\downarrow & & \downarrow \\
\hat{\mathcal{O}}_C & \xrightarrow{\hat{\Phi}} & \hat{\mathcal{O}}_C'
\end{array}
\]
commutes (3.5.6). It follows immediately from the definitions that, if we write \( i : C \to \hat{C} \) and \( i' : C' \to \hat{C}' \) to mean the canonical open immersions (8.3.2), then \( i'(C') \subset G(\hat{\varphi}) \), and the diagram

(8.5.3.2)

\[
\begin{array}{ccc}
C' & \xrightarrow{\Phi} & C \\
\downarrow i & & \downarrow i' \\
G(\hat{\varphi}) & \xrightarrow{\Phi} & \hat{C}
\end{array}
\]

commutes. Finally, if we let \( X = \text{Proj}(\mathscr{S}) \) and \( X' = \text{Proj}(\mathscr{S}') \), and if \( j : X \to \hat{C} \) and \( j' : X' \to \hat{C}' \) are the canonical closed immersions (8.3.2), then it follows from the definition of these immersions that \( j'(G(\varphi)) \subset G(\hat{\varphi}) \), and that the diagram

(8.5.3.3)

\[
\begin{array}{ccc}
G(\varphi) & \xrightarrow{\text{Proj}(\varphi)} & X \\
\downarrow j' & & \downarrow j \\
G(\hat{\varphi}) & \xrightarrow{\Phi} & \hat{C}
\end{array}
\]

commutes.

**Proposition (8.5.4).** — If \( \hat{E} \) (resp. \( \hat{E}' \)) is the pointed projective cone defined by \( \mathscr{S} \) (resp. by \( \mathscr{S}' \)), then \( \hat{\Phi}^{-1}(\hat{E}) \subset \hat{E}' \); furthermore, if \( p : \hat{E} \to X \) and \( p' : \hat{E}' \to X' \) are the canonical retractions, then \( p'(\hat{\Phi}^{-1}(\hat{E})) \subset G(\hat{\varphi}) \), and the diagram

(8.5.4.1)

\[
\begin{array}{ccc}
\hat{\Phi}^{-1}(\hat{E}) & \xrightarrow{\hat{\Phi}} & \hat{E} \\
\downarrow p' & & \downarrow p \\
G(\varphi) & \xrightarrow{\text{Proj}(\varphi)} & X
\end{array}
\]

commutes. If \( \text{Proj}(\varphi) \) is everywhere defined, then so too is \( \hat{\Phi} \), and we have that \( \hat{\Phi}^{-1}(\hat{E}) = \hat{E}' \).

**Proof.** The first claim follows from the commutativity of Diagrams (8.5.1.3) and (8.5.3.2), and the two following claims from the definition of the canonical retractions (8.3.5) and the definition of \( \hat{\varphi} \). To see that \( \hat{\Phi} \) is everywhere defined whenever \( \text{Proj}(\varphi) \) is, we can restrict to the case where \( Y = \text{Spec}(A) \) and \( Y' = \text{Spec}(A') \) are affine, and where \( \mathscr{S} = \mathcal{S} \) and \( \mathscr{S}' = \mathcal{S}' \); the hypothesis is that, when \( f \) runs over the set of homogeneous elements of \( S_+ \), the radical in \( S'_+ \) of the ideal generated in \( S'_+ \) by the \( \varphi(f) \) is \( S'_+ \) itself; we thus immediately conclude that the radical in \( (S'|z)_+ \) of the ideal generated by \( z \) and the \( \varphi(f) \) is \( (S'|z)_+ \) itself, whence our claim; this also shows that \( \hat{E}' \) is the union of the \( \hat{C}'_{\varphi(f)} \), and hence equal to \( \hat{\Phi}^{-1}(\hat{E}) \). \( \square \)

**Corollary (8.5.5).** — Whenever \( \text{Proj}(\varphi) \) is everywhere defined, the inverse image under \( \hat{\Phi} \) of the underlying space of the part at infinity (resp. of the vertex prescheme) of \( \hat{C}' \) is the underlying space of the part at infinity (resp. of the vertex prescheme) of \( \hat{C} \).

**Proof.** This follows immediately from (8.5.4) and (8.5.2), taking into account the equalities (8.3.4.1) and (8.3.4.2). \( \square \)
8.6. A canonical isomorphism for pointed cones

Let $Y$ be a prescheme, $\mathcal{S}$ a quasi-coherent positively-graded $\mathcal{O}_Y$-algebra such that $\mathcal{S}_0 = \mathcal{O}_Y$, and let $X$ be the Y-scheme $\text{Proj}(\mathcal{S})$. We are going to apply the results of (8.5) to the case where $Y' = X$, and $q : X \to Y$ is the structure morphism; let

\[
\mathcal{S}_X = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n)
\]

which is a quasi-coherent graded $\mathcal{O}_X$-algebra, with multiplication defined by means of the canonical homomorphisms (3.2.6.1)

\[
\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \to \mathcal{O}_X(m + n)
\]

whose associativity is ensured by the commutative diagram in (2.5.11.4). Let $\mathcal{S}'$ be the quasi-coherent positively-graded $\mathcal{O}_X$-subalgebra $\mathcal{S}_X^+ = \bigoplus_{n \geq 0} \mathcal{O}_X(n)$ of $\mathcal{S}_X$.

Finally, consider the canonical $q$-morphism

\[
a : \mathcal{S} \to \mathcal{S}_X^+
\]

defined in (3.3.2.3) as a homomorphism $\mathcal{S} \to q_*(\mathcal{S}_X)$, but which clearly sends $\mathcal{S}$ to $q_*(\mathcal{S}_X)$. Write

\[
C_X = \text{Spec}(\mathcal{S}_X), \quad \hat{C}_X = \text{Proj}(\mathcal{S}_X^+[\mathbf{Z}]), \quad X' = \text{Proj}(\mathcal{S}_X^+)
\]

and denote by $E_X$ and $\hat{E}_X$ the corresponding pointed affine and pointed projective cones (respectively); denote the canonical morphisms defined in (8.3) by $\epsilon_X : X \to C_X, i_X : C \to \hat{C}_X, j_X : X' \to \hat{C}_X, p_X : \hat{E}_X \to X'$, and $\pi_X : E_X \to X'$.

**Proposition (8.6.2).** — The structure morphism $u : X' \to X$ is an isomorphism, and the morphism $\text{Proj}(a)$ is everywhere defined and identical to $u$. The morphism $\text{Proj}(\hat{a}) : \hat{C}_X \to \hat{C}$ is everywhere defined, and its restrictions to $\hat{E}_X$ and $E_X$ are isomorphisms to $\bar{E}$ and $E$ (respectively). Finally, if we identify $X'$ with $X$ via $u$, then the morphisms $p_X$ and $\pi_X$ are identified with the structure morphisms of the $X$-preschemes $\hat{E}_X$ and $E_X$.

**Proof.** We can clearly restrict to the case where $Y = \text{Spec}(A)$ is affine, and $\mathcal{S} = \hat{S}$; then $X$ is the union of affine open subsets $X_f$, where $f$ runs over the set of homogeneous elements of $S_+$, with the ring of each $X_f$ being $S(f)$. It follows from (8.2.7.1) that

\[
\Gamma(X_f, \mathcal{S}_X^+) = S_f^+.
\]

So $u^{-1}(X_f) = \text{Proj}(S_f^+)$. But if $f \in S_d$ ($d > 0$), then $\text{Proj}(S_f^+)$ is canonically isomorphic to $\text{Proj}((S_f^+)^{[d]})$ (2.4.7), and we also know that $(S_f^+)^{[d]} = (S^{[d]})^f_+$ can be identified with $S(f)[T]$ (2.2.1) by the map $T \mapsto f/1$; we thus conclude (3.1.7) that the structure morphism $u^{-1}(X_f) \to X_f$ is an isomorphism, whence the first claim. To prove the second, note that the restriction $u^{-1}(X_f) \cap G(a) \to X = \text{Proj}(S)$ of $\text{Proj}(a)$ corresponds to the canonical map $x \mapsto x/1$ from $S$ to $S_f^+$ (2.6.2); we thus deduce, first of all, that $\bar{G}(a) = X'$, and then, taking into account the fact that $u^{-1}(X_f) = (u^{-1}(X_f))_{f/1}$, that it follows from (2.8.1.1) that the image of $u^{-1}(X_f)$ under $\text{Proj}(a)$ is contained in $X_f$, and the restriction of $\text{Proj}(a)$ to $u^{-1}(X_f)$, thought of as a morphism to $X_f = \text{Spec}(S(f))$, is indeed identical to that of $u$. Finally, applying (8.3.5.4) to $p_X$ instead of $p$, we see that $p_X^{-1}(u^{-1}(X_f)) = \text{Spec}((S_f^+)_f^{[d]})$, and this open subset is, by (8.5.1.1), the inverse image under $\text{Proj}(\hat{a})$ of $p^{-1}(X_f) = \text{Spec}(S_f^+)$ (8.3.5.3). Taking (8.2.3.2) into account, the restriction of $\text{Proj}(\hat{a})$ to $p_X^{-1}(u^{-1}(X_f))$ corresponds to the isomorphism inverse to (8.2.7.2), restricted to $S_f^{\leq}$, whence the third claim; the last claim is evident by definition.

We note also that it follows from the commutative diagram in (8.5.3.2) that the restriction to $C_X$ of $\text{Proj}(\hat{a})$ is exactly the morphism $\text{Spec}(a)$. \qed

**Corollary (8.6.3).** — Considered as $X$-schemes, $\hat{E}_X$ is canonically isomorphic to $\text{Spec}(\mathcal{S}_X^\leq)$, and $E_X$ to $\text{Spec}(\mathcal{S}_X)$.
PROOF. Since we know that the morphisms $p_X$ and $\pi_X$ are affine ((8.3.5) and (8.3.6)), it suffices (given (1.3.1)) to prove the corollary in the case where $Y = \text{Spec}(A)$ is affine and $\mathscr{S} = \mathfrak{S}$. The first claim follows from the existence of the canonical isomorphisms (8.2.7.2) $(S_f^\mathfrak{S})_{f/1} \sim \to S_f^\mathfrak{S}$ and from the fact that these isomorphisms are compatible with the map sending $f$ to $fg$ (where $f$ and $g$ are homogeneous in $S_+$). Similarly, applying (8.3.6.2) to $\pi_X$ instead of $\pi$, we see that $\pi^{-1}_X(u^{-1}(X_f)) = \text{Spec}((S_f^\mathfrak{S})_{f/1})$ for $f$ homogeneous in $S_+$, and the second claim then follows from the existence of the canonical isomorphisms (8.2.7.2) $(S_f^\mathfrak{S})_{f/1} \sim \to S_f$.

We can then say that $\hat{C}_X$, thought of as an $X$-scheme, is given by gluing the affine $X$-schemes $C_X = \text{Spec}(\mathscr{S}_X^\mathfrak{S})$ and $\hat{E}_X = \text{Spec}(\mathscr{S}_X^\mathfrak{S})$ over $X$, where the intersection of the two affine $X$-schemes is the open subset $E_X = \text{Spec}(\mathscr{S}_X)$.

**Corollary (8.6.4).** — Assume that $\mathcal{O}_X(1)$ is an invertible $\mathcal{O}_X$-module, and that $\mathscr{S}$ is isomorphic to $\bigoplus_{n \in \mathbb{Z}} (\mathcal{O}_X(1))^{\otimes n}$ (which will be the case, in particular, whenever $\mathscr{S}$ is generated by $\mathscr{S}_1$ ((3.2.5) and (3.2.7))). Then the pointed projective cone $\hat{E}$ can be identified with the rank-1 vector bundle $\mathcal{V}(\mathcal{O}_X(-1))$ on $X$, and the pointed affine cone $E$ with the subscheme of this vector bundle induced on the complement of the null section. With this identification, the canonical retraction $\hat{E} \to X$ is identified with the structure morphism of the $X$-scheme $\mathcal{V}(\mathcal{O}_X(-1))$. Finally, there exists a canonical $Y$-morphism $\mathcal{V}(\mathcal{O}_X(1)) \to C$, whose restriction to the complement of the null section of $\mathcal{V}(\mathcal{O}_X(1))$ is an isomorphism from this complement to the pointed affine cone $E$.

**Proof.** If we write $\mathcal{L} = \mathcal{O}_X(1)$, then $\mathscr{S}_X^\mathfrak{S}$ is identical to $S_{\mathcal{O}_X(1)}(\mathcal{L})$, and so $\hat{E}_X$ is canonically identified with $\mathcal{V}(\mathcal{L}^{-1})$, by (8.6.3), and $C_X$ with $\mathcal{V}(\mathcal{L})$. The morphism $\mathcal{V}(\mathcal{L}) \to C$ is the restriction of $\text{Proj}(\mathcal{F})$, and the claims of the corollary are then particular cases of (8.6.2).

We note that the inverse image under the morphism $\mathcal{V}(\mathcal{O}_X(1)) \to C$ of the underlying space of the vertex prescheme of $C$ is the underlying space of the null section of $\mathcal{V}(\mathcal{O}_X(1))$ (8.5.5); but, in general, the corresponding subschemes of $C$ and of $\mathcal{V}(\mathcal{O}_X(1))$ are not isomorphic. This problem will be studied below.

### 8.7. Blowing up based cones

**8.7.1.** Under the conditions of (8.6.1), we have, writing $r = \text{Proj}(\mathcal{F})$, a commutative diagram

$$
\begin{array}{ccc}
X & \overset{j_X}{\longrightarrow} & \hat{C}_X \\
\downarrow q & & \downarrow r \\
Y & \overset{j_Y}{\longrightarrow} & \hat{C}
\end{array}
$$

(8.7.1)

by (8.5.1.3) and (8.5.3.2); furthermore, the restriction of $r$ to the complement $\hat{C}_X - i_X(\varepsilon_X(x))$ of the null section is an isomorphism to the complement $\hat{C} - i(\varepsilon(Y))$ of the null section, by (8.6.2). If we suppose, to simplify things, that $Y$ is affine, that $\mathcal{S}$ is of finite type and generated by $\mathcal{S}_1$, and that $X$ is projective over $Y$ and $\hat{C}_X$ projective over $X$ (5.5.1), then $\hat{C}_X$ is projective over $Y$ (5.5.5, ii), and a fortiori over $\hat{C}$ (5.5.5, v). We then have a projective $Y$-morphism $r : \hat{C}_X \to \hat{C}$ (whose restriction to $C_X$ is a projective $Y$-morphism $C_X \to C$) that contracts $X$ to $Y$ (II) and that induces an isomorphism when we restrict to the complements of $X$ and $Y$. We thus have a connection between $C_X$ and $C$, analogous to that which exists between a blow-up prescheme and the original prescheme (8.1.3). We will effectively show that $C_X$ can be identified with the homogeneous spectrum of a graded $\mathcal{O}_C$-algebra.

**8.7.2.** Keeping the notation of (8.6.1), consider, for all $n > 0$, the quasi-coherent ideal

$$
\mathcal{I}_n = \bigoplus_{m \geq n} \mathcal{I}_m
$$

of the graded $\mathcal{O}_Y$-algebra $\mathcal{S}$. It is clear that

$$
\mathcal{I}_0 = \mathcal{S}, \quad \mathcal{I}_n \subset \mathcal{I}_m \quad \text{for } m \leq n
$$

(8.7.2)

$$
\mathcal{I}_n \mathcal{I}_m \subset \mathcal{I}_{n+m}.
$$

(8.7.2.3)
Consider the $\mathcal{O}_C$-module associated to $\mathcal{I}_n$, which is a quasi-coherent ideal of $\mathcal{O}_C = \mathcal{F} (1.4.4)$

(8.7.2.4) \[ \mathcal{I}_n = (\mathcal{I}_n)^{\sim}. \]

We thus deduce, from (8.7.2.2) and (8.7.2.3), using (1.4.4) and (1.4.8.1), the analogous formulas

(8.7.2.5) \[ \mathcal{I}_0 = \mathcal{O}_C, \quad \mathcal{I}_n \subseteq \mathcal{I}_n \] for $m \leq n$

(8.7.2.6) \[ \mathcal{I}_n \mathcal{I}_{[n]} \subseteq \mathcal{I}_{[m+n]}. \]

We are thus in the setting of (8.1.1), which leads us to introduce the quasi-coherent graded $\mathcal{O}_C$-algebra

(8.7.2.7) \[ \mathcal{I}^\sim = \bigoplus_{n \geq 0} \mathcal{I}_n = \left( \bigoplus_{n \geq 0} \mathcal{I}_n \right)^{\sim}. \]

**Proposition (8.7.3).** — There is a canonical $C$-isomorphism

(8.7.3.1) \[ h : CX \xrightarrow{\sim} \text{Proj}(\mathcal{I}^\sim). \]

**Proof.** Suppose first of all that $Y = \text{Spec}(A)$ is affine, so that $\mathcal{I} = \mathcal{S}$, with $S$ a positively-graded $A$-algebra, and $C = \text{Spec}(S)$. Definition (8.2.7.4) then shows, with the notation of (8.2.6), that $\mathcal{I}^\sim = (S^\sim)^{\sim}$. To define (8.7.3.1), consider a homogeneous element $f \in S_{d}(d > 0)$ and the corresponding element $f^\sim \in S^\sim (8.2.6)$; the $S$-isomorphism in (8.2.7.3) then defines a $C$-isomorphism

(8.7.3.2) \[ \text{Spec}(S^\sim_f) \xrightarrow{\sim} \text{Spec}(S^\sim_{g^\sim}). \]

But with the notation of (8.6.2), if $v : CX \to X$ is the structure morphism, then it follows from (8.6.2.1) that $v^{-1}(X_f) = \text{Spec}(S^\sim_{f^\sim})$. We also have that $\text{Spec}(S^\sim_{g^\sim}) = D_+(f^\sim)$, which means that (8.7.3.2) defines an isomorphism $v^{-1}(X_f) \to D_+(f^\sim)$. Furthermore, if $g \in S_e (e > 0)$, then the diagram

\[
\begin{array}{ccc}
v^{-1}(X_f) & \xrightarrow{\sim} & D_+(f^\sim) \\
\downarrow & & \downarrow \\
v^{-1}(X_f) & \xrightarrow{\sim} & D_+(f^\sim)
\end{array}
\]

commutes, by definition of the isomorphism in (8.2.7.3). Finally, by definition, $S_+$ is generated by the homogeneous $f$, and so it follows from (8.2.10, iv) and from (2.3.14) that the $D_(f^\sim)$ form a cover of $\text{Proj}(S^\sim)$, and that the $v^{-1}(X_f)$ form a cover of $CX$, since the $X_f$ form a cover of $X$; in this case, we have thus defined the isomorphism (8.7.3.1).

To prove (8.7.3) in the general case, it suffices to show that, if $U$ and $U'$ are affine open subsets of $Y$, given by rings $A$ and $A'$ (respectively), and such that $U' \subseteq U$, then, setting $\mathcal{I}|U = \mathcal{S}$ and $\mathcal{I}|U' = \mathcal{S}'$, the diagram

(8.7.3.3) \[
\begin{array}{ccc}
C_{U'} & \longrightarrow & \text{Proj}(S^\sim) \\
\downarrow & & \downarrow \\
C_U & \longrightarrow & \text{Proj}(S^\sim)
\end{array}
\]

commutes. But $S$ is canonically identified with $S \otimes A A'$, and so $S^\sim$ is canonically identified with $S^\sim \otimes S S' = S^\sim \otimes A A'$; thus $\text{Proj}(S^\sim) = \text{Proj}(S^\sim) \times_U U'$ (2.8.10); similarly, if $X = \text{Proj}(S)$ and $X' = \text{Proj}(S')$, then $X' = X \times_U U'$ and $\mathcal{I}|X' = \mathcal{I}|X \otimes \mathcal{O}_U U'$ (3.5.4), or, equivalently, $\mathcal{I}|X' = j^*(\mathcal{I}|X)$, where $j$ is the projection $X' \to X$. We then (1.5.2) have that $C_{U'} = C_U X' = C_U \times_U U'$, and the commutativity of (8.7.3.3) is then immediate. □
Remark (8.7.4). — (i) The end of the proof of (8.7.3) can be immediately generalised in the following way. Let \( g : Y' \to Y \) be a morphism, \( \mathcal{I}' = g^*(\mathcal{I}) \), and \( X' = \text{Proj}(\mathcal{I}') \); then we have a commutative diagram

\[
\begin{array}{ccc}
C_{X'} & \longrightarrow & \text{Proj}(\mathcal{I}') \\
\downarrow & & \downarrow \\
C_X & \longrightarrow & \text{Proj}(\mathcal{I})
\end{array}
\]

Now let \( \varphi : \mathcal{I}'' \to \mathcal{I} \) be a homomorphism of graded \( \mathcal{O}_Y \)-algebras such that, if we write \( X'' = \text{Proj}(\mathcal{I}'') \), then \( u = \text{Proj}(\varphi) : X \to X'' \) is everywhere defined; we also have a \( Y \)-morphism \( v : C \to C'' \) (with \( C'' = \text{Spec}(\mathcal{O}_Y) \)) such that \( \varphi(v) = \varphi \), and, since \( \varphi \) is a homomorphism of graded algebras, \( \varphi \) induces a \( \nu \)-morphism of graded algebras \( \psi : \mathcal{I}'' \to \mathcal{I} \). Furthermore, it follows from (8.2.10, iv) and from the hypothesis on \( \varphi \) that \( \text{Proj}(\psi) \) is everywhere defined. Finally, taking (3.5.6.1) into account, there is a canonical \( \nu \)-morphism \( \mathcal{I}'' \to \mathcal{I} \), whence (1.5.6) a morphism \( w : C'' \to C \). With this in mind, the diagram

\[
\begin{array}{ccc}
C_{X''} & \rightsquigarrow & \text{Proj}(\mathcal{I}'') \\
\downarrow w & & \downarrow \text{Proj}(\psi) \\
C_X & \rightsquigarrow & \text{Proj}(\mathcal{I})
\end{array}
\]

is commutative, as we can immediately verify by restricting to the case where \( Y \) is affine.

(ii) Note that, by (8.7.2.5) and (8.7.2.6), we have \( \mathcal{I}''_{\mathfrak{m}} \subset \mathcal{I}_{\mathfrak{m}} \subset \mathcal{I}_1 \) for all \( m > 0 \). But, by definition, \( \mathcal{I}_1 = (\mathcal{I}^+) \), and so \( \mathcal{I}_1 \) defines the closed subscheme \( \mathcal{E}(Y) \) in \( C \) (1.4.10) and (8.3.2)); we thus conclude that, for all \( m > 0 \), the support of \( \mathcal{O}_C / \mathcal{I}_m \) is contained in the underlying space of the vertex prescheme \( \mathcal{E}(Y) \); on the inverse image of the pointed affine cone \( E \), the structure morphism \( \text{Proj}(\mathcal{I}^2) \to C \) thus restricts to an isomorphism (by (8.7.3) and (8.7.1)). Furthermore, by canonically identifying \( C \) with an open subset of \( \hat{C} \) (8.3.3), we can clearly extend the ideals \( \mathcal{I}_m \) of \( \mathcal{O}_C \) to ideals \( \mathcal{I}_m \) of \( \mathcal{O}_{\hat{C}} \), by asking for it to agree with \( \mathcal{O}_{\hat{C}} \) on the open subset \( \tilde{E} \) of \( \hat{C} \). If we define \( \mathcal{T} = \bigoplus_{m \geq 0} \mathcal{I}_m \), which is a quasi-coherent graded \( \mathcal{O}_{\hat{C}} \)-algebra, we can extend the isomorphism (8.7.3.1) to a \( \hat{C} \)-isomorphism

\[
\mathcal{C}_X \xrightarrow{\sim} \text{Proj}(\mathcal{T}).
\]

Indeed, over \( \tilde{E} \), it follows from the above that \( \text{Proj}(\mathcal{T}) \) is canonically identified with \( \hat{E} \), and we thus define the isomorphism (8.7.4.3) over \( \hat{E} \) by asking for it to agree with the canonical isomorphism \( \hat{E}_X \to \hat{E} \) (8.6.2); it is clear that this isomorphism and (8.7.3.1) then agree over \( \tilde{E} \).

Corollary (8.7.5). — Suppose that there exists some \( n_0 > 0 \) such that

\[
\mathcal{I}_{n+1} = \mathcal{I}_1 \mathcal{I}_n \quad \text{for } n \geq n_0.
\]

Then the vertex subscheme (7) of \( C_X \) (isomorphic to \( X \)) is the inverse image under the canonical morphism \( r : C_X \to C \) of the vertex subscheme of \( C \) (isomorphic to \( Y \)). Conversely, if this property is true, and if we further assume that \( Y \) is Noetherian and that \( \mathcal{I} \) is of finite type, then there exists some \( n_0 > 0 \) such that (8.7.5.1) holds true.

Proof. The first claim being local on \( Y \), we can assume that \( Y = \text{Spec}(A) \) is affine, so that \( \mathcal{I} = S \), with \( S \) a positively-graded \( A \)-algebra. The claim then follows from (8.2.12), since \( \text{Proj}(S \otimes S) = C_X \times C \mathcal{E}(Y) \) (by the identification in (8.7.3.1)), or, in other words, since this prescheme is the inverse image of \( \mathcal{E}(Y) \) in \( C_X \) (1.4.4.1). The converse also follows from (8.2.12) whenever \( Y \) is Noetherian affine and \( S \) is of finite type. If \( Y \) is Noetherian (but not necessarily affine) and \( \mathcal{I} \) is of finite type, then there exists a finite cover of \( Y \) by Noetherian affine open subsets \( U_i \), and we then deduce from the above that, for all \( i \), there exists an integer \( n_i \) such that \( \mathcal{I}_{n+1}|_U = (\mathcal{I}_1|_U)(\mathcal{I}_n|_U) \) for \( n \geq n_i \); the largest of the \( n_i \) then ensures that (8.7.5.1) holds true. \( \square \)
(8.7.6). Now consider the C-prescheme $Z$ given by blowing up the vertex subprescheme $\epsilon(Y)$ in the affine cone $C$; by Definition (8.1.3), it is exactly the prescheme $\text{Proj}(\bigoplus_{n \geq 0} \mathcal{S}^n)$; the canonical injection

$$(8.7.6.1) \quad \iota : \bigoplus_{n \geq 0} \mathcal{S}^n \hookrightarrow \mathcal{S}^n$$

defines (by the identification in (8.7.3)) a canonical dominant $C$-morphism

$$(8.7.6.2) \quad G(\iota) \rightarrow Z$$

where $G(\iota)$ is an open subset of $C_X$ (3.5.1); note that it could be the case that $G(\iota) \neq C_X$, as shown by the example where $Y = \text{Spec}(K)$, with $K$ a field, and $\mathcal{S} = \bar{S}$, with $S = K[y]$, where $y$ is an indeterminate of degree 2; if $R_n$ denotes the set $(S_+)^n$, thought of as a subset of $S_{[n]} = S^n$, then $S^n$ is not the radical in $S^+_n$ of the ideal generated by the union of the $R_n$ (cf. (2.3.14)).

**Corollary (8.7.7).** — Assume that there exists some $n_0 > 0$ such that

$$(8.7.7.1) \quad \mathcal{S} = \mathcal{S}_n^1 \quad \text{for } n \geq n_0.$$  

Then the canonical morphism (8.7.6.2) is everywhere defined, and is an isomorphism $C_X \xrightarrow{\sim} Z$. Conversely, if this property is true, and if we further assume that $Y$ is Noetherian and that $\mathcal{S}$ is of finite type, then there exists some $n_0$ such that (8.7.7.1) holds true.

**Proof.** The first claim is local on $Y$, and thus follows from (8.2.14); the converse follows similarly, arguing as in (8.7.5). □

**Remark (8.7.8).** — Since condition (8.7.7.1) implies (8.7.5.1), we see that, whenever it holds true, not only can $C_X$ be identified with the prescheme given by blowing up the vertex (identified with $Y$) of the affine cone $C$, but also the vertex (identified with $X$) of $C_X$ can be identified with the closed subprescheme given by the inverse image of the vertex $Y$ of $C$. Furthermore, hypothesis (8.7.7.1) implies that, on $X = \text{Proj}(\mathcal{S})$, the $\mathcal{O}_X$-modules $\mathcal{O}_X(n)$ are invertible ((3.2.5) and (3.2.9)), and that $\mathcal{O}_X(n) = \mathcal{L}^\oplus_n$ with $\mathcal{L} = \mathcal{O}_X(1)$ ((3.2.7) and (3.2.9)); by Definition (8.6.1.1), $C_X$ is thus the vector bundle $V(\mathcal{L})$ on $X$, and its vertex is the null section of this vector bundle.

### 8.8. Ample sheaves and contractions

(8.8.1). Let $Y$ be a prescheme, $f : X \rightarrow Y$ a separated and quasi-compact morphism, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module that is ample relative to $f$. Consider the positively-graded $\mathcal{O}_Y$-algebra

$$(8.8.1.1) \quad \mathcal{S} = \mathcal{O}_Y \oplus \bigoplus_{n \geq 1} f_*(\mathcal{L}^\oplus_n)$$

which is quasi-coherent (I, 9.2.2, a). There is a canonical homomorphisms of graded $\mathcal{O}_X$-algebras

$$(8.8.1.2) \quad \tau : f^*(\mathcal{S}) \rightarrow \bigoplus_{n \geq 0} \mathcal{L}^\oplus_n$$

which, in degrees $\geq 1$, agrees with the canonical homomorphism $\sigma : f^*(f_*(\mathcal{L}^\oplus_n)) \rightarrow \mathcal{L}^\oplus_n$ (0, 4.4.3), and is the identity in degree 0. The hypothesis that $\mathcal{L}$ is $f$-ample then implies ((4.6.3) and (3.6.1)) that the corresponding $Y$-morphism

$$r = r_{\mathcal{L}, f} : X \rightarrow P = \text{Proj}(\mathcal{S})$$

is everywhere defined and is a dominant open immersion, and that

$$(8.8.1.3) \quad r^*(\mathcal{O}_P(n)) = \mathcal{L}^\oplus_n \quad \text{for all } n \in \mathbb{Z}.$$  

**Proposition (8.8.2).** — Let $C = \text{Spec}(\mathcal{S})$ be the affine cone defined by $\mathcal{S}$; if $\mathcal{L}$ is $f$-ample, then there exists a canonical $Y$-morphism

$$(8.8.2.1) \quad g : V = \text{V}(\mathcal{L}) \rightarrow C$$
such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{j} & V(L) \\
\downarrow f & & \downarrow \pi \\
Y & \xrightarrow{\epsilon} & C \\
\end{array}
\]

\[
\begin{array}{ccc}
& & X \\
& \downarrow g & \downarrow f \\
C & \xrightarrow{\psi} & Y \\
\end{array}
\]

commutes, where \(\psi\) and \(\pi\) are the structure morphisms, and \(j\) and \(\epsilon\) the canonical immersions sending \(X\) and \(Y\) (respectively) to the null section of \(V(L)\) and the vertex prescheme of \(C\) (respectively). Furthermore, the restriction of \(g\) to \(V(L) - j(X)\) is an open immersion

\[
V(L) - j(X) \rightarrow E = C - \epsilon(Y)
\]

into the pointed affine cone \(E\) corresponding to \(\mathcal{S}\).

**Proof.** With the notation of (8.8.1), let \(\mathcal{S}_p^\# = \bigoplus_{n \geq 0} \mathcal{O}_p(n)\) and \(C_p = \text{Spec}(\mathcal{S}_p^\#)\). We know (8.6.2) that there is a canonical morphism \(h = \text{Spec}(\alpha) : C_p \rightarrow C\) such that the diagram

\[
\begin{array}{ccc}
C_p & \xrightarrow{p} & P \\
\downarrow h & & \downarrow \mathcal{P} \\
C & \xrightarrow{\psi} & Y \\
\end{array}
\]

commutes; furthermore, if \(\mathcal{P} : P \rightarrow C_p\) is the canonical immersion, then the diagram

\[
\begin{array}{ccc}
P & \xrightarrow{p} & C_p \\
\downarrow \mathcal{P} & & \downarrow h \\
Y & \xrightarrow{\epsilon} & C \\
\end{array}
\]

commutes (8.7.1.1), and, finally, the restriction of \(H\) to the pointed affine cone \(E_p\) is an isomorphism \(E_p \sim E\) (8.6.2). It follows from (8.8.1.4) that

\[
r^*(\mathcal{S}_p^\#) = S_{\mathcal{O}_X}(L)
\]

and so we have a canonical \(P\)-morphism \(q : V(L) \rightarrow C_p\), with the commutative diagram

\[
\begin{array}{ccc}
V(L) & \xrightarrow{\pi} & X \\
\downarrow q & & \downarrow r \\
C_p & \rightarrow & P \\
\end{array}
\]

identifying \(V(L)\) with the product \(C_p \times_P X\) (1.5.2); since \(r\) is an open immersion, so too is \(q\) (I, 4.3.2). Furthermore, the restriction of \(q\) to \(V(L) - j(X)\) sends this prescheme to \(E_p\), by (8.5.2), and the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{j} & V(L) \\
\downarrow r & & \downarrow q \\
P & \xrightarrow{\epsilon_P} & C_p \\
\end{array}
\]

is commutative (since it is a particular case of (8.5.1.3)). The claims of (8.8.2) immediately follow from these facts, by taking \(g\) to be the composite morphism \(h \circ q\).

**Remark (8.8.3).** Assume further that \(Y\) is a Noetherian prescheme, and that \(f\) is a proper morphism. Since \(r\) is then proper (5.4.4), and thus closed, and since it is also a dominant open immersion, \(r\) is necessarily an isomorphism \(X \sim P\). Furthermore, we will see, in Chapter III (III, 2.3.5.1), that \(\mathcal{S}\) is then necessarily an \(\mathcal{O}_Y\)-algebra of finite type. It then follows that \(\mathcal{S}^8\) is an \(\mathcal{S}^8\)-algebra of finite type (8.2.10, i) and (8.7.2.7); since \(C_p\) is \(C\)-isomorphic to \(\text{Proj}(\mathcal{S}^\#)\) (8.7.3), we see that the morphism \(h : C_p \rightarrow C\) is projective; since the morphism \(r\) is an isomorphism, so too is \(q : V(L) \rightarrow C_p\), and we thus conclude that the morphism \(g : V(L) \rightarrow C\) is projective. Furthermore, since the restriction of
$h$ to $E_P$ is an isomorphism to $E$, and since $q$ is an isomorphism, the restriction (8.8.2.3) of $g$ is an isomorphism $V(L) - j(X) \sim E$.

If we further assume that $L$ is very ample for $f$, then, as we will also see in Chapter III (III, 2.3.5.1), there exists some integer $n_0 > 0$ such that $\mathcal{S}_n = \mathcal{S}_n^\prime$ for $n \geq n_0$. We then conclude, by (8.7.7), that $V(L)$ can be identified with the prescheme $Z$ given by blowing up the vertex prescheme (identified with $Y$) in the affine cone $C$, and that the null section of $V(L)$ (identified with $Y$) is the inverse image of the vertex subscheme $Y$ of $C$.

Some of the above results can in fact be proven even without the Noetherian hypothesis:

**Corollary (8.8.4).** Let $Y$ be a prescheme (resp. a quasi-compact scheme), $f : X \to Y$ a proper morphism, and $L$ an invertible $O_X$-module that is ample relative to $f$. Then the morphism in (8.8.2.1) is proper (resp. projective), and its restriction (8.8.2.3) is an isomorphism.

**Proof.** To prove that $g$ is proper, we can restrict to the case where $Y$ is affine, and it then suffices to consider the case where $Y$ is a quasi-compact scheme. The same arguments as in (8.8.3) first of all show that $r$ is an isomorphism $X \sim P$; then $q$ is also an isomorphism, and, since the restriction of $h$ to $E_P$ is an isomorphism $E_P \sim E$, we have already seen that (8.8.2.3) is an isomorphism. It remains only to prove that $g$ is projective.

Since $f$ is of finite type, by hypothesis, we can apply (3.8.5) to the homomorphism $\tau$ from (8.8.1.2): there is an integer $d > 0$ and a quasi-coherent $O_Y$-submodule $\mathcal{E}$ of finite type of $\mathcal{I}_d$ such that, if $\mathcal{S}$ is the $O_Y$-subalgebra of $\mathcal{S}$ generated by $\mathcal{E}$, and $\tau' = \tau \circ q'\circ (\phi)$ (where $\phi$ is the canonical injection $\mathcal{S} \rightarrow \mathcal{I}'$), then $r' = r \circ \tau' \circ \phi$ is an immersion

$$X \rightarrow P' = \text{Proj}(\mathcal{S})$$

Furthermore, since $g$ is projective, $r'$ is also a dominant immersion (3.7.6); the same argument as for $r$ then shows that $r'$ is a surjective closed immersion; since $r'$ factors as $X \xrightarrow{\iota} \text{Proj}(\mathcal{S}) \xrightarrow{\Phi} \text{Proj}(\mathcal{S}')$, where $\Phi = \text{Proj}(\phi)$, we thus conclude that $\Phi$ is also a surjective closed immersion. But this implies that $\Phi$ is an isomorphism; we can restrict to the case where $Y = \text{Spec}(A)$ is affine, and $\mathcal{S} = \mathcal{I}$ and $\mathcal{S}' = \mathcal{I}'$, with $S$ a graded $A$-algebra and $S'$ a graded subalgebra of $S$. For every homogeneous element $t \in S'$, we have that $S'(t)$ is a subring of $S(t)$; if we return to the definition of $\text{Proj}(\phi)$ (2.8.1), we see that it suffices to prove that, if $B$ is a subring of a ring $B$, and the morphism $\text{Spec}(B) \to \text{Spec}(B')$ corresponding to the canonical injection $B' \to B$ is a closed immersion, then this morphism is necessarily an isomorphism; but this follows from (I, 4.2.3). Furthermore, $\Phi^*(\mathcal{O}_P(n)) = \mathcal{O}_P(n)$ (3.5.2, ii) and (3.5.4)), and so $\tau^*(\mathcal{O}_P(n))$ is isomorphic to $\mathcal{L} \otimes n$ (4.6.3). Let $\mathcal{L}'' = \mathcal{S}(d)$, so that (3.1.8, i) $X$ is canonically identified with $P'' = \text{Proj}(\mathcal{S}'')$, and $\mathcal{L}'' = \mathcal{L} \otimes d$ with $\mathcal{O}_P(1)$ (3.2.9, ii).

Now, if $C'' = \text{Spec}(\mathcal{S}'')$, then $\mathcal{S}_n = \mathcal{S}_n \otimes \mathcal{O}_P(n)$ can be identified with $\mathcal{O}_{C''}$, and thus $\mathcal{O}_{C''}$ is isomorphic to $\text{Proj}(\mathcal{S}'')$; by the definition of $\mathcal{S}''$, we know that $\mathcal{S}'$ is generated by $\mathcal{S}_0^\prime$, and that $\mathcal{S}_1^\prime$ is of finite type over $\mathcal{S}_0^\prime = \mathcal{S}''$ (8.2.10, i and iii), and so $\text{Proj}(\mathcal{S}_1^\prime)$ is projective over $C''$ (5.5.1). Consider the diagram

\[
\begin{align*}
\mathcal{V}(\mathcal{L}) & \xrightarrow{g} \text{Spec}(\mathcal{S}) = C \\
\mathcal{V}(\mathcal{L}'') & \xrightarrow{g''} \text{Spec}(\mathcal{S}'') = C''
\end{align*}
\]

where $g$ and $g''$ correspond, by (1.5.6), to the canonical $j$-morphisms

$$\mathcal{I} \rightarrow \bigoplus_{n \geq 0} \mathcal{L} \otimes n$$

and

$$\mathcal{I}'' \rightarrow \bigoplus_{n \geq 0} \mathcal{L}'' \otimes n$$

(3.3.2.3) (see (8.8.5) below), and $v$ and $u$ to the inclusion morphisms $\mathcal{I}'' \rightarrow \mathcal{S}$ and $\bigoplus_{n \geq 0} \mathcal{L} \otimes nd \rightarrow \bigoplus_{n \geq 0} \mathcal{L} \otimes n$ (respectively); it is immediate (3.3.2) that this diagram is commutative. We have just seen that $g''$ is a projective morphism; we also know that $u$ is a finite morphism. Since the question is local on $X$, we can assume that $X$ is affine of ring $A$, and that $\mathcal{L} = \mathcal{O}_X$; everything then reduces to noting that the ring $A[T]$ is a module of finite type over its subring $A[T^d]$ (with $T$ an indeterminate). Since $Y$ is a quasi-compact scheme, and since $C''$ is affine over $Y$, we know that $C''$ is also a quasi-compact
Consider again the situation in (8.8.1). We will see that the morphism $g : V(\mathcal{L}) \to C$ can be also be defined in a way that works for any invertible (but not necessarily ample) $\mathcal{O}_X$-module $\mathcal{L}$. For this, consider the $f$-morphism

(8.8.5.1)

$$\tau^* : \mathcal{I} \to \bigoplus_{n \geq 0} \mathcal{L}^\otimes n$$

corresponding to the morphism $\tau$ from (8.8.1.2). This induces (1.5.6) a morphism $g' : V \to C$ such that, if $\pi : V \to X$ and $\psi : C \to Y$ are the structure morphisms, the diagrams

(8.8.5.2)

$$X \underset{\pi}{\leftarrow} V \underset{\psi}{\rightarrow} C \quad \text{and} \quad X \underset{\psi}{\rightarrow} V \underset{\pi}{\leftarrow} C$$

commute ((8.5.1.2) and (8.5.1.3)). We will show that (if we assume that $\mathcal{L}$ is $f$-ample) the morphisms $g$ and $g'$ are identical.

Since the questions is local on $Y$, we can assume that $Y = \text{Spec}(A)$ is affine, and (by (8.8.1.3)) identify $X$ with an open subset of $P = \text{Proj}(S)$, where $S = A \oplus \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^\otimes n)$; we then deduce, by (8.8.1.4), that $\Gamma(X, \mathcal{O}_P(n)) = \Gamma(X, \mathcal{L}^\otimes n)$ for all $n \in \mathbb{Z}$. Taking into account the definition of $h = \text{Spec}(a)$, where $a$ is the canonical $p$-morphism $S \to \mathcal{I}_P$ (8.6.1.2), we have to show that the restriction to $X$ of $a^*: p^*(S) \to \mathcal{I}_P$ is identical to $\tau$. Taking (0, 4.4.3) into account, it suffices to show that, if we compose the canonical homomorphism $\alpha_n : S_n \to \Gamma(P, \mathcal{O}_P(n))$ with the restriction homomorphism $\Gamma(P, \mathcal{O}_P(n)) \to \Gamma(X, \mathcal{O}_P(n)) = \Gamma(X, \mathcal{L}^\otimes n)$, then we obtain the identity, for all $n > 0$; but this follows immediately from the definition of the algebra $S$ and of $\alpha_n$ (6.2.6).

**Proposition (8.8.6).** — Assume (with the notation of (8.8.5)) that, if we write $f = (f_0, \lambda)$, then the homomorphism $\lambda : \mathcal{O}_Y \to f_0(\mathcal{O}_X)$ is bijective; then:

(i) if we write $g = (g_0, \mu)$, then $\mu : \mathcal{O}_C \to g_0(\mathcal{O}_V)$ is an isomorphism; and

(ii) if $X$ is integral (resp. locally integral and normal), then $C$ is integral (resp. normal).

**Proof.** Indeed, the $f$-morphism $\tau^*$ is then an isomorphism

$$\tau^* : \mathcal{I} = \psi_*(\mathcal{O}_C) \to f_0(\mathcal{O}_V) = \psi_*(\mathcal{O}_V))$$

and the $Y$-morphism $g$ can be considered as that for which the homomorphism $\mathcal{O}(g)$ (1.1.2) is equal to $\tau^*$. To see that $\mu$ is an isomorphism of $\mathcal{O}_C$-modules, it suffices (1.4.2) to see that $\mathcal{O}(\mu) : \psi_* (\mathcal{O}_C) \to \psi_* (\mathcal{O}_V)$ is an isomorphism. But, by Definition (1.1.2), we have that $\mathcal{O}(\mu) = \mathcal{O}(g)$, whence the conclusion of (i).

To prove (ii), we can restrict to the case where $Y$ is affine, and so $\mathcal{I} = S$, with $S = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^\otimes n)$; the hypothesis that $X$ is integral implies that the ring $S$ is integral (I, 7.4.4), and thus so too is $C$ (I, 5.1.4). To show that $C$ is normal, we will use the following lemma:

**Lemma (8.8.6.1).** — Let $Z$ be a normal integral prescheme. Then the ring $\Gamma(Z, \mathcal{O}_Z)$ is integral and integrally closed.

**Proof.** It follows from (I, 8.2.1.1) that $\Gamma(Z, \mathcal{O}_Z)$ is the intersection, in the field of rational functions $R(Z)$, of the integrally closed rings $\mathcal{O}_z$ over all $z \in Z$. □

With this in mind, we first show that $V$ is locally integral and normal; for this, we can restrict to the case where $X = \text{Spec}(A)$ is affine, with ring $A$ integral and integrally closed (6.3.8), and where $\mathcal{L} = \mathcal{O}_X$. Since then $V = \text{Spec}(A[T])$, and $A[T]$ is integral and integrally closed [Jaf60, p. 99], this proves our claim. For every affine open subset $U$ of $C$, $g^{-1}(U)$ is quasi-compact, since the morphism $g$ is quasi-compact; since $V$ is locally integral, the connected components of $g^{-1}(U)$ are open integral preschemes in $g^{-1}(U)$, and thus finite in number, and, since $V$ is normal, these preschemes are also normal (6.3.8). Then $\Gamma(U, \mathcal{O}_C)$, which is equal to $\Gamma(g^{-1}(U), \mathcal{O}_V)$, by (i), is the direct sum (8.6.6.1) of finitely-many integral and integrally closed rings (8.8.6.1), which proves that $C$ is normal (6.3.4). □
8.9. Grauert’s ampleness criterion: statement

We intend to show that the properties proven in (8.8.2) characterise $f$-ample $\mathcal{O}_X$-modules, and, more precisely, to prove the following criterion:

Theorem (8.9.1). — (Grauert’s criterion). Let $Y$ be a prescheme, $p : X \to Y$ a separated and quasi-compact morphism, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. For $\mathcal{L}$ to be ample relative to $p$, it is necessary and sufficient for there to exist a $Y$-prescheme $C$, a $Y$-section $\epsilon : Y \to C$ of $C$, and a $Y$-morphism $q : V(\mathcal{L}) \to C$, satisfying the following properties:

(i) the diagram

\[
\begin{array}{ccc}
X & \rightarrow & V(\mathcal{L}) \\
p \downarrow & & \downarrow q \\
Y & \rightarrow & C \\
\end{array}
\]

commutes, where $j$ is the null section of the vector bundle $V(\mathcal{L})$; and

(ii) the restriction of $q$ to $V(\mathcal{L}) - j(X)$ is a quasi-compact open immersion

$V(\mathcal{L}) - j(X) \rightarrow X$

whose image does not intersect $\epsilon(Y)$.

Note that, if $C$ is separated over $Y$, we can, in condition (ii), remove the hypothesis that the open immersion is quasi-compact; to see that this property (of quasi-compactness) is in fact a consequence of the other conditions, we can restrict to the case where $Y$ is affine, and the claim then follows from (I, 5.5.1)i and (I, 5.5.10). We can also remove the same hypothesis if we assume that $X$ is Noetherian, since then $V$ is also Noetherian, and the claim follows from (I, 6.3.5).

Corollary (8.9.2). — If the morphism $p : X \to Y$ is proper, then we can, in the statement of Theorem (8.9.1), assume that $q$ is proper, and replace “open immersion” by “isomorphism”.

In a more suggestive manner, we can say (whenever $p : X \to Y$ is proper) that $\mathcal{L}$ is ample relative to $p$ if and only if we can “contract” the null section of the vector bundle $V(\mathcal{L})$ to the base prescheme $Y$.

An important particular case is that where $Y$ is the spectrum of a field, and where the operation of “contraction” consists of contract the null section $V(\mathcal{L})$ to a single point.

(8.9.3). The necessity of the conditions in Theorem (8.9.1) and Corollary (8.9.2) follow immediately from (8.8.2) and (8.8.4).

To show that the conditions of (8.9.1) suffices, consider a slightly more general situation. For this, let (with the notation of (8.8.2))

\[J' = \bigoplus_{n \geq 0} \mathcal{L} \otimes \mathbb{N}\]

and

\[V = V(\mathcal{L}) = \text{Spec}(J').\]

The closed subscheme $j(X)$, null section of $V(\mathcal{L})$, is defined by the quasi-coherent sheaf of ideals $\mathcal{J} = (J'_{+})$ of $\mathcal{O}_V$ (1.4.10). This $\mathcal{O}_V$-module is invertible, since this property is local on $X$, and this reduces to remarking that the ideal $TA[T]$ in a ring of polynomials $A[T]$ is a free cyclic $A[T]$-module. Furthermore, it is immediate (again, because the question is local on $X$) that

\[\mathcal{L} = j^*(\mathcal{J})\]

and

\[j_*(\mathcal{L}) = \mathcal{J} / \mathcal{J}^2.\]

Now, if

\[\pi : V(\mathcal{L}) \rightarrow X\]

is the structure morphism, then $\pi_* (\mathcal{J}) = J'_+$ and $\pi_* (\mathcal{J} / \mathcal{J}^2) = \mathcal{L}$; there are thus canonical homomorphisms $\mathcal{L} \rightarrow \pi_* (\mathcal{J}) \rightarrow \mathcal{L}$, the first being the canonical injection $\mathcal{L} \rightarrow J'_+$, and the second the canonical projection from $J'_+$ to $J'_1 = \mathcal{L}$, and their composition being the identity. We can also
canonically embed $\pi_*$ into the product $\prod_{n \geq 1} \mathcal{L}^{\otimes n}$ into the product $\prod_{n \geq 1} \mathcal{L}^{\otimes n} = \lim_{\leftarrow n} \pi_*(\mathcal{J} / \mathcal{J}^{n+1})$ (since $\pi_*(\mathcal{J} / \mathcal{J}^{n+1}) = \mathcal{L} \oplus \mathcal{L}^{\otimes 2} \oplus \ldots \oplus \mathcal{L}^{\otimes n}$), and we thus have canonical homomorphisms

\[ \mathcal{L} \rightarrow \lim_{\leftarrow n} \pi_*(\mathcal{J} / \mathcal{J}^{n+1}) \rightarrow \mathcal{L} \]

whose composition is the identity.

With this in mind, the generalisation of (8.9.1) that we are going to prove is the following:

**Proposition (8.9.4).** — Let $Y$ be a prescheme, $V$ a $Y$-prescheme, and $X$ a closed subprescheme of $V$ defined by an ideal $\mathcal{J}$ of $\mathcal{O}_V$, which is an invertible $\mathcal{O}_V$-module; if $j : X \rightarrow V$ is the canonical injection, then let

\[ \mathcal{L} = j^*(\mathcal{J}) = \mathcal{J} \otimes_{\mathcal{O}_V} \mathcal{O}_X, \]

so that $j_* (\mathcal{L}) = \mathcal{J} / \mathcal{J}^2$. Assume that the structure morphism $p : X \rightarrow Y$ is separated and quasi-compact, and that the following conditions are satisfied:

(i) there exists a $Y$-morphism $\pi : V \rightarrow X$ of finite type such that $\pi \circ j = 1_X$, and so $\pi_*(\mathcal{J} / \mathcal{J}^2) = \mathcal{L}$;

(ii) there exists a homomorphism of $\mathcal{O}_X$-modules $\varphi : \mathcal{L} \rightarrow \lim_{\leftarrow n} \pi_*(\mathcal{J} / \mathcal{J}^{n+1})$ such that the composition

\[ \mathcal{L} \xrightarrow{\varphi} \lim_{\leftarrow n} \pi_*(\mathcal{J} / \mathcal{J}^{n+1}) \xrightarrow{\alpha} \pi_*(\mathcal{J} / \mathcal{J}^2) = \mathcal{L} \]

(where $\alpha$ is the canonical homomorphism) is the identity;

(iii) there exists a $Y$-prescheme $C$, a $Y$-section $\epsilon$ of $C$, and a $Y$-morphism $q : V \rightarrow C$ such that the diagram

\[ \begin{array}{ccc}
X & \xrightarrow{i} & V \\
p \downarrow & & \downarrow q \\
Y & \xrightarrow{\epsilon} & C
\end{array} \]

commutes; and

(iv) the restriction of $q$ to $W = V - j(X)$ is a quasi-compact open immersion into $C$, whose image does not intersect $\epsilon(Y)$.

Then $\mathcal{L}$ is ample relative to $p$.

**8.10. Grauert’s ampleness criterion: proof**

**Lemma (8.10.1).** — Let $\pi : V \rightarrow X$ be a morphism, $j : X \rightarrow V$ an $X$-section of $V$ that is also a closed immersion, and $\mathcal{J}$ a quasi-coherent ideal of $\mathcal{O}_V$ that defines the closed subscheme of $V$ associated to $j$. Then the following hold true.

(i) For all $n \geq 0$, $\pi_*(\mathcal{O}_V / \mathcal{J}^{n+1})$ and $\pi_*(\mathcal{J} / \mathcal{J}^{n+1})$ are quasi-coherent $\mathcal{O}_X$-modules, and $\pi_*(\mathcal{O}_V / \mathcal{J}) = \mathcal{O}_X$ and $\pi_*(\mathcal{J} / \mathcal{J}^2) = j^*(\mathcal{J})$.

(ii) If $X = \{ \xi \} = \text{Spec}(k)$, where $k$ is a field, then $\lim_{\leftarrow n} \pi_*(\mathcal{O}_V / \mathcal{J}^{n+1})$ is isomorphic to the separated completion of the local ring $\mathcal{O}_{j(\xi)}$ for the $m_{j(\xi)}$-adic topology.

(iii) Assume that $\mathcal{J}$ is an invertible $\mathcal{O}_V$-module (which implies that $\mathcal{L} = j^*(\mathcal{J}) = \pi_*(\mathcal{J} / \mathcal{J}^2)$ is an invertible $\mathcal{O}_X$-module), and that there exists a homomorphism $\varphi : \mathcal{L} \rightarrow \lim_{\leftarrow n} \pi_*(\mathcal{J} / \mathcal{J}^{n+1})$ such that the composition $\mathcal{L} \xrightarrow{\varphi} \lim_{\leftarrow n} \pi_*(\mathcal{J} / \mathcal{J}^{n+1}) \xrightarrow{\alpha} \pi_*(\mathcal{J} / \mathcal{J}^2)$ (where $\alpha$ is the canonical homomorphism) is the identity. If we write $\mathcal{F} = \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$, then $\varphi$ canonically induces an isomorphism of $\mathcal{O}_X$-algebras from the completion $\widehat{\mathcal{F}}$ of $\mathcal{F}$ relative to its canonical filtration (the completion being isomorphic to the product $\prod_{n \geq 0} \mathcal{L}^{\otimes n}$) to $\lim_{\leftarrow n} \pi_*(\mathcal{O}_V / \mathcal{J}^{n+1})$.

**Proof.** Note first of all that the support of the $\mathcal{O}_V$-module $\mathcal{O}_V / \mathcal{J}^{n+1}$ is $j(X)$, and the support of $\mathcal{O}_V / \mathcal{J}^{n+1}$ is contained in $j(X)$. In the case of (iii), $j(X)$ is a closed point $j(\xi)$ of $V$, and, by definition, $\pi_*(\mathcal{O}_V / \mathcal{J}^{n+1})$ is the fibre of $\mathcal{O}_V / \mathcal{J}^{n+1}$ at the point $j(\xi)$, or, equivalently, setting $C = \mathcal{O}_{j(\xi)}$, and denoting by $m$ the maximal ideal of $C$, the $C$-module $C/m^{n+1}$; claim (ii) is then evident.

To prove (i), note that the question is local on $X$; we can thus restrict to the case where $X$ is affine. Let $U$ be an affine open subset of $V$; then $j(X) \cap U$ is an affine open subset of $j(X)$, so $U_0 = \pi(j(X) \cap U)$, which is isomorphic to it, is an affine open subset of $X$; for every affine
open subset $W_0 \subset U_0$ in $X$, $W = \pi^{-1}(W_0) \cap U$ is an affine open subset of $V$, since $X$ is a scheme (I, 5.5.10); in particular, $U' = U \cap \pi^{-1}(U_0)$ is an affine open subset of $V$, and clearly $\pi(U') = U_0$ and $j(U_0) = j(X) \cap U$. Then, by definition, $\Gamma(W_0, \pi_* (\mathcal{O}_V / J^{n+1})) = \Gamma(\pi^{-1}(W_0), \mathcal{O}_V / J^{n+1})$, but since every point of $\pi^{-1}(W_0)$ not belonging to $j(W_0)$ has an open neighbourhood in $\pi^{-1}(W_0)$ not intersecting $j(X)$, and in which $\mathcal{O}_V / J^{n+1}$ is thus zero, it is clear that the sections of $\mathcal{O}_V / J^{n+1}$ over $\pi^{-1}(W_0)$ and over $W$ are in bijective correspondence. In other words, if $\pi'$ is the restriction of $\pi$ to $U'$, then the $(\mathcal{O}_X | U_0)$-modules $\pi_* (\mathcal{O}_V / J^{n+1}) | U_0$ and $\pi'_* (\mathcal{O}_V / J^{n+1}) | U'$ are identical. Since $U'$ and $U_0$ are affine, and since the $U_0$ cover $X$, we thus conclude (I, 1.6.3) that $\pi_* (\mathcal{O}_V / J^{n+1})$ is quasi-coherent, and the proof is identical for $\pi_* (\mathcal{J} / J^{n+1})$.

Finally, to prove (iii), note that $\mathcal{J}$ is exactly $S_{\mathcal{O}_X} (\mathcal{L})$; so $\varphi$ canonically induces a homomorphism of $\mathcal{O}_X$-algebras $\varphi : \mathcal{J} \to \varprojlim \pi_* (\mathcal{O}_V / J^{n+1})$ (1.7.4); furthermore, this homomorphism sends $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X$ to $\varprojlim \pi_* (\mathcal{O}_V / J^{n+1})$, and is thus continuous for the topologies considered, and indeed then extends to a homomorphism $\hat{\varphi} : \hat{\mathcal{J}} \to \varprojlim \pi_* (\mathcal{O}_V / J^{n+1})$. To see that this is indeed an isomorphism, we can, as in the proof of (i), restrict to the case where $X = \text{Spec}(A)$ and $V = \text{Spec}(B)$ are affine, with $\mathcal{J} = \mathfrak{J}$, where $\mathfrak{J}$ is an ideal of $B$; there is an injection $A \to B$ corresponding to $\pi$ that identifies $A$ with a subring of $B$ that is complementary to $B$, and $\mathcal{L}$ (resp. $\pi_* (\mathcal{O}_V / J^{n+1})$) is the quasi-coherent $\mathcal{O}_X$-module associated to the $A$-module $L = \mathfrak{J} / \mathfrak{J}^2$ (resp. $B / \mathfrak{J}^n$). Since $\mathcal{J}$ is an invertible $\mathcal{O}_V$-module, we can further assume that $\mathfrak{J} = Bt$, where $t$ is not a zero divisor in $B$. From the fact that $B = A \oplus Bt$, we deduce that, for all $n > 0$,

$$B = A \oplus At \oplus At^2 \oplus \ldots \oplus At^n \oplus Bt^{n+1}$$

and so there exists a canonical $A$-isomorphism from the ring of formal series $A[[T]]$ to $C = \varprojlim B / \mathfrak{J}^{n+1}$ that sends $T$ to $t$. We also have that $L = At$, where $t$ is the class of $t$ modulo $Bt^2$, and the homomorphism $\varphi$ sends, by hypothesis, $\bar{t}$ to an element $t' \in C$ that is congruent to $t$ modulo $Ct^2$. Thus we deduce, by induction on $n$, that

$$A \oplus At' \oplus \ldots \oplus At^n \oplus Ct^{n+1} = A \oplus At \oplus \ldots \oplus At^n \oplus Ct^{n+1}$$

which proves that the homomorphism $\hat{\varphi}$ does indeed correspond to an isomorphism from $\varprojlim L^\otimes n$ to $C$.

**Lemma (8.10.2).** Under the hypotheses of Lemma (8.10.1), let $g : X' \to X$ be a morphism, write $V' = V \times_X X'$, and let $\pi' : V' \to X'$ and $g : V' \to V$ be the canonical projections, so that we have the commutative diagram

$$\begin{array}{ccc}
V & \xrightarrow{g'} & V' \\
\pi & \downarrow & \pi' \\
X & \xrightarrow{g} & X'
\end{array}$$

Then $j' = j \times X'$ is an $X'$-section of $V'$ that is also a closed immersion, and $\mathcal{J}' = g'^*(\mathcal{J}) \mathcal{O}_{V'}$ is the quasi-coherent ideal of $\mathcal{O}_{V'}$ that defines the closed subscheme of $V'$ associated to $j'$. Furthermore, $\pi'_* (\mathcal{O}_{V'} / J'^{n+1}) = g'^* (\pi_* (\mathcal{O}_V / J^{n+1}))$. Finally, $\mathcal{J}'$ is an $\mathcal{O}_{V'}$-module that is canonically isomorphic to $g'^*(\mathcal{J})$, and is, in particular, invertible if $\mathcal{J}$ is an invertible $\mathcal{O}_V$-module.

**Proof.** The fact that $j'$ is a closed immersion follows from (I, 4.3.1), and it is an $X'$-section of $V'$ by functoriality of extension of the base prescheme. Furthermore, if $Z$ (resp. $Z'$) is the closed subscheme of $V$ (resp. $V'$) associated to $j$ (resp. $j'$), then $Z' = g'^{-1}(Z)$ (I, 4.3.1), and the second claim then follows from (I, 4.4.5). To prove the other claims, we see, as in (8.10.1), that we can restrict to the case where $X$, $V$, and $X'$ (and thus also $V'$) are affine; we keep the notation from the proof of (8.10.1), and let $X' = \text{Spec}(A')$. Then $V' = \text{Spec}(B')$, where $B' = B \otimes_A A'$, and $\mathcal{J}' = \mathfrak{J}'$, where $\mathfrak{J}' = \text{Im}(\mathfrak{J} \otimes_A A')$. Then $B' / \mathfrak{J}'^{n+1} = (B / \mathfrak{J}^{n+1}) \otimes_A A'$; furthermore, since $\mathfrak{J}$ is a direct factor (as an $A$-module) of $B$, $\mathfrak{J} \otimes_A A'$ is a direct factor (as an $A'$-module) of $B'$, and is thus canonically identified with $\mathfrak{J}'$.

**Corollary (8.10.3).** Assume that the hypotheses of Lemma (8.10.1) are satisfied, and assume further that $\pi$ is of finite type, and that $\mathcal{J}$ is an invertible $\mathcal{O}_V$-module. Then, for all $x \in X$, the local ring at the point
\[ j(x) \text{ of the fibre } \pi^{-1}(x) \text{ is a regular (thus integral) ring of dimension } 1, \text{ whose completion is isomorphic to the formal series ring } k(x)[[T]] \text{ (where } T \text{ is an indeterminate); furthermore, there exists exactly one irreducible component of } \pi^{-1}(x) \text{ that contains } j(x). \]

**Proof.** Since \( \pi^{-1}(x) = V \times_X \text{Spec}(k(x)) \), we are led, by (8.10.2), to the case where \( X \) is the spectrum of a field \( K \). Since \( \pi \) is of finite type (I, 6.4.3, iv), \( \mathcal{O}_{j(x)} \) is a Noetherian local ring, and thus separated for the \( \mathfrak{m}_{j(x)} \)-adic topology (0, 7.3.5); it follows from (8.10.1, ii and iii) that the completion of this ring is isomorphic to \( K[[T]] \), and so \( \mathcal{O}_{j(x)} \) is regular and of dimension 1 ([CC, p. 17-01, th. 1]); finally, since \( \mathcal{O}_{j(x)} \) is integral, \( j(x) \) belongs to exactly one of the (finitely many) irreducible components of \( V \) (I, 5.1.4).

**Corollary (8.10.4).** — Suppose that the hypotheses of Lemma (8.10.1) are satisfied, and assume further that \( \mathcal{J} \) is an invertible \( \mathcal{O}_V \)-module. Let \( W = V - j(X) \); for every quasi-coherent ideal \( \mathcal{K} \) of \( \mathcal{O}_X \), let \( \mathcal{K}_V = \pi^*(\mathcal{K}) \mathcal{O}_V \) and \( \mathcal{K}_W = \mathcal{K}_V|W \). Then \( \mathcal{K}_V \) is the largest quasi-coherent ideal of \( \mathcal{O}_V \) whose restriction to \( W \) is \( \mathcal{K}_W \).

**Proof.** Indeed, we see as in (8.10.1) that the question is local on \( X \) and \( V \); we can thus reuse the notation from the proof of (8.10.1), with \( \mathfrak{J} = \mathfrak{B}_t \), where \( t \) is not a zero divisor in \( B \). Furthermore, we have \( W = \text{Spec}(B_t) \) and \( \mathcal{K} = \mathfrak{A}_t \), where \( \mathfrak{A} \) is an ideal of \( A \); whence \( \mathfrak{p}^* (\mathcal{K}) \mathcal{O}_V = (\mathfrak{A} B_t)^{-1} \) (I, 1.6.9), \( \mathcal{K}_W = (\mathfrak{A} B_t)^{-1} \), and the largest ideal of \( B \) whose canonical image in \( B_t \) is \( \mathfrak{A} B_t \) is the inverse image of \( \mathfrak{A} B_t \), that is, the set of \( s \in B \) such that, for some integer \( n > 0 \), we have \( t^n s \in \mathfrak{A} B_t \). We have to show that this last relation implies that \( s \in \mathfrak{A} B \), or again that the canonical image of \( t \) is not a zero divisor in \( B/\mathfrak{A} B = (A/\mathfrak{A}) \otimes_A B_t \), which follows from (8.10.2) applied to \( X' = \text{Spec}(A/\mathfrak{A}) \).

**Corollary (8.10.5).** — Suppose that the hypotheses of (8.10.3) are satisfied; let \( W = V - j(X) \), \( x \) be a point of \( X \), \( \mathcal{K} \) a quasi-coherent ideal of \( \mathcal{O}_X \), and \( z \) the generic point of the irreducible component of \( \pi^{-1}(x) \) that contains \( j(x) \) (8.10.3).

(i) Let \( g \) be a section of \( \mathcal{O}_V \) over \( V \) such that \( g|W \) is a section of \( \mathcal{K}_W \) over \( W \) (using the notation from (8.10.4)). Then \( g \) is a section of \( \mathcal{K}_V \); if further \( g(z) \neq 0 \), and if, for every integer \( m > 0 \), we denote by \( g^m_x \) the germ at the point \( x \) of the canonical image \( g_m \) of \( g \) in \( \Gamma (X, \pi_*(\mathcal{O}_V/\mathcal{J}^{m+1})) \), then there exists an integer \( m > 0 \) such that the image of \( g^m_x \) in \( (\pi_*(\mathcal{O}_V/\mathcal{J}^{m+1}))_x \otimes_{\mathcal{O}_x} k(x) \)

is \( \neq 0 \).

(ii) Suppose further that the conditions of (8.10.1, iii) are fulfilled. Then, if there exists a section \( g \) of \( \mathcal{K}_V \) over \( V \) such that \( g(z) \neq 0 \), then there exists an integer \( n \geq 0 \) and a section \( f \) of \( \mathcal{K}, \mathcal{L}^{\otimes n} = \mathcal{K} \otimes \mathcal{L}^{\otimes n} \subset \mathcal{L}^{\otimes n} \) such that \( f(x) \neq 0 \). If \( g \) is a section of \( \mathcal{J} \), we can take \( n > 0 \).

**Proof.**

(i) Since the ideal of \( \mathcal{O}_W \) generated by \( g|W \) is contained in \( \mathcal{K}_W \) by hypothesis, the ideal of \( \mathcal{O}_V \) generated by \( g \) is contained in \( \mathcal{K}_V \) by (8.10.4), or, in other words, \( g \) is a section of \( \mathcal{K}_V \). To prove the second claim of (i), we can again assume that \( X \) and \( V \) are affine, and reuse the notation from (8.10.1); the fibre \( \pi^{-1}(x) \) is then affine of ring \( B' = B \otimes_A k(x) \), and there exists in \( B' \) an element \( t' \) which is not a zero divisor and is such that \( B' = k(x) \oplus B't' \). Since \( j(x) \) is a specialisation of \( z \) and since \( g(z) \neq 0 \), we necessarily have that \( g(j(x)) \neq 0 \). But \( \mathcal{O}_{j(x)} \) is a separated local ring (8.10.3), and thus embeds into its completion, and the image of \( g \) in this completion is thus not null. But this completion is isomorphic to \( \lim_{\longrightarrow} (B'/B't'^{m+1}) \) (8.10.3); if \( g' = g \otimes 1 \in B' \), then there exists an integer \( m \) such that \( g' \notin B't'^{m+1} \), or, again, the image \( g^m_x \) of \( g' \) in \( B'/B't'^{m+1} \) is not null. But since \( g^m_x \) is exactly the image of \( g^m_x \), our claim is proved.

(ii) By (8.10.1, iii), \( \pi_*(\mathcal{O}_V/\mathcal{J}^{m+1}) \) is isomorphic to the direct sum of the \( \mathcal{L}^{\otimes k} \) for \( 0 \leq k \leq m \); we denote by \( f_k \) the section of \( \mathcal{L}^{\otimes k} \) over \( X \) that is the component of the element of \( \bigoplus_{k=0}^m \Gamma (X, \mathcal{L}^{\otimes k}) \) which corresponds to \( g^m \) by this isomorphism. Choosing \( m \) as in (i), there is thus an index \( k \) such that \( f_k(x) \neq 0 \), by (i). To see that \( f_k \) is a section of \( \mathcal{X}_x \mathcal{L}^{\otimes k} \), it suffices to consider, as above, the case where \( X \) and \( V \) are affine, and this follows immediately from the fact that \( g \in \mathfrak{A} B \) (with the notation from (8.10.4)). The final claim follows from the fact that the hypothesis \( g \in \Gamma (V, \mathcal{J}) \) implies that \( f_0 = 0 \).
(8.10.6). Proof of (8.9.4). The question is local on $Y$ (4.6.4); since $\varepsilon$ is a $Y$-section, we can thus replace $C$ by an affine open neighbourhood $U$ of a point of $\varepsilon(Y)$ such that $\varepsilon(Y) \cap U$ is closed in $U$. In other words, we can assume that $C$ is affine, and that $Y$ is a closed subscheme of $C$ (and thus also affine) defined by a quasi-coherent sheaf $I$ of ideals of $O_C$. Since $p$ is separated and quasi-compact, $X$ is thus a quasi-compact scheme, and we are reduced to proving that $\mathcal{L}$ is ample (4.6.4). By criterion (4.5.2, a)), we must thus prove the following: for every quasi-coherent ideal $\mathcal{K}$ of $O_X$ and every point $x \in X$ not belonging to the support of $O_X/\mathcal{K}$, there exists an integer $n > 0$ and a section $f$ of $\mathcal{L} \otimes O_X^{\otimes n}$ over $X$ such that $f(x) \neq 0$.

For this, set

$$\mathcal{K}_V = \pi^*(\mathcal{K})O_V$$

$$\mathcal{K}_W = \mathcal{K}_V|W$$

where $\mathcal{K}$ is a quasi-coherent ideal of $O_C$. Furthermore, since, by hypotheses, $q^{-1}(Y) \subset j(X)$, and since $Y$ is defined by the ideal $I$, the restriction to $W$ of $q^*(I)O_V$ is identical to that of $O_V$, and so $\mathcal{K}_W$ is also the restriction to $W$ of $q^*(I)O_V$, and we can thus suppose that $\mathcal{K}_C \subset \mathcal{I}$, whence

$$\mathcal{K}_V \subset q^*(\mathcal{I})O_V \subset \mathcal{K}$$

taking into account (I, 4.4.6) and the commutativity of (8.9.4.1). Furthermore, we deduce from (8.10.4) that

$$(8.10.6.2)$$

$$\mathcal{K}_V \subset \mathcal{K}_V.$$  

With this in mind, it follows from (8.10.3) that $j(x)$ belongs to exactly one irreducible component of $\pi^{-1}(x)$; let $z$ be the generic point of this component, and let $z' = q(z)$. By (8.10.5), the proof will be finished (taking (8.10.6.1) and (8.10.6.2) into account) if we show the existence of a section $g$ of $\mathcal{K}_V$ over $V$ such that $g(z) \neq 0$. But, by hypothesis, $\mathcal{K}$ has a restriction equal to that of $O_X$ in an open neighbourhood of $x$; also, it follows from (8.10.3) that $z \neq j(x)$, and so $z \in W$, and thus $(\mathcal{K}_W)_z = O_{V,z}$, whence, by definition, $(\mathcal{K}_C)_z = O_{C,z}$. Since $C$ is affine, there is thus a section $g'$ of $\mathcal{K}_C$ over $C$ such that $g'(z') \neq 0$, and by taking $g$ to be the section of $\mathcal{K}_V'$ corresponding canonically to $g'$, we indeed have $g(z) \neq 0$, which finishes the proof.

Remark (8.10.7). — We ignore the question of whether or not condition (ii) in (8.9.4) is superfluous or not. In any case, the conclusion does not hold if we do not assume the existence of a $Y$-morphism $\pi : V \to X$ such that $\pi \circ j = 1_X$; we briefly point out how we can indeed construct a counterexample, whose details will not be developed until later on. We take $Y = \text{Spec}(k)$, where $k$ is a field, and $C = \text{Spec}(A)$, where $A = k[T_1, T_2]$, and the $Y$-section $\varepsilon$ corresponding to the augmentation homomorphism $A \to k$. We denote by $C'$ the scheme induced by $C$ by blowing up the closed point $a = \varepsilon(Y)$ of $C$; if $D$ is the inverse image of $a$ in $C'$, we consider in $D$ a closed point $b$, and we denote by $V$ the scheme induced by $C'$ by blowing up $b$; $X$ is the closed subscheme of $V$ given by the inverse image of $a$ by the structure morphism $q : V \to C$. We now show that $X$ is the union of two irreducible components, $X_1$ and $X_2$, where $X_1$ is the inverse image of $b$ in $V$. It is immediate that the ideal $\mathcal{J}$ of $O_V$ that defines $X$ is again invertible, we can show that $j^*(\mathcal{J}) = \mathcal{L}$ (where $j$ is the canonical injection $X \to V$) is not ample, by considering the “degree” of the inverse image of $\mathcal{L}$ in $X_1$, which would be $> 0$ if $\mathcal{L}$ were ample, but we can show (by an elementary intersection calculation) that it is in fact equal to 0. 

\[\square\]
8.11. Uniqueness of contractions

Lemma (8.11.1). — Let $U$ and $V$ be preschemes, and $h = (h_0, \lambda) : U \to V$ a surjective morphism. Suppose that

1. $\lambda : \mathcal{O}_V \to h_*(\mathcal{O}_U) = (h_0)_*(\mathcal{O}_U)$ is an isomorphism;
2. the underlying space of $V$ can be identified with the quotient of the underlying space of $U$ by the relation $h_0(x) = h_0(y)$ (a condition which always holds whenever the morphism $h$ is open or closed, or, a fortiori when $h$ is proper.)

Then, for every prescheme $W$, the map

$$(8.11.1.1) \quad \text{Hom}(V, W) \longrightarrow \text{Hom}(U, W)$$

that, to each morphism $v = (v_0, v)$ from $V$ to $W$, associates the morphism $u = v \circ h = (u_0, \mu)$, is a bijection from $\text{Hom}(V, W)$ to the set of $u$ such that $u_0$ is constant on every fibre $h_0^{-1}(x)$.

Proof. It is clear that, if $u = v \circ h$, so that $u_0 = v_0 \circ h_0$, then $u_0$ is constant on every set $h_0^{-1}(x)$. Conversely, if $u$ has this property, we will show that there exists exactly one $v \in \text{Hom}(V, W)$ such that $u = v \circ h$. The existence and uniqueness of the continuous map $v_0 : V \to W$ such that $u_0 = v_0 \circ h_0$ follows from the hypotheses, since $h_0$ can be identified with the canonical map from $U$ to $U/R$. We can also, replacing $V$ by some isomorphic prescheme if necessary, suppose that $\lambda$ is the identity; by hypothesis, $\mu$ is then a homomorphism $\mu : \mathcal{O}_W \to (v_0)_*(\mathcal{O}_U) = (v_0)_*((h_0)_*(\mathcal{O}_U))$ such that the corresponding homomorphism $\mu^0 : u_0^*(\mathcal{O}_W) \to \mathcal{O}_U$ is local on every fibre. Since $(v_0)_*((h_0)_*(\mathcal{O}_U)) = (v_0)_*(\mathcal{O}_V)$, we necessarily have that $v = u$, and everything then reduces to showing that the corresponding homomorphism $\nu = v_0^*(\mathcal{O}_W) \to \mathcal{O}_V$ is local on every fibre. But every $y \in V$ is of the form $h_0(x)$ for some $x \in U$; let $z = v_0(y) = u_0(x)$. Then (0, 3.5.5) the homomorphism $\mu^0$ factors as

$$\mu^0 : \mathcal{O}_z \xrightarrow{v_0^0} \mathcal{O}_y \xrightarrow{\lambda^0} \mathcal{O}_x.$$ 

By hypothesis, $\lambda^0$ and $\mu^0$ are local homomorphisms; thus $\lambda^0$ sends every invertible element of $\mathcal{O}_y$ to an invertible element of $\mathcal{O}_x$; if $v_0^0$ sent a non-invertible element of $\mathcal{O}_z$ to an invertible element of $\mathcal{O}_y$, then $\mu^0$ would send this element of $\mathcal{O}_z$ to an invertible element of $\mathcal{O}_x$, contradicting the hypothesis, whence the lemma.

Corollary (8.11.2). — Let $U$ be an integral prescheme, and $V$ a normal prescheme; then every morphism $h : U \to V$ that is universally closed, birational, and radicial, is also an isomorphism.

Proof. If $h = (h_0, \lambda)$, then it follows from the hypotheses that $h_0$ is injective and closed, and that $h_0(0)$ is dense in $V$, and so $h_0$ is a homeomorphism from $U$ to $V$. To prove the corollary, it will suffice to show that $\lambda : \mathcal{O}_V \to (h_0)_*(\mathcal{O}_U)$ is an isomorphism: we can then apply (8.11.1), which proves that the map (8.11.1.1) is bijective (the fibres $h_0^{-1}(x)$ each consisting of a single point); thus $h$ will be an isomorphism. The question clearly being local on $V$, we can suppose that $V = \text{Spec}(A)$ is affine, of an integral and integrally closed ring (8.8.6.1); $h$ then corresponds (I, 2.2.4) to a homomorphism $\varphi : A \to \Gamma(U, \mathcal{O}_U)$, and everything reduces to showing that $\varphi$ is an isomorphism. But, if $K$ is the field of fractions of $A$, then $\Gamma(U, \mathcal{O}_U)$ has, by hypothesis, $K$ as its field of fractions, and $A$ is a subring of $\Gamma(U, \mathcal{O}_U)$, with $\varphi$ being the canonical injection (I, 8.2.7). Since the morphism $h$ satisfies the hypotheses of (7.3.11), $\Gamma(U, \mathcal{O}_U)$ is a subring of the integral closure of $A$ in $K$, and is thus identical to $A$ by hypothesis.

Remark (8.11.3). — We will see in chapter III (III, 4.4.11) that, whenever $V$ is a locally Noetherian prescheme, every morphism $h : U \to V$ that is proper and quasi-finite (in particular, every morphism satisfying the hypotheses of (8.11.2)) is necessarily finite. The conclusion of (8.11.2) then follows in this case from (6.1.15).

(8.11.4). We will now see that, in Grauert’s criterion, we can often prove that the prescheme $C$ and the “contraction” $q$ are determined in an essentially unique manner.

Lemma (8.11.5). — Let $Y$ be a prescheme, $p : X \to Y$ a proper morphism, $\mathcal{L}$ a $p$-ample invertible $\mathcal{O}_X$-module, $C$ a $Y$-prescheme, $\varepsilon : Y \to C$ a $Y$-section, and $q : V = V(\mathcal{L}) \to C$ a $Y$-morphism, all such that
the diagram in (8.9.1.1) commutes. Suppose further that, if \( p = (p_0, \theta) \), then \( \theta : \mathcal{O}_Y \to p_*(\mathcal{O}_X) \) is an isomorphism. Let \( \mathcal{I}' = \bigoplus_{n \geq 0} p_*(\mathcal{L}^n) \) and \( C' = \text{Spec}(\mathcal{I}') \), and let \( q' : \mathcal{V}(\mathcal{L}) \to C' \) be the canonical Y-morphism (8.8.5). Then there exists exactly one Y-morphism \( u : C' \to C \) such that \( q = u \circ q' \).

**Proof.** The hypothesis on \( \theta \) implies, in particular, that \( p \) is surjective; since, by (8.8.4), the restriction of \( q' \) to \( \mathcal{V}(\mathcal{L}) - j(X) \) is an isomorphism to \( C' - \epsilon'(Y) \) (where \( \epsilon \) is the vertex section of \( C' \)), it follows from (8.8.4) that \( q' \) is proper and surjective; furthermore, by (8.8.6), if we let \( q' = (q_0, \tau) \), then \( \tau : \mathcal{O}_C \to q'_*(\mathcal{O}_Y) \) is an isomorphism. We are thus in a situation where we can apply (8.11.1), and we will have proven the lemma if we show that \( q \) is constant on every fibre \( q^{-1}(z') \), where \( z' \in C' \). But this condition is trivially satisfied for \( z' \notin \epsilon'(Y) \). If \( z' \in \epsilon'(Y) \), then there exists exactly one \( y \in Y \) such that \( z' = \epsilon'(y) \), and, by commutativity of (8.8.5.2) and the fact that \( q' \) sends \( \mathcal{V}(\mathcal{L}) - j(X) \) to \( C' - \epsilon'(Y) \), \( q^{-1}(z') = j(p^{-1}(y)) \); the commutativity of the diagram in (8.9.1.1) then proves our claim. \( \square \)

**Corollary (8.11.6).** — Under the hypotheses of (8.11.5), suppose further that \( q \) is proper, and that the restriction of \( q \) to \( \mathcal{V}(\mathcal{L}) - j(X) \) is an isomorphism to \( C - \epsilon(Y) \). Then the morphism \( u \) is universally closed, surjective, and radical, and its restriction to \( C' - \epsilon'(Y) \) is an isomorphism to \( C - \epsilon(Y) \).

**Proof.** Since \( q' \) is an isomorphism from \( \mathcal{V}(\mathcal{L}) - j(X) \) to \( C' - \epsilon'(Y) \) (8.8.4), the last claim follows immediately from the fact that \( q = u \circ q' \). Furthermore, the commutativity of the diagrams in (8.8.5.2) and (8.9.1.1) shows that the restriction of \( u \) to the closed subscheme \( \epsilon'(Y) \) of \( C' \) is an isomorphism to the closed subscheme \( \epsilon(Y) \) of \( C \), from which we immediately deduce that, for all \( z' \in \epsilon'(Y) \), if \( z = u(z') \), then \( u \) defines an isomorphism from \( k(z) \) to \( k(z') \). These remarks prove that \( u \) is bijective and radical; furthermore, if \( \psi : C \to Y \) and \( \psi' : C' \to Y \) are the structure morphisms, then \( \psi' = \psi \circ u \), and, since \( \psi' \) is separated (1.2.4), so too is \( u \) (I, 5.5.1, v). We have already seen, in the proof of (8.11.5), that \( q' \) is surjective; since \( q = u \circ q' \) is proper, we finally conclude, from (5.4.3) and (5.4.9), that \( u \) is universally closed. \( \square \)

**Proposition (8.11.7).** — Let \( Y \) be a prescheme, \( X \) an integral prescheme, \( p : X \to Y \) a proper morphism, \( \mathcal{L} \) a p-ample invertible \( \mathcal{O}_X \)-module, \( C \) a normal Y-prescheme, \( \epsilon : Y \to C \) a Y-section, and \( q : \mathcal{V}(\mathcal{L}) \to C \) a Y morphism, all such that the diagram in (8.9.1.1) commutes. Suppose further that, if \( p = (p_0, \theta) \), then \( \theta : \mathcal{O}_Y \to p_*(\mathcal{O}_X) \) is an isomorphism. Let \( \mathcal{I}' = \bigoplus_{n \geq 0} p_*(\mathcal{L}^n) \) and \( C' = \text{Spec}(\mathcal{I}') \), and let \( q' : \mathcal{V}(\mathcal{L}) \to C' \) be the canonical Y-morphism (8.8.5). Then the unique Y-morphism \( u : C' \to C \) such that \( q = u \circ q' \) is an isomorphism.

**Proof.** It follows from (8.8.6) that \( C' \) is integral; since \( C \) is a homeomorphism of the underlying subspaces \( C' \to C \) (\( u \) being bijective and closed, by (8.11.6)), \( C \) is irreducible, thus integral, and, since the restriction of \( u \) to a non-empty open subset of \( C' \) is an isomorphism to an open subset of \( C, u \) is birational. Since \( C \) is assumed to be normal, it suffices to apply (8.11.2) to obtain the conclusion. \( \square \)

**Remark (8.11.8).** —

(i) The hypothesis that \( C \) is normal implies that \( X \) is also normal. Indeed, \( C' = \text{Spec}(\mathcal{I}') \) is then normal, being isomorphic to \( C \), and integral, by (8.8.6); we thus conclude that \( \text{Proj}(\mathcal{I}') \) is normal. Indeed, the question is local on \( Y \); if \( Y \) is affine, with \( \mathcal{I}' = S' \), then the ring \( S' = \Gamma(C', \mathcal{I}') \) is integral and integrally closed (8.8.6.1), and so, for every homogeneous element \( f \in S' \), the graded ring \( S'_f \) is integral and integrally closed [SZ60, t. I, p. 257 and 261], and thus so too is the ring \( S'_{(f)} \) of its degree-zero terms, because the intersection of \( S'_f \) with the field of fractions of \( S'_{(f)} \) is equal to \( S'_{(f)} \); this proves our claim (6.3.4). Finally, since \( X \) is isomorphic to an open subscheme of \( \text{Proj}(\mathcal{I}') \) (8.8.1), \( X \) is indeed normal. We can thus express (8.11.7) in the following form: If \( X \) is integral and normal, and \( p = (p_0, \theta) : X \to Y \) is a proper morphism such that \( \theta : \mathcal{O}_Y \to p_*(\mathcal{O}_X) \) is an isomorphism, then, for every p-ample \( \mathcal{O}_X \)-module \( \mathcal{L} \), there exists exactly one way of contracting the null section of \( V = \mathcal{V}(\mathcal{L}) \) to obtain a normal Y-scheme \( C \) and a proper Y-morphism \( q : V \to C \).

(ii) When \( p \) is proper, the hypothesis \( p_*(\mathcal{O}_X) = \mathcal{O}_Y \) can be considered as an auxiliary hypothesis, not really restricting the generality of the result. Indeed, if it is not satisfied, then it suffices to replace \( Y \) with the scheme \( Y' = \text{Spec}(p_*(\mathcal{O}_X)) \), and to consider \( X \) as a \( Y' \)-scheme. We will return to this general method in chapter III, § 4.
8.12. Quasi-coherent sheaves on based cones

(8.12.1). Let us use the hypotheses and notation of (8.3.1). Let \( \mathcal{M} \) be a quasi-coherent graded \( \mathcal{O} \)-module; to avoid any confusion, we denote by \( \mathcal{M} \) the quasi-coherent \( \mathcal{O}_{\mathcal{C}} \)-module associated to \( \mathcal{M} \) (1.4.3) when \( \mathcal{M} \) is considered as a non-graded \( \mathcal{I} \)-module, and by \( \mathcal{P} \mathcal{O}_{\mathcal{I}}(\mathcal{M}) \) the quasi-coherent \( \mathcal{O}_{\mathcal{X}} \)-module associated to \( \mathcal{M} \), \( \mathcal{M} \) being considered this time as a graded \( \mathcal{I} \)-module (in other words, the \( \mathcal{O}_{\mathcal{X}} \)-module denoted by \( \mathcal{M} \) in (3.2.2)). In addition, we set

\[
\mathcal{M}_X = \mathcal{P} \mathcal{O}_{\mathcal{I}}(\mathcal{M}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{P} \mathcal{O}_{\mathcal{I}}(\mathcal{M}(n));
\]

the quasi-coherent graded \( \mathcal{O}_{\mathcal{X}} \)-algebra \( \mathcal{I} \) being defined by (8.6.1.1), \( \mathcal{P} \mathcal{O}_{\mathcal{I}}(\mathcal{M}) \) is equipped with a structure of a (quasi-coherent) graded \( \mathcal{I}_\mathcal{X} \)-module, by means of the canonical homomorphisms (3.2.6.1)

\[
\mathcal{O}_{\mathcal{X}}(m) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{P} \mathcal{O}_{\mathcal{I}}(\mathcal{M}(n)) \rightarrow \mathcal{P} \mathcal{O}_{\mathcal{I}}(\mathcal{I}(m) \otimes \mathcal{M}(n)) \rightarrow \mathcal{P} \mathcal{O}_{\mathcal{I}}(\mathcal{M}(m + n)),
\]

the verification of the axioms of sheaves of modules being done using the commutative diagram in (2.5.11.4).

If \( Y = \text{Spec}(A) \) is affine, \( \mathcal{I} = \tilde{\mathcal{S}} \), and \( \mathcal{M} = \tilde{M} \), where \( S \) is a graded \( A \)-algebra and \( M \) is a graded \( S \)-module, then, for every homogeneous element \( f \in S_+ \), we have

\[
\Gamma(X_f, \mathcal{P} \mathcal{O}_{\mathcal{I}}(\tilde{M})) = M_f
\]

by the definitions and (8.2.9.1).

Now consider the quasi-coherent graded \( \mathcal{I} \)-module

\[
\tilde{\mathcal{M}} = \mathcal{M} \otimes \mathcal{I} \mathcal{I}
\]

(\( \mathcal{I} \) being defined by (8.3.1.1)); this induces a quasi-coherent graded \( \mathcal{O}_\mathcal{C} \)-module \( \mathcal{P} \mathcal{O}_{\mathcal{I}}(\tilde{\mathcal{M}}) \), which we will also denote by

\[
\mathcal{M}^{\square} = \mathcal{P} \mathcal{O}_{\mathcal{I}}(\tilde{\mathcal{M}}).
\]

It is clear (3.2.4) that \( \mathcal{M}^{\square} \) is an additive functor which is exact in \( \mathcal{M} \), commuting with direct sums and with inductive limits.

**Proposition (8.12.2).** — With the notation of (8.3.2), we have canonical functorial isomorphisms

\[
i^*(\mathcal{M}^{\square}) \cong \tilde{\mathcal{M}}, \quad j^*(\mathcal{M}^{\square}) \cong \mathcal{P} \mathcal{O}_{\mathcal{I}}(\mathcal{M}).
\]

Indeed, \( i^*(\mathcal{M}^{\square}) \) is canonically identified with \( (\tilde{\mathcal{M}} \otimes (z - 1)) \mathcal{M} \) on \( \text{Spec}(\mathcal{I} / (z - 1)) \mathcal{I} \) by (3.2.3); the first of the canonical isomorphisms (8.12.2.1) is then immediately induced (1.4.1) by the canonical isomorphism \( \mathcal{M} \otimes (z - 1) \mathcal{M} \cong \tilde{\mathcal{M}} \). The canonical immersion \( j : X \rightarrow \mathcal{C} \) corresponds to the canonical homomorphism \( \mathcal{I} \rightarrow \mathcal{I} \) with kernel \( z \mathcal{I} \) (8.3.2); the second homomorphism (8.12.2.1) is the particular case of the canonical homomorphism (3.5.2, ii), since here we have \( \mathcal{M} \otimes \mathcal{I} \mathcal{I} = \mathcal{M} \); to verify that this is an isomorphism, we can restrict to the case where \( Y = \text{Spec}(A) \) is affine, \( \mathcal{I} = \tilde{\mathcal{S}} \), and \( \mathcal{M} = \tilde{M} \) by appealing to (2.8.8), the proof that, for all homogeneous \( f \) in \( S_+ \), the preceding homomorphism, restricted to \( X_f \), restricts to an isomorphism, is then immediate.

By an abuse of language, we again say, thanks to the existence of the first isomorphism (8.12.2.1), that \( \mathcal{M}^{\square} \) is the projective closure of the \( \mathcal{O}_{\mathcal{X}} \)-module \( \tilde{\mathcal{M}} \) (it being implicit that the data of the \( \mathcal{O}_{\mathcal{C}} \)-module \( \tilde{\mathcal{M}} \) includes the grading of the \( \mathcal{I} \)-module \( \mathcal{M} \)).

(8.12.3). With the notation of (8.3.5), we have a canonical functorial homomorphism

\[
p^*(\mathcal{P} \mathcal{O}_{\mathcal{I}}(\mathcal{M})) \rightarrow \mathcal{M}^{\square} \mathcal{I} \mathcal{E}.
\]

Indeed, this is a particular case of the homomorphism \( \nu^* \) defined more generally in (3.5.6). If \( Y = \text{Spec}(A) \) is affine, \( \mathcal{I} = \tilde{\mathcal{S}} \), and \( \mathcal{M} = \tilde{M} \), then, by appealing to (2.8.8), the restriction of (8.12.3.1) to \( p^{-1}(X_f) = \tilde{C}_f \) (for some homogeneous \( f \) in \( S_+ \)) corresponds to the canonical homomorphism

\[
M_f \otimes S(f)S_f \leq \rightarrow M_f^\leq
\]

taking into account (8.2.3.2) and (8.2.5.2).
(8.12.4). Let us place ourselves in the settings of (8.5.1), and assume its hypotheses and keep its notation. It follows from (1.5.6) that, for every quasi-coherent graded $\mathcal{S}$-module $\mathcal{M}$, we have, on one hand, a canonical homomorphism

\[ \Phi^* (\mathcal{M}) \xrightarrow{\sim} (q^* (\mathcal{M}) \otimes q^*(\mathcal{S}) \mathcal{S})^{-} \]  

of $\mathcal{O}_C$-modules; on the other hand, (3.5.6) implies the existence of a canonical $\text{Proj}(q)$-morphism

\[ \text{Proj}(q_0, \mathcal{M}) \longrightarrow (\mathcal{R} \mathcal{S}_0 (q^* (\mathcal{M})) \otimes q^*(\mathcal{S}) \mathcal{S}) \mathcal{S} | G(q) \]

and also of a canonical $\tilde{\Phi}$-morphism

\[ \text{Proj}(q_0, \mathcal{M}) \longrightarrow (\mathcal{R} \mathcal{S}_0 (q^* (\mathcal{M})) \otimes q^*(\mathcal{S}) \mathcal{S}) \mathcal{S} | G(q). \]

(8.12.5). Consider now the setting of (8.6.1), with the same notation; we thus take $Y' = X$, the morphism $q : X \rightarrow Y$ being the structure morphism, and $q$ the canonical $q$-morphism (8.6.1.2). We then have a canonical isomorphism

\[ q^* (\mathcal{M}) \otimes q^*(\mathcal{S}) \mathcal{S} \overset{\sim}{\longrightarrow} \mathcal{M}^\wedge \]

by setting $\mathcal{M}^\wedge = \bigoplus_{n \geq 0} \mathcal{R} \mathcal{S}_0 (\mathcal{M}(n))$. We can indeed restrict to the case where $Y = \text{Spec}(A)$ is affine, $\mathcal{S} = \tilde{\mathcal{S}}$, and $\mathcal{M} = \tilde{\mathcal{M}}$, and define the isomorphism (8.12.5.1) on each of the affine open subsets $X_f$ (where $f$ is homogeneous in $S_+$), by verifying the compatibility with taking a homogeneous multiple of $f$. But the restriction to $X_f$ of the left-hand side of (8.12.5.1) is $\tilde{M}' = ((M \otimes_A S(f)) \otimes_{S(f)} S_S^f)^\wedge$ by (8.6.2.1); since we have a canonical isomorphism from $M \otimes_A S(f)$ to $M \otimes_S (S \otimes_A S(f))$, we have an induced isomorphism from $\tilde{M}'$ to $(M \otimes_S S_S^f)^\wedge$, and the latter is canonically isomorphic, by (8.2.9.1), to the restriction to $X_f$ of the right-hand side of (8.12.5.1), and satisfies the required compatibility conditions.

Replacing $\mathcal{M}$ by $\tilde{\mathcal{M}}$, $\mathcal{S}$ by $\tilde{\mathcal{S}}$, and $\mathcal{S} \mathcal{S}$ by $(\mathcal{S} \mathcal{S})^\wedge$ in the previous argument, we similarly have a canonical isomorphism

\[ q^* (\mathcal{M}) \otimes q^*(\mathcal{S}) (\mathcal{S} \mathcal{S})^\wedge \overset{\sim}{\longrightarrow} (\mathcal{M}^\wedge)^\wedge. \]

If we recall (8.6.2) that the structure morphism $u : \text{Proj}(\mathcal{S} \mathcal{S}) \rightarrow X$ is an isomorphism, then we deduce, first of all, from the above, that we have a canonical $u$-isomorphism

\[ \text{Proj}(q_0, \mathcal{M}) \overset{\sim}{\longrightarrow} \text{Proj}(q_0, (\mathcal{M}^\wedge)) \]

as a particular case of (8.12.4.2). We note that, with the notation from the proof of (8.6.2), this reduces to seeing that the canonical homomorphism $M(f) \otimes_{S(f)} (S_S^f)^{\vdash} \rightarrow (M(f)^{\vdash})^{\vdash}$ is an isomorphism whenever $f \in S_+$, which is immediate.

Secondly, the isomorphism (8.12.5.2) gives us, by this time applying (8.12.4.3) to the canonical morphism $r = \text{Proj}(\tilde{\mathcal{M}}) : \tilde{\mathcal{C}}_X \rightarrow \tilde{\mathcal{C}}$, a canonical $r$-morphism

\[ \mathcal{M}^\square \longrightarrow (\mathcal{M}^\wedge)^\square. \]

Recall now (8.6.2) that the restrictions of $r$ to the pointed cones $\tilde{E}_X$ and $E_X$ are isomorphisms to $\tilde{E}$ and $E$ (respectively). Furthermore:

**Proposition (8.12.6).** — The restrictions to $\tilde{E}_X$ and $E_X$ of the canonical $r$-morphism (8.12.5.4) are isomorphisms

\[ (8.12.6.1) \quad \mathcal{M}^\square | \tilde{E} \overset{\sim}{\longrightarrow} (\mathcal{M}^\wedge)^\square | \tilde{E}_X \]

\[ (8.12.6.2) \quad \mathcal{M}^\wedge | \tilde{E} \overset{\sim}{\longrightarrow} (\mathcal{M}^\wedge)^\wedge | E_X. \]

**Proof.** We restrict to the case where $Y$ is affine, as in the proof of (8.6.2) (whose notation we adopt); by reducing to definitions (2.8.8), we have to show that the canonical homomorphism

\[ \tilde{M}(f) \otimes_{S(f)} (S_S^f)^\wedge |_{(f/1)} \longrightarrow (M \otimes_S S_S^f)^\wedge |_{(f/1)} \]
is an isomorphism; but, by (8.2.3.2) and (8.2.5.2), the left-hand side is canonically identified with \( M_f^\leq \otimes_{S_f^\leq} (S_f^>)^\leq_{f/1} \), and thus with \( M_f^\leq \), by (8.2.7.2), and the right-hand side with \( (M_f^>)^\leq_{f/1} \), and thus also with \( M_f^\leq \), by (8.2.9.2), whence the conclusion concerning (8.12.6.1); (8.12.6.2) then follows from (8.12.6.1) and (8.12.2.1).

**Corollary (8.12.7).** With the identifications of (8.6.3), the restriction of \( (\mathcal{M}_X^>)^\leq \) to \( \tilde{E}_X \) can be identified with \( (\mathcal{M}_X^>)^\leq \tilde{\Omega}_X \), and the restriction of \( (\mathcal{M}_X^>)^\leq \) to \( E_X \) with \( \tilde{\mathcal{H}}_X \).

**Proof.** We can restrict to the affine case, and this follows from the identification of \( (M_f^>)^\leq_{f/1} \) with \( M_f^\leq \) and of \( (M_f^>)^\leq_{f/1} \) with \( M_f \) (8.2.9.2).

**Proposition (8.12.8).** Under the hypotheses of (8.6.4), the canonical homomorphism (8.12.3.1) is an isomorphism.

**Proof.** Taking into account the fact that \( \text{Proj}(\mathcal{S}_X^\leq) \to X \) is an isomorphism (8.6.2), and the isomorphisms (8.12.5.4) and (8.12.6.1), we are led to proving the corresponding proposition for the canonical homomorphism \( p_X^*(\text{Proj}_0(\mathcal{M}_X^>)^\leq) \to (\mathcal{M}_X^>)^\leq |E_X \) or, in other words, we can restrict to the case where \( \mathcal{S}_f \) is an invertible \( \tilde{\mathcal{O}}_Y \)-module, and where \( \mathcal{S} \) is generated by \( \mathcal{S}_f \). With the notation of (8.12.3), we then have, for some \( f \in S_1 \), that \( S_f^\leq = S_{(f)}[1/f] \), and the canonical homomorphism \( M_{(f)} \otimes_{S_{(f)}} S_f^\leq \to M_f^\leq \) is an isomorphism, by the definition of \( M_f^\leq \).

(8.12.9). Consider now the quasi-coherent \( \mathcal{S} \)-modules

\[
\mathcal{M}[n] = \bigoplus_{m \geq n} \mathcal{M}_m
\]

and (with the notation of (8.7.2)) the graded quasi-coherent \( \mathcal{S}^\leq \)-module

\[
\mathcal{M}^\leq = \left( \bigoplus_{n \geq 0} \mathcal{M}[n] \right) ^\leq
\]

We have seen (8.7.3) that there exists a canonical \( C \)-isomorphism \( h : C_X \to \text{Proj}(\mathcal{S}^\leq) \). Furthermore:

**Proposition (8.12.10).** There exists a canonical \( h \)-isomorphism

\[
\text{Proj}_0(\mathcal{M}^\leq) \to \mathcal{M}_X
\]

**Proof.** We argue as in (8.7.3), this time using the existence of the di-isomorphism (8.2.9.3) instead of (8.2.7.3). We leave the details to the reader.

### 8.13. Projective closures of subsheaves and closed subschemes

(8.13.1). With hypotheses and notation as in (8.12.1), consider a *not-necessarily graded* quasi-coherent sub-\( \mathcal{S} \)-module \( \mathcal{N} \) of \( \mathcal{M} \). We can then consider the quasi-coherent \( \tilde{\mathcal{O}}_C \)-module \( \tilde{\mathcal{N}} \) associated to \( \mathcal{N} \), which is a sub-\( \tilde{\mathcal{O}}_C \)-module of \( \tilde{\mathcal{M}} \). We have seen elsewhere (8.2.9.1) that \( \tilde{\mathcal{M}} \) can be identified with the restriction of \( \mathcal{M}^\square \) to \( C \). Since the canonical injection \( i : C \to \tilde{C} \) is an affine morphism (8.3.2), and *a fortiori* quasi-compact, the canonical extension \( \tilde{\mathcal{N}}^\leq \), the largest sub-\( \tilde{\mathcal{O}}_C \)-module contained in \( \mathcal{M}^\square \) and inducing \( \tilde{\mathcal{N}} \) on \( C \), is a quasi-coherent \( \tilde{\mathcal{O}}_C \)-module (I, 9.4.2). We will give a more explicit description by using a graded \( \mathcal{S} \)-module.

(8.13.2). For this, consider, for every integer \( n \geq 0 \), the homomorphism \( \bigoplus_{i \leq n} \mathcal{M}_i \to \mathcal{M} \) which, for every open \( U \) of \( Y \), sends the family

\[
(s_i) \in \bigoplus_{i \leq n} \Gamma(U, \mathcal{M}_i)
\]

to the section \( \sum s_i \in \Gamma(U, \mathcal{M}) \). Denote by \( \mathcal{N}'_n \) the inverse image of \( \mathcal{N} \) by this homomorphism, which is a quasi-coherent sub-\( \mathcal{S} \)-module of \( \bigoplus_{i \leq n} \mathcal{M}_i \). Now consider the homomorphism \( \bigoplus_{i \leq n} \mathcal{M}_i \to \mathcal{M} = \mathcal{M}[z] \) which sends \( (s_i) \) to the section \( \sum s_i z^{n-i} \in \Gamma(U, \mathcal{M}_n) \), and let \( \mathcal{N}_n \) be the image of
We can thus already suppose that \( \mathcal{N} = \bigoplus_{n \geq 0} \mathcal{N}_n \) is a (quasi-coherent) sub-\( \mathcal{F} \)-module of \( \mathcal{M} \); we say that \( \mathcal{N} \) is induced from \( \mathcal{N}' \) by homogenisation, via the “homogenising variable” \( z \). We note that, if \( \mathcal{N} \) is already a graded sub-\( \mathcal{F} \)-module of \( \mathcal{M} \), then \( \mathcal{N} \) can be identified with the direct sum of the components \( \mathcal{N}_n \) of degree \( n \geq 0 \) in \( \mathcal{N} = \mathcal{N}[z] \).

**Proposition (8.13.3).** — The \( \mathcal{O}_C \)-module \( \mathcal{N}(\mathcal{N}) \) is the canonical extension \( (\mathcal{N}) \) of \( \mathcal{N} \) to \( \mathcal{C} \).

**Proof.** The question is local on \( Y \) and \( \mathcal{C} \) by the definition of the canonical extension (I, 9.4.1). We can thus already suppose that \( Y = \text{Spec}(A) \) is affine, with \( \mathcal{F} = \mathcal{S}, \mathcal{M} = \mathcal{M}, \) and \( \mathcal{N} = \mathcal{N} \), where \( N \) is a non-necessarily-graded sub-\( S \)-module of \( M \). Furthermore (8.3.2.6), \( \mathcal{C} \) is a union of affine opens \( \mathcal{C}_z = C \) and \( \mathcal{C}_f = \text{Spec}(S^f) \) (with \( f \) homogeneous in \( S_+ \)). It thus suffices to show that: (1) the restriction of \( \mathcal{N}(\mathcal{N}) \) to \( C \) is \( \mathcal{N} \); (2) the restriction of \( \mathcal{N}(\mathcal{N}) \) to each \( \mathcal{C}_f \) is the canonical extension of the restriction of \( \mathcal{N} \) to \( C \cap \mathcal{C}_f = \text{Spec}(S_f) \) (8.3.2.6). For the first point, note that \( \mathcal{N}(\mathcal{N}) \mid C \) can be identified with \( (\mathcal{N}(z)) \) (8.3.4); but \( \mathcal{N}(z) \) is canonically identified (2.2.5) with the image of \( \mathcal{N} \) in \( \mathcal{M}/(z - 1)\mathcal{M} \), and by the canonical isomorphism of the latter with \( M \) (8.2.5), this image can be identified with \( N \), by the definition of \( \mathcal{N} \) given in (8.13.2).

To prove the second point, note that the injection \( i : C \cap \mathcal{C}_f \to \mathcal{C} \) corresponds to the canonical injection \( S^f \to S_f \) (8.3.2.6); we also have that \( \Gamma(\mathcal{C}_f, \mathcal{M}^f) = M^f \), that \( \Gamma(\mathcal{C}_f, \mathcal{N}(\mathcal{N})) = N \), and, by (8.12.2.1), that \( \Gamma(\mathcal{C}_f, i^*(\mathcal{M}^f)) = M_f \). Taking (I, 9.4.2) into account, we are thus led to showing that \( \mathcal{N}(f) \subset \mathcal{M}/(z - 1)\mathcal{M} \) is canonically identified with the inverse image of \( N_f \) under the canonical injection \( M^f \to M_f \). Indeed, let \( d = \deg(f) > 0 \), and suppose that an element \( (\sum_{k \leq m} x_k)/f^m \) of \( M_f \) (with \( x_k \in M_k \)) is of the form \( y/f^m \) with \( y \in N \). By multiplying \( y \) and the \( x_k \) by one single suitable \( f^h \), we can already assume that \( \sum_{k \leq m} x_k = y \). But in the identification of (8.2.5.2), \( (\sum_{k \leq m} x_k)/f^m \) corresponds to \( \sum_{k \leq m} x_k z^{nd-k}/f^m \), and this is indeed an element of \( \mathcal{N}(f) \), since \( \sum_{k \leq m} x_k \in N \); the converse is evident. \[ \square \]

**Remark (8.13.4).** —

(i) The most important case of application of (8.13.3) is that where \( \mathcal{M} = \mathcal{F} \), with \( \mathcal{N} \) then being an arbitrary quasi-coherent sheaf of ideals \( \mathcal{F} \) of \( \mathcal{O}_C \) (1.4.3), corresponding bijectively to a closed subscheme \( Z \) of \( C \). Then the canonical extension \( \mathcal{F} \) of \( \mathcal{F} \) is the quasi-coherent sheaf of ideals of \( \mathcal{O}_C \) that defines the closure \( Z \) of \( Z \) in \( \mathcal{C} \) (I, 9.5.10); Proposition (8.13.3) gives a canonical way of defining \( Z \) by using a graded ideal in \( \mathcal{F} = \mathcal{F}[z] \).

(ii) Suppose, to simplify things, that \( Y \) is affine, and adopt the notation from the proof of (8.13.3). For every non-zero \( x \in \mathcal{N} \), let \( d(x) \) be the largest degree of the homogeneous components \( x_i \) of \( x \) in \( M \); by definition, \( \mathcal{N} \) is the submodule of \( M \) consisting of 0 and elements of the form \( h(x, k) = z^k \sum_{i \leq d(x)} x_i z^{d(x)-i} \) (for integral \( k \geq 0 \)); it is thus generated, as a module over \( \mathcal{S} = S[z] \), by the elements of the form

\[ h(x, 0) = \sum_{i \leq d(x)} x_i z^{d(x)-i}. \]

We say that \( h(x, 0) \) is induced from \( x \) by homogenisation via the “homogenising variable” \( z \). But since \( h(x, 0) \) does not depend additively on \( x \) (nor \( \text{a fortiori} \), \( S \)-linearly), we will refrain from believing (even when \( M = S \)) that the \( h(x, 0) \) form a system of generators of the graded \( S \)-module \( \mathcal{N} \) when we let \( x \) run over a system of generators of the \( S \)-module \( N \). This is, however, the case (considered only in elementary algebraic geometry) when \( N \) is a free cyclic \( S \)-module, since, if \( t \) is a basis of \( N \), then \( h(t, 0) \) generates the \( \mathcal{S} \)-module \( \mathcal{N} \).
8.14. Supplement on sheaves associated to graded \(\mathcal{I}\)-modules

(8.14.1). Let \(Y\) be a prescheme, \(\mathcal{I}\) a positively-graded quasi-coherent \(O_Y\)-algebra, \(X = \text{Proj}(\mathcal{I})\), and \(q : X \to Y\) the structure morphism (which is separated, by (3.1.3)). Using the notation of (8.12.1), we have defined a functor \(\mathcal{M}_X = \text{Proj}(\mathcal{I})\) in \(\mathcal{M}\), from the category of graded quasi-coherent \(\mathcal{I}\)-modules to the category of graded quasi-coherent \(\mathcal{I}_X\)-modules; it is further clear (3.2.4) that this is an additive and exact functor, commuting with inductive limits.

Note, furthermore, that it follows immediately from the definition (8.12.1.1) that we have

\[(8.14.1)\]
\[
\text{Proj}(\mathcal{M}(n)) = (\text{Proj}(\mathcal{M}))(n) \quad \text{for all } n \in \mathbb{Z}.
\]

(8.14.2). We will first extend the canonical homomorphisms \(\lambda\) and \(\mu\), defined in (3.2.6), to \(\mathcal{I}_X\)-modules of the form \(\text{Proj}(\mathcal{I})\). For this, note that, for any \(m \in \mathbb{Z}\) and \(n \in \mathbb{Z}\), we have, by (2.1.2.1), a canonical homomorphism of \(\mathcal{O}_X\)-modules

\[
(8.14.2.1) \lambda_{mn} : \text{Proj}_0((\text{Hom}_\mathcal{I}(\mathcal{M}, \mathcal{N}))(n - m)) \to \text{Hom}_{\mathcal{O}_X}(\text{Proj}_0(\mathcal{M}(m)), \text{Proj}_0(\mathcal{N}(n)))
\]

for any graded quasi-coherent \(\mathcal{I}\)-modules \(\mathcal{M}\) and \(\mathcal{N}\). This induces a homomorphism

\[
(8.14.2.2) \mu_k : \text{Proj}_0((\text{Hom}_\mathcal{I}(\mathcal{M}, \mathcal{N}))(k)) \to (\text{Hom}_{\mathcal{O}_X}(\text{Proj}(\mathcal{M}), \text{Proj}(\mathcal{N})))
\]

given by sending every \(u \in \Gamma(U, \text{Proj}_0((\text{Hom}_\mathcal{I}(\mathcal{M}, \mathcal{N}))(k))))\) to the homomorphism \(\mu_k(u)\), of degree \(k\), of graded \(\mathbb{Z}\)-modules \(\Gamma(U, \text{Proj}_0(\mathcal{M})) \to \Gamma(U, \text{Proj}_0(\mathcal{N}))\) (where \(U\) is open in \(X\)) which, in each \(\Gamma(U, \text{Proj}_0(\mathcal{M}(m))))\), agrees with \(\mu_{m,m+k}(u)\); furthermore, by returning to the definition of the \(\mu_{mn}\) (2.5.12.1), we immediately see that \(\mu_k(u)\) is in fact a homomorphism of degree \(k\) of graded \(\Gamma(U, \mathcal{I}_X)\)-modules, and, furthermore, that the \(\mu_k\) define a homomorphism of graded \(\mathcal{I}_X\)-modules

\[
(8.14.2.3) \text{Proj}(\text{Hom}_\mathcal{I}(\mathcal{M}, \mathcal{N})) \to \text{Hom}_{\mathcal{O}_X}(\text{Proj}(\mathcal{M}), \text{Proj}(\mathcal{N})).
\]

Similarly, taking the associativity diagram (2.5.11.4) into account, the homomorphisms (8.14.2.1) give a homomorphism of graded \(\mathcal{I}_X\)-modules

\[
(8.14.2.4) \lambda : \text{Proj}(\mathcal{M}) \otimes_{\mathcal{I}_X} \text{Proj}(\mathcal{N}) \to \text{Proj}(\mathcal{M} \otimes \mathcal{N}).
\]

Proposition (8.14.3). The homomorphism (8.14.2.4) is bijective; so too is (8.14.2.3) whenever the graded \(\mathcal{I}\)-module \(\mathcal{M}\) admits a finite presentation (3.1.1).

Proof. The question is clearly local on \(X\) and \(Y\); we can thus suppose that \(Y = \text{Spec}(A)\) is affine, with \(\mathcal{I} = \mathcal{S}\), \(\mathcal{M} = M\), and \(\mathcal{N} = N\), where \(S\) is a positively-graded \(A\)-algebra, and \(M\) and \(N\) are graded \(S\)-modules. If \(f\) is a homogeneous element of \(S_+\), then the homomorphisms (8.14.2.1) and (8.14.2.2), restricted to the affine open \(D_+(f)\), correspond to the canonical homomorphisms (2.5.11.1) and (2.5.12.1):

\[
(8.14.2.5) M(m)(f) \otimes_{S(f)} N(n)(f) \to (M \otimes S N)(m + n)(f)
\]

\[
(\text{Hom}_S(M, N))(n - m)(f) \to \text{Hom}_{S(f)}(M(m)(f), N(n)(f)).
\]

If we refer to the definitions of these homomorphisms, we thus see (taking (8.2.9.1) into account) that the restriction of (8.14.2.4) to \(D_+(f)\) corresponds to the canonical homomorphism

\[
(8.14.2.6) M_f \otimes_{S_f} N_f \to (M \otimes S N)_f
\]

defined in (0, 1.3.4), and we know that this latter homomorphism is an isomorphism. Similarly, the restriction of (8.14.2.3) to \(D_+(f)\) corresponds to the canonical homomorphism (0, 1.3.5)

\[
(8.14.2.7) (\text{Hom}_S(M, N))_f \to \text{Hom}_{S_f}(M_f, N_f)
\]

taking into account the fact that, since \(M\) is of finite type, the module \(\text{Hom}_S(M, N)\), the direct sum of the subgroups consisting of homogeneous homomorphisms of \(S\)-modules (2.1.2), agrees with the set of all homomorphisms \(M \to N\) of \(S\)-modules. The hypothesis that \(M\) admits a finite presentation then implies (0, 1.3.5) that the canonical homomorphism in question is indeed an isomorphism. □

Proposition (8.14.4). If \(U\) is a quasi-compact open of \(X\), then there exists an integer \(d\) such that, for every integer \(n\) that is a multiple of \(d\), \(O_X(n)|_U\) is invertible, with its inverse being \(O_X(−n)|_U\).
Proof. Since \( q(U) \) is quasi-compact, it is covered by a finite number of affine opens \( V_i \), and so every \( x \in U \) is contained in some affine open of the form \( D_+(f) \), where \( f \) is a homogeneous element of degree \( > 0 \) of one of the rings \( \Gamma(V_i, \mathcal{O}) \). Since \( U \) is quasi-compact, we can cover it by a finite number of such opens \( D_+(f_j) \); let \( d \) be a common multiple of the degrees of the \( f_j \). This \( d \) satisfies the desired property, by (2.5.17).

(8.14.5). With the hypotheses and notation of (8.14.1), we defined, in (3.3.2), canonical homomorphisms of \( \mathcal{O}_Y \)-modules

\[
\alpha_n : M_n \longrightarrow q_+ (\mathcal{Q}_0^+ (M_n)) \quad (n \in \mathbb{Z}).
\]

Generalising the notation of (3.3.1), we set, for every graded \( \mathcal{O}_X \)-module \( \mathcal{F} \),

\[
\Gamma_+ (\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} q_+ (\mathcal{F}_n).
\]

In particular, \( \Gamma (\mathcal{F}_X) = \bigoplus_{n \in \mathbb{Z}} q_+ (\mathcal{O}_X (n)) \) is the graded \( \mathcal{O}_Y \)-algebra denoted by \( \Gamma_+ (\mathcal{O}_X) \) in (3.3.1.2); it is clear that \( \Gamma (\mathcal{F}) \) is a graded \( \Gamma_+ (\mathcal{O}_X) \)-algebra (0.4.2.2). When we take \( M = \mathcal{F} \) in the homomorphisms (8.14.5.1), we obtain the homomorphism of graded \( \mathcal{O}_Y \)-algebras

\[
\alpha : \mathcal{F} \longrightarrow \Gamma (\mathcal{F}_X)
\]

previously defined in (3.3.2), and which makes \( \Gamma_+ (\mathcal{F}) \) a graded \( \mathcal{O}_X \)-module; the homomorphisms (8.14.5.1) then define a homomorphism (of degree 0) of graded \( \mathcal{O}_X \)-modules

\[
\alpha : M \longrightarrow \Gamma_+ (\mathcal{Q}_0^+ (M)).
\]

(8.14.6). In general, for a graded quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \), it is not certain that the graded \( \mathcal{O}_X \)-module \( \Gamma_+ (\mathcal{F}) \) will necessarily be quasi-coherent. Consider an open \( X' \) of \( X \) such that the restriction \( q' : X' \rightarrow Y \) of \( q \) to \( X' \) is a quasi-compact morphism. Since \( q' \) is further separated, \( q_+ (\mathcal{F}) \) is then a quasi-coherent \( \mathcal{O}_Y \)-module for every graded quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{F}' \) (I, 9.2.2, b). We set

\[
\mathcal{F}_X' = \mathcal{F}_X |_{X'} = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X (n) |_{X'}
\]

and, for every graded \( \mathcal{O}_X \)-module \( \mathcal{F}' \),

\[
\Gamma_+ (\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} q_+ (\mathcal{F}_n).
\]

The previous remark then shows that, if \( \mathcal{F}' \) is a quasi-coherent \( \mathcal{O}_X \)-module, then \( \Gamma_+ (\mathcal{F}') \) is a graded quasi-coherent \( \mathcal{O}_X \)-module (I, 9.6.1).

We note also that the canonical injection \( j : X' \rightarrow X \) is quasi-compact, because \( q' = q \circ j \) is quasi-compact and \( q \) is separated (I, 6.6.4, v). Then \( \mathcal{F} = j_* (\mathcal{F}') \) is a graded quasi-coherent \( \mathcal{O}_X \)-module for every graded quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{F}' \), and it follows from the previous definitions that

\[
\Gamma_+ (\mathcal{F}') = \Gamma_+ (\mathcal{F}).
\]

With the same hypotheses on \( X' \), for every graded quasi-coherent \( \mathcal{O}_X \)-module \( M \), we set

\[
\mathcal{Q}_0^+ (M) = \mathcal{Q}_0^+ (M) |_{X'}
\]

which is a graded quasi-coherent \( \mathcal{O}_X \)-module. The canonical homomorphism

\[
\mathcal{Q}_0^+ (M) \longrightarrow j_* (\mathcal{Q}_0^+ (M))
\]

(0, 4.4.3) thus gives a canonical homomorphism \( \Gamma_+ (\mathcal{Q}_0^+ (M)) \rightarrow \Gamma_+ (\mathcal{Q}_0^+ (M)) \) of graded \( \mathcal{O}_X \)-modules, and, by composition with (8.14.5.4), we obtain a functorial canonical homomorphism (of degree 0) of graded quasi-coherent \( \mathcal{O}_X \)-modules

\[
\alpha' : M \longrightarrow \Gamma_+ (\mathcal{Q}_0^+ (M)).
\]

(8.14.7). Keeping the hypotheses on \( X' \) from (8.14.6), let \( \mathcal{F}' \) be a graded quasi-coherent \( \mathcal{O}_X \)-module such that \( \mathcal{Q}_0^+ (\Gamma_+ (\mathcal{F}')) \) is also a graded quasi-coherent \( \mathcal{O}_X \)-module. We will define a functorial canonical homomorphism (of degree 0) of graded \( \mathcal{O}_X \)-modules

\[
\beta' : \mathcal{Q}_0^+ (\Gamma_+ (\mathcal{F}')) \longrightarrow \mathcal{F}'.
\]
Suppose first of all that $Y = \text{Spec}(A)$ is affine, and that $\mathcal{S} = \overline{S}$, where $S$ is a positively-graded $A$-algebra; then $\Gamma'_*(\mathcal{F}') = \overline{M}$, where $M = \bigoplus n \in \mathbb{Z}\Gamma(X', \mathcal{F}_n)$ is a graded $S$-module. Let $f \in S_d$ be such that $D_+(f) \subset X'$; by definition (2.6.2), $a_d(f)$ restricted to $D_+(f)$ is the section of $\mathcal{O}_X(d)$ over $D_+(f)$ corresponding to the element $f/1$ of $(S(d))_f$, and is thus invertible; thus so too is $a_d(f^n)$ for every $n > 0$. From this, we immediately conclude that we have defined an $\overline{\mathcal{S}}$-homomorphism (of degree 0) of graded modules proving (with the notation of (8.14.7)) that the homomorphism $\beta : M_f \to \Gamma(D_+(f), \mathcal{F}')$ by sending each element $z/f^n \in M_f$ (where $z \in M$) to the section $(z|D_+(f))(a_d(f^n)|D_+(f))^{-1}$ of $\mathcal{F}'$ over $D_+(f)$. Furthermore, we have a commutative diagram corresponding to (2.6.4.1), whence the definition of $\beta'$ in this case. To pass to the general case, we must consider an $A$-algebra $A'$, the graded $A'$-algebra $S' = S \odot_A A'$, and use the commutative diagram analogous to (2.8.13.2); we leave the details to the reader.

**Proposition (8.14.8).** — If $X'$ is an open of $X = \text{Proj}(\mathcal{S})$ such that $q' : X' \to Y$ is quasi-compact, then the homomorphism $\beta'$ defined in (8.14.7) is bijective.

**Proof.** We can clearly restrict to the case where $Y$ is affine, and everything then reduces to proving (with the notation of (8.14.7)) that the homomorphism $\beta_f : M_f \to \Gamma(D_+(f), \mathcal{F}')$ is an isomorphism. But replacing $f$ by one of its powers changes neither $D_+(f)$ nor $\beta_f$; since $X'$ is quasi-compact by hypothesis, we can always assume, by (8.14.4), that the sheaf $\mathcal{O}_X(d)$ is invertible. Since $X'$ is a scheme (because $q'$ is separated), the proposition is then exactly (I, 9.3.1). □

**Corollary (8.14.9).** — Under the hypotheses of (8.14.8), every graded quasi-coherent $\mathcal{S}_X$-module is isomorphic to a graded $\mathcal{S}_X$-module of the form $\text{Proj}^f(\mathcal{M})$, where $\mathcal{M}$ is a graded quasi-coherent $\mathcal{S}$-module. Further, if $\mathcal{F}'$ is of finite type, and if we assume that $Y$ is a quasi-compact scheme, or a prescheme whose underlying space is Noetherian, then we can assume that $\mathcal{M}$ is of finite type.

**Proof.** The proof starting from (8.14.8) follows exactly the same route as the proof of (3.4.5) starting from (3.4.4), and we leave the details to the reader. □

**Proposition (8.14.10).** — Under the hypotheses of (8.14.7), let $\mathcal{M}$ be a graded quasi-coherent $\mathcal{S}$-module, and $\mathcal{F}'$ a graded quasi-coherent $\mathcal{S}_X$-module; the composite homomorphisms

\[
(8.14.10.1) \quad \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{F}'(\mathcal{M}) \xrightarrow{\mathcal{S} \otimes \beta_f} \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{F}'(\Gamma'_*(\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{M})) \xrightarrow{\beta'} \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{F}'(\mathcal{M})
\]

\[
(8.14.10.2) \quad \Gamma'_*(\mathcal{F}') \xrightarrow{\beta'_f} \Gamma'_*(\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{F}'(\mathcal{M})) \xrightarrow{\Gamma'_*(\beta_f)} \Gamma'_*(\mathcal{F}')
\]

are the identity isomorphisms.

**Proof.** The question is local on $Y$, and the proof follows as in (2.6.5); we leave the details to the reader. □

**Remark (8.14.11).** — In chapter III (III, 2.3.1), we will see that, when $Y$ is locally Noetherian, and $\mathcal{S}$ is a graded quasi-coherent $\mathcal{O}_Y$-algebra of finite type (in which case we can take $X' = X$), then the homomorphism $a$ (8.14.5.4) is (TN)-bijective for every graded quasi-coherent $\mathcal{S}$-module $\mathcal{M}$ satisfying condition (TF).

**Remark (8.14.12).** — The situation described in (8.14.4) is a particular case of the following. Let $X$ be a ringed space, and $\mathcal{F}$ a (positively- and negatively-) graded $\mathcal{O}_X$-algebra; suppose that there exists an integer $d > 0$ such that $\mathcal{S}_d$ and $\mathcal{S}_{-d}$ are invertible, with the canonical homomorphism

\[
(8.14.12.1) \quad \mathcal{S}_d \otimes \mathcal{O}_X \mathcal{S}_{-d} \to \mathcal{O}_X
\]

being an isomorphism (such that $\mathcal{S}_{-d}$ is identified with $\mathcal{S}_d^{-1}$). We then say that the graded $\mathcal{O}_X$-algebra $\mathcal{S}$ is periodic, of period $d$. This nomenclature stems from the following property: under the preceding hypotheses, for every graded $\mathcal{S}$-module $\mathcal{F}$, the canonical homomorphism

\[
(8.14.12.1) \quad \mathcal{S}_d \otimes \mathcal{F}_n \to \mathcal{F}_{n+d}
\]

is an isomorphism for all $n \in \mathbb{Z}$. Indeed, the question is local on $X$, and we can assume that $\mathcal{S}_d$ has an invertible section $s$ over $X$, with its inverse $s'$ being a section of $\mathcal{S}_{-d}$. The homomorphism $\mathcal{F}_{n+d} \to \mathcal{S}_d \otimes \mathcal{F}_n$, which sends each section $z \in \Gamma(U, \mathcal{F}_{n+d})$ to the section $(s|U) \otimes (s'|U)z$ of
\( \mathcal{S}_d \otimes F_n \) over \( U \), is then the inverse of (8.14.12.2), whence our claim. This induces, for all \( k \in \mathbb{Z} \), a canonical isomorphism
\[
(\mathcal{S}_d)^{\otimes k} \otimes F_n \sim \mathcal{F}_{n+kd}.
\]
Then the data of a graded \( \mathcal{S} \)-module \( F \) is equivalent to the data of \( \mathcal{S}_0 \)-modules \( F_i \) \( (0 \leq i \leq d-1) \) and canonical homomorphisms
\[
\mathcal{S}_i \otimes F_j \rightarrow F_{i+j} \quad \text{for } 0 \leq i, j \leq d-1
\]
(setting \( \mathcal{F}_{i+j} = \mathcal{S}_d \otimes \mathcal{S}_0 \mathcal{F}_{i+j-d} \) whenever \( i+j \geq d \)). Of course, for these homomorphisms to give a well-defined \( \mathcal{S} \)-module structure on the direct sum of the \((\mathcal{S}_d)^{\otimes k} \otimes F_i \) \( (k \in \mathbb{Z}, 0 \leq i \leq d-1) \), they should satisfy some associativity conditions that we will not explain.

In the case where \( d = 1 \) (which is the one considered in (3.3)), we can thus say that the category of graded \( \mathcal{S} \)-modules (resp. quasi-coherent \( \mathcal{S} \)-modules if \( X \) is a prescheme and \( \mathcal{S} \) is quasi-coherent) is equivalent to the category of arbitrary \( \mathcal{S}_0 \)-modules (resp. quasi-coherent \( \mathcal{S}_0 \)-modules); it is in this way that we can think of the results of this paragraph as generalising those of §3. Furthermore, we see that, under suitable finiteness conditions, the results of this paragraph (along with (8.14.11)) reduces, in some sense, the study of graded quasi-coherent algebras on a prescheme, and graded modules "modulo (TN)" on such algebras, to the study of the particular case where the algebras in question are periodic (and where condition (TN) for \( \mathcal{M} \) (3.4.2) thus implies that \( \mathcal{M} = 0 \)).

**Remark (8.14.13).** — Under the hypotheses of (8.14.1), let \( d \) be an integer \( > 0 \); we have defined a canonical \( Y \)-isomorphism \( h \) from \( X \to X^{(d)} = \text{Proj}(\mathcal{S}^{(d)}) \) (3.1.8). For every graded quasi-coherent \( \mathcal{S} \)-module \( \mathcal{M} \) and every integer \( k \) such that \( 0 \leq k \leq d-1 \), we also have (with the notation of (3.1.1)) a canonical \( h \)-isomorphism
\[
(\text{Proj}(\mathcal{M}))^{(d,k)} \sim (\text{Proj}(\mathcal{M}^{(d)})^{(d,k)}).
\]
Suppose, first of all, that \( Y = \text{Spec}(A) \) is affine, \( \mathcal{S} = \widetilde{S} \), and \( \mathcal{M} = \widetilde{M} \), where \( S \) is a positively-graded \( A \)-algebra, and \( M \) a graded \( S \)-module. We know, for every \( f \in S_e \) \( (e > 0) \), that \( h \) sends \( D_+(f) \) to \( D_+(f^d) \), and corresponds to the canonical isomorphism \( S((f^d)) \rightarrow S((f)) \) (2.2.2). The restriction of (8.14.13.1) to \( D_+(f^d) \) then corresponds to the canonical di-isomorphism \( M_{(f^d)} \rightarrow M_{(f)} \) restricted to the elements of \( M_{(f)} \), whose degree is congruent to \( k \) (modulo \( d \)). We leave to the reader the task of showing that these isomorphisms are compatible with passing from \( f \) to some homogeneous multiple \( g_f \), and then that there is an analogous compatibility with passing from \( S \) to a graded \( A' \)-algebra \( S' = S \otimes_A A' \), where \( A' \) is some \( A \)-algebra. In particular, this gives us an \( h \)-isomorphism
\[
(\mathcal{S}^{(d)})_{X^{(d)}} \sim (\mathcal{S}^{(d)})_{X^{(d)}}
\]
that respects the multiplicative structures of both the source and the target, and that, thanks to (8.14.13.1), becomes an \( h \)-di-isomorphism from a graded \( (\mathcal{S}^{(d)})_{X^{(d)}} \)-module to a graded \( (\mathcal{S}^{(d)})_{X^{(d)}} \)-module. Similarly, we have an \( h \)-isomorphism
\[
\text{Proj}(\mathcal{M}^{(d,k)}(n)) \sim \mathcal{O}_X(nud+k),
\]
which completes the result of (3.2.9, ii).

The isomorphism in (8.14.13.1) immediately induces an isomorphism of graded \( \mathcal{S}^{(d)} \)-modules
\[
\Gamma_s^{(d)}(\text{Proj}(\mathcal{M}^{(d,k)})) \sim \Gamma_s((\text{Proj}(\mathcal{M}))^{(d,k)})
\]
where \( \Gamma_s^{(d)} \) corresponds to the structure morphism \( q^{(d)} : X^{(d)} \rightarrow Y \); it can be immediately verified that the canonical homomorphism \( \alpha \) (8.14.5.4), and the analogous homomorphism \( \alpha^{(d)} \) for \( X^{(d)} \), make the following diagram commute:
\[
\begin{array}{ccc}
\Gamma_s^{(d)}(\text{Proj}(\mathcal{M}^{(d,k)})) & \rightarrow & \Gamma_s((\text{Proj}(\mathcal{M}))^{(d,k)})
\end{array}
\]
where we proceed by supposing that \( Y \) is affine and then calculating the restrictions of the images under \( \alpha^{(d)} \) and \( \alpha \) of some single element of \( M^{(d,k)} \) to the open subsets \( D_+(f^d) \) and \( D_+(f) \) (using the same notation as above).
Proposition (8.14.14). — Let $Y$ be a quasi-compact prescheme, $\mathcal{I}$ a graded quasi-coherent $\mathcal{O}_Y$-algebra of finite type, and $\mathcal{M}$ a graded quasi-coherent $\mathcal{I}$-module satisfying condition (TF); let $X = \text{Proj}(\mathcal{I})$. Then $\mathcal{O}_X$ is a periodic graded $\mathcal{O}_X$-algebra (8.14.12), and there exists some period $d$ of $\mathcal{O}_X$ such that the $(\text{Proj}(\mathcal{M}))^{(d,k)}$ $(0 \leq k \leq d - 1)$ are $(\mathcal{O}_X)^{(d)}$-modules of finite type.

Proof. Indeed, (3.1.10) proves that there exists some $d$ such that $\mathcal{O}^{(d)}$ is generated by $\mathcal{O}_d = (\mathcal{O}^{(d)})_1$, with the latter being an $\mathcal{O}_0$-module of finite type. To prove the first claim, we can thus, by (8.14.13.2), restrict to the case where $d = 1$, and the proposition then follows from (3.2.7). Furthermore, taking (8.14.13.1) into account, the second claim is a consequence of (2.1.6, iii) and (3.4.3). \qed
CHAPTER III

Cohomological study of coherent sheaves (EGA III)

build hack [CC]

SUMMARY

§1. Cohomology of affine schemes.
§2. Cohomological study of projective morphisms.
§3. Finiteness theorem for proper morphisms.
§5. An existence theorem for coherent algebraic sheaves.
§6. Local and global Tor functors; Künnett formula.
§8. The duality theorem for projective bundles.
§9. Relative cohomology and local cohomology; local duality.
§10. Relations between projective cohomology and local cohomology. Formal completion technique along a divisor.
§11. Global and local Picard groups.

This chapter gives the fundamental theorems concerning the cohomology of coherent algebraic sheaves, with the exception of theorems explaining the theory of residues (duality theorems), which will be the subject of a later chapter. Amongst all those included here, there are essentially six fundamental theorems, and each one is the subject of one of the first six chapters. These results will prove to be essential tools in all that follows, even in questions which are not truly cohomological in their nature; the reader will see the first such examples starting from §4. §7 gives some more technical results, but ones which are constantly used in applications. Finally, in §§8–11, we will develop certain results, related to the duality of coherent sheaves, that are particularly important for applications, and which can be explained even before the introduction of the full general theory of residues.

The content of §§1 and 2 is due to J.-P. Serre, and the reader will observe that we have had only to follow (FAC). §8 and 9 are equally inspired by (FAC) (the changes necessitated by the different contexts, however, being less evident). Finally, as we said in the Introduction, §4 should be considered as the formalisation, in modern language, of the fundamental “invariance theorem” of Zariski’s “theory of holomorphic functions”.

We draw attention to the fact that the results of n°3.4 (and the preliminary propositions of (0, 13.4 to 13.7)) will not be used in what follows Chapter III, and can thus be skipped in a first reading.

§1. COHOMOLOGY OF AFFINE SCHEMES

1.1. Review of the exterior algebra complex

(1.1.1). Let $A$ be a ring, $f = (f_i)_{1 \leq i \leq r}$ a system of $r$ elements of $A$. The exterior algebra complex $K_\bullet(f)$ corresponding to $f$ is a chain complex $(G, I, 2.2)$ defined in the following way: the graded $A$-module $K_\bullet(f)$ is equal to the exterior algebra $\wedge(A')$, graded in the usual way, and the boundary map is the interior multiplication $i_f$ by $f$ considered as an element of the dual $(A')^\vee$; we recall that $i_f$ is an
antiderivation of degree $-1$ of $\wedge (A^r)$, and if $(e_i)_{1 \leq i \leq r}$ is the canonical basis of $A^r$, then we have $i_q(e_i) = f_i$; the verification of the condition $i_q \circ i_q = 0$ is immediate.

An equivalent definition is the following: for each $i$, we consider a chain complex $K_i(f_i)$ defined as follows: $K_0(f_i) = K_1(f_i) = A$, $K_n(f_i) = 0$ for $n \neq 0, 1$: the boundary map is defined by the condition that $d_1 : A \to A$ is multiplication by $f_i$. We then take $K_*(f)$ to be the tensor product $K_0(f_1) \otimes K_0(f_2) \otimes \cdots \otimes K_0(f_r)$ (G, I, 2.7) with its total degree; the verification of the isomorphism from this complex to the complex defined above is immediate.

(1.1.2). For every $A$-module $M$, we define the chain complex

$K_*(f, M) = K_*(f) \otimes_A M$

and the cochain complex (G, I, 2.2)

$K^*(f, M) = \text{Hom}_A(K_*(f, M))$.

If $g$ is a $k$-cochain of this latter complex, and if we set

$g(i_1, \ldots, i_k) = g(e_{i_1} \wedge \cdots \wedge e_{i_k})$,

then $g$ identifies with an alternating map from $[1, r]^k$ to $M$, and it follows from the above definitions that we have

$(1.1.2.3) \quad d_k^* g(i_1, i_2, \ldots, i_{k+1}) = \sum_{h=1}^{k+1} (-1)^{h-1} f_{i_h} g(i_1, \ldots, \hat{i}_h, \ldots, i_{k+1})$.

(1.1.3). From the above complexes, we deduce as usual the homology and cohomology $A$-modules (G, I, III 83 2.2)

$(1.1.3.1) \quad H_*(f, M) = H_*(K_*(f, M))$,

$(1.1.3.2) \quad H^*(f, M) = H^*(K^*(f, M))$.

We define an $A$-isomorphism $K_*(f, M) \simeq K^*(f, M)$ by sending each chain $z = \sum(e_{i_1} \wedge \cdots \wedge e_{i_k}) \otimes z_{i_1, \ldots, i_k}$ to the cochain $g_z$ such that $g_z(j_1, \ldots, j_{r-k}) = \varepsilon z_{i_1, \ldots, i_k}$, where $(j_k)_{1 \leq h \leq r-k}$ is the strictly increasing sequence complementary to the strictly increasing sequence $(i_k)_{1 \leq h \leq r-k}$ in $[1, r]$ and $\varepsilon = (-1)^v$, where $v$ is the number of inversions of the permutation $i_1, \ldots, i_k, j_1, \ldots, j_{r-k}$ of $[1, r]$. We verify that $g_{dz} = d(g_z)$, which gives an isomorphism

$(1.1.3.3) \quad H^i(f, M) \simeq H_{r-i}(f, M) \text{ for } 0 \leq i \leq r$.

In this chapter, we will especially consider the cohomology modules $H^*(f, M)$.

For a given $f$, it is immediate (G, I, 2.1) that $M \mapsto H^*(f, M)$ is a cohomological functor (T, II, 2.1) from the category of $A$-modules to the category of graded $A$-modules, zero in degrees $< 0$ and $> r$. In addition, we have

$(1.1.3.4) \quad H^0(f, M) = \text{Hom}_A(A/(f), M)$,

denoting by $(f)$ the ideal of $A$ generated by $f_1, \ldots, f_r$; this follows immediately from (1.1.2.3), and it is clear that $H^0(f, M)$ identifies with the submodule of $M$ killed by $(f)$. Similarly, we have by (1.1.2.3) that

$(1.1.3.5) \quad H^r(f, M) = M/\left(\sum_{i=1}^{r} f_i M\right) = (A/(f)) \otimes_A M$.

We will use the following known result, which we will recall a proof of to be complete:

Proposition (1.1.4). — Let $A$ be a ring, $f = (f_i)_{1 \leq i \leq r}$ a finite family of elements of $A$, and $M$ an $A$-module. If, for $1 \leq i \leq r$, the scaling $z \mapsto f_i \cdot z$ on $M_{i-1} = M/(f_1 M + \cdots + f_{i-1} M)$ is injective, then we have $H^i(f, M) = 0$ for $i \neq r$.

It suffices to prove that $H_i(f, M) = 0$ for all $i > 0$ according to (1.1.3.3). We argue by induction on $r$, the case $r = 0$ being trivial. Set $f' = (f_i)_{1 \leq i \leq r-1}$; this family satisfies the conditions in the statement, so if we set $L_* = K_*(f', M)$, then we have $H_i(L_*) = 0$ for $i > 0$ by hypothesis, and $H_0(L_*_{-1}) = M_{-1}$ by virtue of (1.1.3.3) and (1.1.3.5). To abbreviate, set $K_* = K_*(f, M)$, $K_0 = K_1 = A$, $d_1 : K_1 \to K_0$ multiplication by $f_1$; we have by definition (1.1.1) that $K_0(f, M) = K_0 \otimes_A L_*$. We have the following lemma:
Lemma (1.1.4.1). — Let $K_\bullet$ be a chain complex of free $A$-modules, zero except in dimensions 0 and 1. For every chain complex $L_\bullet$ of $A$-modules, we have an exact sequence

$$0 \longrightarrow H_0(K_\bullet \otimes H_p(L_\bullet)) \longrightarrow H_p(K_\bullet \otimes L_\bullet) \longrightarrow H_1(K_\bullet \otimes H_{p-1}(L_\bullet)) \longrightarrow 0$$

for every index $p$.

This is a particular case of an exact sequence of low-order terms of the Künneth spectral sequence (M, XVII, 5.2 (a) and G, I, 5.5.2); it can be proved directly as follows. Consider $K_0$ and $K_1$ as chain complexes (zero in dimensions $\neq 0$ and $\neq 1$ respectively); we then have an exact sequence of complexes

$$0 \longrightarrow K_0 \otimes L_\bullet \longrightarrow K_\bullet \otimes L_\bullet \longrightarrow K_1 \otimes L_\bullet \longrightarrow 0,$$

to which we can apply the exact sequence in homology

$$\cdots \longrightarrow H_{p+1}(K_1 \otimes L_\bullet) \longrightarrow H_p(K_0 \otimes L_\bullet) \longrightarrow H_p(K_1 \otimes L_\bullet) \longrightarrow H_p(K_\bullet \otimes H_{p-1}(L_\bullet)) \longrightarrow \cdots.$$

But it is evident that $H_p(K_0 \otimes L_\bullet) = K_0 \otimes H_p(L_\bullet)$ and $H_p(K_1 \otimes L_\bullet) = K_1 \otimes H_p(L_\bullet)$ for all $p$; in addition, we verify immediately that the operator $\partial : K_1 \otimes H_p(L_\bullet) \to K_0 \otimes H_p(L_\bullet)$ is none other than $d_1 \otimes 1$; the lemma thus follows from the above exact sequence and the definition of $H_0(K_\bullet \otimes H_p(L_\bullet))$ and $H_1(K_\bullet \otimes H_{p-1}(L_\bullet))$.

The lemma having been established, the end of the proof of Proposition (1.1.4) is immediate: the induction hypothesis of Lemma (1.1.4.1) gives $H_p(K_\bullet \otimes L_\bullet) = 0$ for $p \geq 2$; in addition if we show that $H_1(K_\bullet, H_0(L_\bullet)) = 0$, then we also deduce from Lemma (1.1.4.1) that $H_2(K_\bullet \otimes L_\bullet) = 0$; but by definition, $H_1(K_\bullet, H_0(L_\bullet))$ is none other than the kernel of the scaling $z \mapsto f_r \cdot z$ on $M_{r-1}$, and as by hypothesis this kernel is zero, this finishes the proof.

(1.1.5). Let $g = (g_i)_{1 \leq i \leq r}$ be a second sequence of $r$ elements of $A$, and set $fg = (f_i g_i)_{1 \leq i \leq r}$. We can define a canonical homomorphism of complexes

$$\phi_g : K_\bullet(fg) \longrightarrow K_\bullet(f)$$

as the canonical extension to the exterior algebra $\wedge(A')$ of the $A$-linear map $(x_1, \ldots, x_r) \mapsto (g_1 x_1, \ldots, g_r x_r)$ from $A'$ to itself. To see that we have a homomorphism of complexes, it suffices to note, in general, that if $u : E \to F$ is an $A$-linear map, and if $x \in F'$ and $y = t u(x) \in E'$, then we have the formula

$$(\wedge u) \circ i_y = i_x \circ (\wedge u);$$

indeed, the two elements are antiderivations of $\wedge F$, and it suffices to check that they coincide on $F$, which follows immediately from the definitions.

When we identify $K_\bullet(f)$ with the tensor product of the $K_\bullet(f_i)$ (1.1.1), $\phi_g$ is the tensor product of the $\phi_{g_i}$, where $\phi_{g_i}$ is the identity in degree 0 and multiplication by $g_i$ in degree 1.

(1.1.6). In particular, for every pair of integers $m$ and $n$ such that $0 \leq n \leq m$, we have homomorphisms of complexes

$$\phi_{f_{m-n}} : K_\bullet(f^n) \longrightarrow K_\bullet(f^m)$$

and as a result, homomorphisms

$$\phi_{f_{m-n}} : K^\bullet(f^m, M) \longrightarrow K^\bullet(f^n, M),$$

$$\phi_{f_{m-n}} : H^\bullet(f^n, M) \longrightarrow H^\bullet(f^m, M).$$

The latter homomorphisms evidently satisfy the transitivity condition $\phi_{f_{m-n}} = \phi_{f_{m-p}} \circ \phi_{f_{p-n}}$ for $p \leq n \leq m$; they therefore define two inductive systems of $A$-modules; we set

$$C^\bullet((f), M) = \lim_{\longrightarrow} K^\bullet(f^n, M),$$

$$H^\bullet((f), M) = H^\bullet(C^\bullet((f), M)) = \lim_{\longrightarrow} H^\bullet(f^n, M),$$

the last equality following from the fact that passing to the inductive limit commutes with the functor $H^\bullet(G, I, 2.1)$. We will later see (1.4.3) that $H^\bullet((f), M)$ does not depend on the ideal $(f)$ of $A$ (and similarly on the $(f)$-pre-adic topology on $A$), which justifies the notations.
It is clear that $M \mapsto C^*(\mathbf{f}, M)$ is an exact $A$-linear functor, and $M \mapsto H^*(\mathbf{f}, M)$ is a cohomological functor.

(1.1.7) Set $\mathbf{f} = (f_i) \in A'$ and $\mathbf{g} = (g_i) \in A'$; denote by $e_\mathbf{g}$ the left multiplication by the vector $\mathbf{g} \in A'$ on the exterior algebra $\Lambda(A')$; we know that we have the homotopy formula

\begin{equation}
1 \mapsto \langle \mathbf{g}, \mathbf{f} \rangle 1
\end{equation}

in the $A$-module $A'$ (1 denotes the identity automorphism of $A'$); this relation also implies that in the complex $K_\bullet(\mathbf{f})$ we have

\begin{equation}
d e_\mathbf{g} + e_\mathbf{g} d = \langle \mathbf{g}, \mathbf{f} \rangle 1.
\end{equation}

If the ideal $\langle \mathbf{f} \rangle$ is equal to $A$, then there exists a $\mathbf{g} \in A'$ such that $\langle \mathbf{g}, \mathbf{f} \rangle = \sum_i g_i f_i = 1$. As a result (G, I, 2.4):

**Proposition (1.1.8).** Suppose that the ideal $\langle \mathbf{f} \rangle$ generated by the $f_i$ is equal to $A$. Then the complex $K_\bullet(\mathbf{f})$ is homotopically trivial, and so are the complexes $K_\bullet(\mathbf{f}, M)$ and $K^\bullet(\mathbf{f}, M)$ for every $A$-module $M$.

**Corollary (1.1.9).** If $\langle \mathbf{f} \rangle = A$, then we have $H^\bullet(\mathbf{f}, M) = 0$ and $H^\bullet((\mathbf{f}), M) = 0$ for every $A$-module $M$.

**Proof.** Indeed, we then have $(\mathbf{f}^n) = A$ for all $n$. □

**Remark (1.1.10).** With the same notations as above, set $X = \text{Spec}(A)$ and $Y$ the closed sub-prescheme of $X$ defined by the ideal $\langle \mathbf{f} \rangle$. We will prove in §9 that $H^\bullet((\mathbf{f}), M)$ is isomorphic to the cohomology $H^\bullet_Y(X, M)$ corresponding to the antifilter $\Phi$ of closed subsets of $Y$ (I, 3.2). We will also show that Proposition (1.2.3) applied to $X$ and to $\mathcal{F} = M$ is a particular case of an exact sequence in cohomology

\[ \cdots \to H^p_Y(X, \mathcal{F}) \to H^p(X, \mathcal{F}) \to H^p(X - Y, \mathcal{F}) \to H^{p+1}_Y(X, \mathcal{F}) \to \cdots. \]

### 1.2. Čech cohomology of an open cover

**Notation (1.2.1).** In this section, we denote:

1. $X$ a prescheme;
2. $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module;
3. $A = \Gamma(X, \mathcal{O}_X)$, $M = \Gamma(X, \mathcal{F})$;
4. $\mathbf{f} = (f_i)_{1 \leq i \leq r}$ a finite system of elements of $A$;
5. $U_i = X_{f_i}$, the open set (0, 5.5.2) of the $x \in X$ such that $f_i(x) \neq 0$;
6. $U = \bigcup_{i=1}^r U_i$;
7. $\mathcal{U}$ the cover $(U_i)_{1 \leq i \leq r}$ of $U$.

(1.2.2) Suppose that $X$ is either a prescheme whose underlying space is Noetherian or a scheme whose underlying space is quasi-compact. We then know (I, 9.3.3) that we have $\Gamma(U_i, \mathcal{F}) = M_{f_i}$. We set

\[ U_{i_0i_1\cdots i_p} = \bigcap_{k=0}^p U_{i_k} = X_{f_{i_0}f_{i_1}\cdots f_{i_p}} \]

(0, 5.5.3); so we also have

\begin{equation}
\Gamma(U_{i_0i_1\cdots i_p}, \mathcal{F}) = M_{f_{i_0}f_{i_1}\cdots f_{i_p}}.
\end{equation}

We have (0, 1.6.1) that $M_{f_{i_0}f_{i_1}\cdots f_{i_p}}$ identifies with the inductive limit $\lim_{\rightarrow n} M_{i_0i_1\cdots i_p}^{(n)}$, where the inductive system is formed by the $M_{i_0i_1\cdots i_p}^{(n)} = M$, the homomorphisms $\phi_{nm} : M_{i_0i_1\cdots i_p}^{(n)} \to M_{i_0i_1\cdots i_p}^{(m)}$ being multiplication by $(f_{i_0}f_{i_1}\cdots f_{i_p})^{n-m}$ for $m \leq n$. We denote by $C^p_\mathcal{U}(M)$ the set of alternating maps from $[1, r]^{p+1}$ to $M$ (all $n$); these $A$-modules also form an inductive system with respect to the $\phi_{nm}$. If $C^p(\mathcal{U}, \mathcal{F})$ is the group of alternating Čech $p$-cochains relative to the cover $\mathcal{U}$, with coefficients in $\mathcal{F}$ (G, II, 5.1), then it follows from the above that we can write

\begin{equation}
C^p(\mathcal{U}, \mathcal{F}) = \lim_{\rightarrow n} C^p_\mathcal{U}(M).
\end{equation}
With the notations of (1.1.2), $C^p_0(M)$ identifies with $K^{p+1}(f, M)$, and the map $\phi_{fm}$ identifies with the map $\phi_{fm}$ defined in (1.1.6). We thus have, for every $p \geq 0$, a canonical functorial isomorphism

\begin{equation} \tag{1.2.2.3} C^p(U, \mathcal{F}) \simeq C^{p+1}((f), M). \end{equation}

In addition, the formula (1.2.2.3) and the definition of the cohomology of a cover (G, II, 5.1) shows that the isomorphisms (1.2.2.3) are compatible with the coboundary maps.

**Proposition (1.2.3).** — If $X$ is a prescheme whose underlying space in Noetherian or a scheme whose underlying space is quasi-compact, then there exists a canonical functorial isomorphism in $\mathcal{F}$

\begin{equation} \tag{1.2.3.1} H^p(U, \mathcal{F}) \simeq H^{p+1}((f), M) \text{ for } p \geq 1. \end{equation}

In addition, we have a functorial exact sequence in $\mathcal{F}$

\begin{equation} \tag{1.2.3.2} 0 \rightarrow H^0((f), M) \rightarrow M \rightarrow H^0(U, \mathcal{F}) \rightarrow H^1((f), M) \rightarrow 0. \end{equation}

**Proof.** The isomorphisms (1.2.3.1) are immediate consequences of what we saw in (1.2.2). On the other hand, we have $C^0(U, \mathcal{F}) = C^1((f), M)$; as a result, $H^0(U, \mathcal{F})$ identifies with the subgroup of 1-cocycles of $C^1((f), M)$; as $M = C^0((f), M)$, the exact sequence (1.2.3.2) is none other than the one given by the definition of the cohomology groups $H^0((f), M)$ and $H^1((f), M)$. 

■

**Corollary (1.2.4).** — Suppose that the $X_i$ are quasi-compact and that there exists $g_i \in \Gamma(U, \mathcal{F})$ such that $\sum g_i(f_i|U) = 1|U$. Then for every quasi-coherent $(\mathcal{O}_X|U)$-module $\mathcal{G}$, we have $H^p(U, \mathcal{G}) = 0$ for $p > 0$; if in addition $U = X$, then the canonical homomorphism (1.2.3.2) $M \rightarrow H^0(U, \mathcal{F})$ is bijective.

**Proof.** As by hypothesis the $U_i = X_i$, are quasi-compact, so is $U$, and we can reduce to the case where $U = X$; the hypothesis then implies that $H^p((f), M) = 0$ for all $p \geq 0$ (1.1.9). The corollary then follows immediately from (1.2.3.1) and (1.2.3.2).

We note that since $H^0(U, \mathcal{F}) = H^0(U, \mathcal{F})$ (G, II, 5.2.2), we have again proved (1.1.9) as a special case.

**Remark (1.2.5).** — Suppose that $X$ is an affine scheme; then the $U_i = X_i = D(f_i)$ are affine open sets, as well as the $U_{i_0i_1\ldots i_p}$ (but $U$ is not necessarily affine). In this case, the functors $\Gamma(X, \mathcal{F})$ and $\Gamma(U_{i_0i_1\ldots i_p}, \mathcal{F})$ are exact in $\mathcal{F}$ (I, 1.3.11). If we have an exact sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ of quasi-coherent $\mathcal{O}_X$-modules, then the sequence of complexes

\[ 0 \rightarrow C^\infty(U, \mathcal{F}') \rightarrow C^\infty(U, \mathcal{F}) \rightarrow C^\infty(U, \mathcal{F}'') \rightarrow 0 \]

is exact, and thus gives an exact sequence in cohomology

\[ \cdots \rightarrow H^p(U, \mathcal{F}) \rightarrow H^p(U, \mathcal{F}) \rightarrow H^p(U, \mathcal{F}'') \rightarrow H^{p+1}(U, \mathcal{F}') \rightarrow \cdots. \]

On the other hand, if we set $M' = \Gamma(X, \mathcal{F}')$ and $M'' = \Gamma(X, \mathcal{F}'')$, then the sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact; as $C^\infty((f), M)$ is an exact functor in $M$, we also have the exact sequence in cohomology

\[ \cdots \rightarrow H^p((f), M') \rightarrow H^p((f), M) \rightarrow H^p((f), M'') \rightarrow H^{p+1}((f), M') \rightarrow \cdots. \]

This being so, as the diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & C^\infty(U, \mathcal{F}') & \rightarrow & C^\infty(U, \mathcal{F}) & \rightarrow & C^\infty(U, \mathcal{F}'') & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & C^\infty((f), M') & \rightarrow & C^\infty((f), M) & \rightarrow & C^\infty((f), M'') & \rightarrow & 0
\end{array}
\]

is commutative, we conclude that the diagrams

\[
\begin{array}{ccc}
H^p(U, \mathcal{F}'') & \rightarrow & H^{p+1}(U, \mathcal{F}') \\
\downarrow & & \downarrow \\
H^{p+1}((f), M'') & \rightarrow & H^{p+2}((f), M')
\end{array}
\]
are commutative for all \( p \) (G, I, 2.1.1).

1.3. Cohomology of an affine scheme

**Theorem (1.3.1).** — Let \( X \) be an affine scheme. For every quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \), we have \( H^p(X, \mathcal{F}) = 0 \) for all \( p > 0 \).

**Proof.** Let \( U \) be a finite cover of \( X \) by the affine open sets \( X_{f_i} = D(f_i) \) \( (1 \leq i \leq r) \); we then know that the ideal of \( A = \Gamma(X, \mathcal{O}_X) \) generated by the \( f_i \) is equal to \( A \). We thus conclude from Corollary (1.2.4) that we have \( H^p(U, \mathcal{F}) = 0 \) for \( p > 0 \). As there are finite covers of \( X \) by affine open sets which are arbitrarily fine (I, 1.1.10), the definition of Čech cohomology (G, II, 5.8) shows that we also have \( H^p(X, \mathcal{F}) = 0 \) for \( p > 0 \). But this also applies to every prescheme \( X_f \) for \( f \in A \) (I, 1.3.6), hence \( H^p(X_f, \mathcal{F}) = 0 \) for \( p > 0 \). As we have \( X_f \cap X_g = X_{fg} \), we deduce that we also have \( H^p(X, \mathcal{F}) = 0 \) for all \( p > 0 \), by virtue of (G, II, 5.9.2). \( \square \)

**Corollary (1.3.2).** — Let \( Y \) be a prescheme, \( f : X \to Y \) an affine morphism (II, 1.6.1). For every quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \), we have \( R^p f_*(\mathcal{F}) = 0 \) for \( p > 0 \).

**Proof.** By definition \( R^p f_*(\mathcal{F}) \) is the \( \mathcal{O}_Y \)-module associated to the presheaf \( U \mapsto H^p(f^{-1}(U), \mathcal{F}) \), where \( U \) varies over the open subsets of \( Y \). But the affine open sets form a basis for \( Y \), and for such an open set \( U \), \( f^{-1}(U) \) is affine (II, 1.3.2), hence \( H^p(f^{-1}(U), \mathcal{F}) = 0 \) by Theorem (1.3.1), which proves the corollary. \( \square \)

**Corollary (1.3.3).** — Let \( Y \) be a prescheme, \( f : X \to Y \) an affine morphism. For every quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \), the canonical homomorphism \( H^p(Y, f_*(\mathcal{F})) \to H^p(X, \mathcal{F}) \) (0, 12.1.3.1) is bijective for all \( p \).

**Proof.** It suffices (by (0, 12.1.7)) to show that the edge homomorphisms \( H^p(X, f_*(\mathcal{F})) \to H^p(Y, f_*(\mathcal{F})) \to H^p(X, \mathcal{F}) \) of the second spectral sequence of the composite functor \( f_* \) are bijective. But the \( E_2 \) term of this spectral sequence is given by \( H^p(f_*\mathcal{F}) = H^q(Y, R^p f_*(\mathcal{F})) \) (G, II, 4.17.1), so it follows from Corollary (1.3.2) that \( H^p(f_*\mathcal{F}) = 0 \) for \( q > 0 \), and the spectral sequence degenerates; hence our assertion (0, 11.1.6). \( \square \)

**Corollary (1.3.4).** — Let \( f : X \to Y \) be an affine morphism, \( g : Y \to Z \) a morphism. For every quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \), the canonical homomorphism \( R^p (g \circ f)_* (\mathcal{F}) \to R^p (g \circ f)_* (\mathcal{F}) \) (0, 12.2.5.1) is bijective for all \( p \).

**Proof.** It suffices to note that, according to Corollary (1.3.3), for every affine open subset \( W \) of \( Z \), the canonical homomorphism \( H^p(g^{-1}(W), f_*(\mathcal{F})) \to H^p(f^{-1}(g^{-1}(W)), \mathcal{F}) \) is bijective; this proves that the homomorphism of presheaves defining the canonical homomorphism \( R^p (g \circ f)_* (\mathcal{F}) \to R^p (g \circ f)_* (\mathcal{F}) \) is bijective (0, 12.2.5). \( \square \)

1.4. Application to the cohomology of arbitrary preschemes

**Proposition (1.4.1).** — Let \( X \) be a scheme, \( \mathcal{U} = (U_a) \) be a cover of \( X \) by affine open sets. For every quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \), the cohomology modules \( H^*(X, \mathcal{F}) \) and \( H^*(\mathcal{U}, \mathcal{F}) \) \( (\text{over } \Gamma(X, \mathcal{O}_X)) \) are canonically isomorphic.

**Proof.** As \( X \) is a scheme, every finite intersection \( V \) of open sets in the cover \( \mathcal{U} \) is affine (I, 5.5.6), so \( H^q(V, \mathcal{F}) = 0 \) for \( q \geq 1 \) by Theorem (1.3.1). The proposition then follows from a theorem of Leray (G, II, 5.4.1). \( \square \)

**Remark (1.4.2).** — We note that the result of Proposition (1.4.1) is still true when the finite intersections of the sets \( U_a \) are affine, even when we do not necessarily assume that \( X \) is a scheme.

**Corollary (1.4.3).** — Let \( X \) be a scheme with quasi-compact underlying space, \( A = \Gamma(X, \mathcal{O}_X) \), and \( f = (f_i)_{1 \leq i \leq r} \) a finite sequence of elements of \( A \) such that the \( X_{f_i} \) (notation of (1.2.1)) are affine. Then (with the notations of (1.2.1)), for every quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \), we have a canonical isomorphism which is functorial in \( \mathcal{F} \)

\[
H^q(U, \mathcal{F}) \simeq H^{q+1}((f), M) \quad \text{for } q \geq 1,
\]

and an exact sequence which is functorial in \( \mathcal{F} \)

\[
0 \to H^q((f), M) \to M \to H^0(U, \mathcal{F}) \to H^1((f), M) \to 0.
\]
1. COHOMOLOGY OF AFFINE SCHEMES

PROOF. This follows immediately from Propositions (1.4.1) and (1.2.3).

(1.4.4) If $X$ is an affine scheme, then it follows from Remark (1.2.5) and Proposition (1.4.1) that for all $q \geq 0$, the diagrams

$$
\begin{array}{ccc}
\mathcal{H}^q(U, \mathcal{F}''') & \xrightarrow{\partial} & \mathcal{H}^{q+1}(U, \mathcal{F}') \\
\downarrow & & \downarrow \\
\mathcal{H}^{q+1}(f, M''') & \xrightarrow{\partial} & \mathcal{H}^{q+2}(f, M')
\end{array}
$$

corresponding to an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of quasi-coherent $\mathcal{O}_X$-modules (with the notations of Remark (1.2.5)) are commutative.

Proposition (1.4.5). — Let $X$ be a quasi-compact scheme, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module, and consider the graded ring $A_\ast = \Gamma_\ast(\mathcal{L})$ (0.5.4.6); then $\mathcal{H}^\ast(\mathcal{F}, \mathcal{L}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}^n(\mathcal{X}, \mathcal{F} \otimes \mathcal{L}^\otimes n)$ is a graded $A_\ast$-module, and for all $f \in A_n$, we have a canonical isomorphism

$$
\mathcal{H}^n(X_f, \mathcal{F}) \simeq (\mathcal{H}^n(\mathcal{F}, \mathcal{L}))(f)
$$
of $(A_n)_{(f)}$-modules.

PROOF. As $X$ is a quasi-compact scheme, we can calculate the cohomology of all the $\mathcal{O}_X$-modules $\mathcal{F} \otimes \mathcal{L}^\otimes n$ using the same finite cover $\mathcal{U} = (U_i)$ consisting of the affine open sets such that the restriction $\mathcal{L}|U_i$ is isomorphic to $\mathcal{O}_X|U_i$ for each $i$ (1.4.1). It is then immediate that the $U_i \cap X_f$ are affine open sets (1.3.6), and we can thus calculate the cohomology $\mathcal{H}^n(X_f, \mathcal{F} \otimes \mathcal{L}^\otimes n)$ using the cover $\mathcal{U}|X_f = (U_i \cap X_f)$ (1.4.1). It is immediate that for all $f \in A_n$, multiplication by $f$ defines a homomorphism $C^\ast(\mathcal{U}, \mathcal{F} \otimes \mathcal{L}^\otimes n) \to C^\ast(\mathcal{U}, \mathcal{F} \otimes \mathcal{L}^\otimes (n+1))$, hence a homomorphism $\mathcal{H}^n(\mathcal{U}, \mathcal{F} \otimes \mathcal{L}^\otimes n) \to \mathcal{H}^n(\mathcal{U}, \mathcal{F} \otimes \mathcal{L}^\otimes (n+1))$, which establishes the first assertion. On the other hand, for a given $f \in A_n$, it follows from (I, 9.3.2) that we have an isomorphism of complexes of $(A_n)_{(f)}$-modules

$$
C^\ast(\mathcal{U} | X_f, \mathcal{F}) \simeq \left( C^\ast \left( \mathcal{U}, \bigoplus_{n \in \mathbb{Z}} \mathcal{F} \otimes \mathcal{L}^\otimes n \right) \right)_{(f)},
$$
taking into account (I, 1.3.9, ii). Passing to the cohomology of these complexes, we induce the isomorphism (1.4.5.1), recalling that the functor $M \mapsto M_{(f)}$ is exact on the category of graded $A_\ast$-modules.

Corollary (1.4.6). — Suppose that the hypotheses of Proposition (1.4.5) are satisfied, and in addition suppose that $\mathcal{L} = \mathcal{O}_X$. If we set $A = \Gamma(X, \mathcal{O}_X)$, then for all $f \in A$, we have a canonical isomorphism $\mathcal{H}^n(X_f, \mathcal{F}) \simeq (\mathcal{H}^n(X, \mathcal{F}))_{(f)}$ of $A_f$-modules.

Corollary (1.4.7). — Let $X$ be a quasi-compact scheme, $f$ an element of $\Gamma(X, \mathcal{O}_X)$.

(i) Suppose that the open set $X_f$ is affine. Then for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, every $i > 0$, and every $\zeta \in \mathcal{H}^i(X_f, \mathcal{F})$, there exists an integer $n > 0$ such that $f^n \zeta = 0$.

(ii) Conversely, suppose that $X_f$ is quasi-compact and that for every quasi-coherent sheaf of ideals $\mathcal{J}$ of $\mathcal{O}_X$ and every $\zeta \in \mathcal{H}^i(X, \mathcal{J})$, there exists an $n > 0$ such that $f^n \zeta = 0$. Then $X_f$ is affine.

PROOF.

(i) If $X_f$ is affine, then we have $\mathcal{H}^i(X_f, \mathcal{F}) = 0$ for all $i > 0$ (1.3.1), so the assertion follows directly from Corollary (1.4.6).

(ii) By virtue of Serre’s criterion (II, 5.2.1), it suffices to prove that for every quasi-coherent sheaf of ideals $\mathcal{K}$ of $\mathcal{O}_X|X_f$, we have $\mathcal{H}^1(X_f, \mathcal{K}) = 0$. As $X_f$ is a quasi-compact open set in a quasi-compact scheme $X$, there exists a quasi-coherent sheaf of ideals $\mathcal{J}$ of $\mathcal{O}_X$ such that $\mathcal{K} = \mathcal{J}|X_f$ (I, 9.4.2). According to Corollary (1.4.6), we have $\mathcal{H}^1(X_f, \mathcal{K}) = (\mathcal{H}^1(X, \mathcal{J}))_{(f)}$, and the hypothesis implies that the right hand side is zero, hence the assertion.

Remark (1.4.8). — We note that Corollary (1.4.7, i) gives a simpler proof of the relation (II, 4.5.13.2).
Lemma (1.4.9). — Let $X$ be a quasi-compact scheme, $\mathcal{U} = (U_i)_{1 \leq i \leq n}$ a finite cover of $X$ by affine open sets, and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. The complex of sheaves $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ defined by the cover $\mathcal{U}$ (G, II, 5.2) is then a quasi-coherent $\mathcal{O}_X$-module.

**Proof.** It follows from the definitions (G, II, 5.2) that $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ is the direct sum of the direct image sheaves of the $\mathcal{F}|_{U_{i_0}\ldots i_p}$ under the canonical injection $U_{i_0}\ldots i_p \to X$. The hypothesis that $X$ is a scheme implies that these injections are affine morphisms (I, 5.5.6), hence the $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ are quasi-coherent (II, 1.2.6).

Proposition (1.4.10). — Let $u : X \to Y$ be a separated and quasi-compact morphism. For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the $R^i u_*(\mathcal{F})$ are quasi-coherent $\mathcal{O}_Y$-modules.

**Proof.** The question is local on $Y$, so we can suppose that $Y$ is affine. Then $X$ is a finite union of affine open sets $U_i$ ($1 \leq i \leq n$); let $\mathcal{U}$ be the cover $(U_i)$. In addition, as $Y$ is a scheme, it follows from (I, 5.5.10) that for every affine open $V \subset Y$, the canonical injection $u^{-1}(V) \to X$ is an affine morphism; we conclude (Proposition (1.4.1) and (G, II, 5.2)) that we have a canonical isomorphism

$$H^\bullet(u^{-1}(V), \mathcal{F}) \simeq H^\bullet(\Gamma(V, \mathcal{K}^\bullet)),$$

where we set $\mathcal{K}^\bullet = u_*(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}))$. According to Lemma (1.4.1) and (I, 9.2.2), $\mathcal{K}^\bullet$ is a quasi-coherent $\mathcal{O}_Y$-module; moreover, it constitutes a complex of sheaves since so is $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$. It then follows from the definition of the cohomology $\mathcal{K}^\bullet(\mathcal{U}, \mathcal{F})$ (G, II, 4.1) that the latter consists of quasi-coherent $\mathcal{O}_Y$-modules (I, 4.1). As (for $V$ affine in $Y$) the functor $\Gamma(V, \mathcal{K}^\bullet)$ is exact in $\mathcal{K}^\bullet$ on the category of quasi-coherent $\mathcal{O}_Y$-modules, we have (G, II, 4.1)

$$(1.4.10.1) \quad H^\bullet(\Gamma(V, \mathcal{K}^\bullet)) = \Gamma(V, \mathcal{K}^\bullet(\mathcal{U}, \mathcal{F})).$$

Finally, we note that it follows from the definition of the canonical homomorphism

$$H^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow H^\bullet(X, \mathcal{F}),$$

given in (G, II, 5.2), that if $V' \subset V$ is a second affine open subset of $Y$, then the diagram

$$
\begin{array}{ccc}
H^\bullet(u^{-1}(V), \mathcal{F}) & \xymatrix{ \longrightarrow \ar[d] & H^\bullet(\Gamma(V, \mathcal{K}^\bullet)) \\
H^\bullet(u^{-1}(V'), \mathcal{F}) & \longrightarrow \ar[d] & H^\bullet(\Gamma(V', \mathcal{K}^\bullet))
\end{array}
$$

is commutative. We thus conclude from the above that the isomorphisms (1.4.10.1) define an isomorphism of $\mathcal{O}_Y$-modules

$$(1.4.10.2) \quad R^i u_*(\mathcal{F}) \simeq \mathcal{K}^\bullet(\mathcal{U}, \mathcal{F}),$$

and as a result, $R^i u_*(\mathcal{F})$ is quasi-coherent.

In addition, it follows from (1.4.10.3), (1.4.10.2), and (1.4.10.1) that:

Corollary (1.4.11). — Under the hypotheses of Proposition (1.4.10), for every affine open set $V$ of $Y$, the canonical homomorphism

$$(1.4.11.1) \quad H^q(u^{-1}(V), \mathcal{F}) \longrightarrow \Gamma(V, R^1 u_*(\mathcal{F}))$$

is an isomorphism for all $q \geq 0$.

Corollary (1.4.12). — Suppose that the hypotheses of Proposition (1.4.10) are satisfied, and in addition suppose that $Y$ is quasi-compact. Then there exists an integer $r > 0$ such that for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and every integer $q > r$, we have $R^q u_*(\mathcal{F}) = 0$. If $Y$ is affine, then we can take for $r$ an integer such that there exists a cover of $X$ consisting of $r$ affine open sets.

**Proof.** As we can cover $Y$ by a finite number of affine open sets, we can reduce to proving the second assertion, by virtue of Corollary (1.4.11). If $\mathcal{U}$ is a cover of $X$ by $r$ affine open sets, then we have $H^q(\mathcal{U}, \mathcal{F}) = 0$ for $q > r$, since the cochains of $C^q(\mathcal{U}, \mathcal{F})$ are alternating; the assertion thus follows from Proposition (1.4.1).
Corollary (1.4.13). — Suppose that the hypotheses of Proposition (1.4.10) are satisfied, and in addition suppose that \(Y = \text{Spec}(A)\) is affine. Then for every quasi-coherent \(\mathcal{O}_X\)-module \(\mathcal{F}\) and every \(f \in A\), we have
\[
\Gamma(Y_f, R^q u_*(\mathcal{F})) = (\Gamma(Y, R^q u_*(\mathcal{F})))_f
\]
up to canonical isomorphism.

**Proof.** This follows from the fact that \(R^q u_*(\mathcal{F})\) is a quasi-coherent \(\mathcal{O}_Y\)-module (I, 1.3.7). \(\Box\)

Proposition (1.4.14). — Let \(f : X \to Y\) be a separated and quasi-compact morphism, \(g : Y \to Z\) an affine morphism. For every quasi-coherent \(\mathcal{O}_X\)-module \(\mathcal{F}\), the canonical homomorphism \(R^p (g \circ f)_* (\mathcal{F}) \to g_* (R^p f_* (\mathcal{F}))\) is bijective for all \(p\).

**Proof.** For every affine open subset \(W\) of \(Z\), \(g^{-1}(W)\) is an affine open subset of \(Y\). The homomorphism of presheaves defining the canonical homomorphism
\[
R^p (g \circ f)_* (\mathcal{F}) \to g_* (R^p f_* (\mathcal{F}))
\]
(0, 12.2.5) is thus bijective by Corollary (1.4.11). \(\Box\)

Proposition (1.4.15). — Let \(u : X \to Y\) be a separated morphism of finite type, \(v : Y' \to Y\) a flat morphism of preschemes (0, 6.7.1); let \(u' = u_*(Y')\) such that we have the commutative diagram
\[
\begin{array}{ccc}
X & \xleftarrow{v'} & X_{(Y')} \\
\downarrow{u} & & \downarrow{u'} \\
Y & \xleftarrow{v} & Y'.
\end{array}
\]

Then for every quasi-coherent \(\mathcal{O}_X\)-module \(\mathcal{F}\), \(R^q u'_*(v'^*(\mathcal{F}))\) is canonically isomorphic to \(R^q u_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'} = v'^*(R^q u_*(\mathcal{F}))\) for all \(q \geq 0\), where \(\mathcal{F}' = v'^*(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'}\).

**Proof.** The canonical homomorphism \(\rho : \mathcal{F} \to v'_*(v'^*(\mathcal{F}'))\) (0, 4.4.3.2) defines by functoriality a homomorphism

\[
R^q u_*(\mathcal{F}) \to R^q u'_*(v'^*(\mathcal{F}')).
\]

On the other hand, we have, by setting \(w = u \circ v' = v \circ u'\), the canonical homomorphisms (0, 12.2.5.1 and 12.2.5.2)

\[
R^q u'_*(v'^*(\mathcal{F}')) \to R^q w_*(\mathcal{F}') \to v_* (R^q u'_*(\mathcal{F}')).
\]

Composing (1.4.15.3) and (1.4.15.2), we have a homomorphism
\[
\psi : R^q u_*(\mathcal{F}) \to v_* (R^q u'_*(\mathcal{F}')),
\]
and finally we obtain a canonical homomorphism (whose definition does not make any assumptions on \(v\))

\[
\psi^\natural : v^* (R^q u_*(\mathcal{F})) \to R^q u'_*(\mathcal{F}'),
\]
and it is necessary to prove that it is an isomorphism when \(v\) is flat. It is clear that the question is local on \(Y\) and \(Y'\), and we can therefore suppose that \(Y = \text{Spec}(A)\) and \(Y' = \text{Spec}(B)\); we will also use the following lemma:

Lemma (1.4.15.5). — Let \(\phi : A \to B\) be a ring homomorphism, \(Y = \text{Spec}(A), X = \text{Spec}(B), f : X \to Y\) the morphism corresponding to \(\phi\), and \(M\) a \(B\)-module. For the \(\mathcal{O}_X\)-module \(M\) to be \(f\)-flat (0, 6.7.1), it is necessary and sufficient for \(M\) to be a flat \(A\)-module. In particular, for the morphism \(f\) to be flat, it is necessary and sufficient for \(B\) to be a flat \(A\)-module.

This follows from the definition (0, 6.7.1) and from (0, 6.3.3), taking into account (I, 1.3.4).

This being so, it follows from (1.4.11.1) and the definitions of the homomorphisms (1.4.15.3) (cf. (0, 12.2.5)) that \(\psi\) then corresponds to the composite morphism
\[
H^q (X, \mathcal{F}) \xrightarrow{\theta_1} H^q (X, v'_*(v'^*(\mathcal{F}'))) \xrightarrow{\theta_2} H^q (X', v'^*(v'^*(\mathcal{F}'))) \xrightarrow{\theta_3} H^q (X', v'^*(\mathcal{F}'))
\]
where \( \rho_q \) and \( \sigma_q \) are the homomorphisms in cohomology corresponding to the canonical morphisms \( \rho \) and \( \sigma : v^*(v'_s(\mathcal{G}')) \rightarrow \mathcal{G}' \), and \( \theta_q \) is the \( \phi \)-morphism (0, 12.1.3.1) relative to the \( O_X \)-module \( v_s'(v^*(\mathcal{F})) \). But by the functoriality of \( \theta_q \), we have the commutative diagram

\[
\begin{array}{ccc}
H^i(X, \mathcal{F}) & \xrightarrow{\rho_i} & H^i(X, v'_s(v^*(\mathcal{F}))) \\
\downarrow \theta_i & & \downarrow s \circ (\rho_i) \\
H^i(X', v^*(\mathcal{F})) & \xrightarrow{v^*(\rho_i)} & H^i(X', v'_s(v^*(\mathcal{F}))))
\end{array}
\]

and as by definition (0, 4.4.3) \( v^*(\rho) \) is the inverse of \( \sigma \), we see that the composite morphism considered above is finally none other than \( \theta_q \); as a result, \( \psi_q \) is the associated \( B \)-homomorphism \( H^i(X, \mathcal{F}) \otimes_A B \rightarrow H^i(X', \mathcal{F}') \). As \( u \) is of finite type, \( X \) is a finite union of affine open sets \( U_i \) \( (1 \leq i \leq r) \); let \( U \) be the cover \( (U_i) \). As \( v \) is an affine morphism, so is \( v' \) (II, 1.6.5, iii), and as a result the \( U'_i = v'^{-1}(U_i) \) form an affine open cover \( U' \) of \( X' \). We then know (0, 12.1.4.2) that the diagram

\[
\begin{array}{ccc}
H^i(U, \mathcal{F}) & \xrightarrow{\theta_i} & H^i(U, \mathcal{F}') \\
\downarrow & & \downarrow \\
H^i(X, \mathcal{F}) & \xrightarrow{\theta_i} & H^i(X, \mathcal{F}')
\end{array}
\]

is commutative, and the vertical arrows are isomorphisms since \( X \) and \( X' \) are schemes (I, 1.4.1). As a result, it suffices to prove that the canonical \( \phi \)-morphism \( \theta_q : H^i(U, \mathcal{F}) \rightarrow H^i(U', \mathcal{F}') \) is such that the associated \( B \)-homomorphism

\[
H^i(U, \mathcal{F}) \otimes_A B \rightarrow H^i(U', \mathcal{F}')
\]

is an isomorphism. For every sequence \( s = (i_k)_{0 \leq k \leq p} \) of \( p + 1 \) indices of \( [1, r] \), set \( U_s = \bigcap_{k=0}^p U_{i_k} \), \( U'_s = \bigcap_{k=0}^p U'_{i_k} = v'^{-1}(U_s) \), \( M_s = \Gamma(U_s, \mathcal{F}) \), and \( M'_s = \Gamma(U'_s, \mathcal{F}') \). The canonical map \( M_s \otimes_A B \rightarrow M'_s \) is an isomorphism (I, 1.6.5), hence the canonical map \( C^p(U, \mathcal{F}) \otimes_A B \rightarrow C^p(U', \mathcal{F}') \) is an isomorphism, by which \( d \otimes 1 \) identifies with the coboundary map \( C^p(U', \mathcal{F}') \rightarrow C^{p+1}(U', \mathcal{F}') \). As \( B \) is a flat A-module, it follows from the definition of the cohomology modules that the canonical map \( H^i(U, \mathcal{F}) \otimes_A B \rightarrow H^i(U', \mathcal{F}') \) is an isomorphism (0, 6.1.1). This result will later be generalized in §6.

**Corollary (1.4.16).** — Let \( A \) be a ring, \( X \) an \( A \)-scheme of finite type, and \( B \) an \( A \)-algebra which is faithfully flat over \( A \). For \( X \) to be affine, it is necessary and sufficient for \( X \otimes_A B \) to be.

**Proof.** The condition is evidently necessary (I, 3.2.2); we show that it is sufficient. As \( X \) is separated over \( A \) and the morphism \( \text{Spec}(B) \rightarrow \text{Spec}(A) \) is flat, it follows from Proposition (1.4.1) that we have

\[
(1.4.16.1) \quad H^i(X \otimes_A B, \mathcal{F}) = H^i(X, \mathcal{F}) \otimes_A B
\]

for every \( i \geq 0 \) and every quasi-coherent \( O_X \)-module \( \mathcal{F} \). If \( X \otimes_A B \) is affine, the left hand side of

\[
(1.4.16.1)
\]

is zero for \( i = 1 \), hence so is \( H^1(X, \mathcal{F}) \) since \( B \) is a faithfully flat \( A \)-module. As \( X \) is a quasi-compact scheme, we finish the proof by Serre’s criterion (II, 5.2.1).

**Proposition (1.4.17).** — Let \( X \) be a prescheme, \( 0 \rightarrow \mathcal{F} \xrightarrow{u} \mathcal{G} \xrightarrow{\nu} \mathcal{H} \rightarrow 0 \) an exact sequence of \( O_X \)-modules. If \( \mathcal{F} \) and \( \mathcal{H} \) are quasi-coherent, then so is \( \mathcal{G} \).

**Proof.** The question is local on \( X \), so we can suppose that \( X = \text{Spec}(A) \) is affine, and it then suffices to prove that \( \mathcal{G} \) satisfies the conditions (d1) and (d2) of (I, 1.4.1) (with \( V = X \)). The verification of (d2) is immediate, because if \( t \in \Gamma(X, \mathcal{G}) \) is zero when restricted to \( D(f) \), then so is its image \( v(t) \in \Gamma(X, \mathcal{H}) \); therefore there exists an \( m \) such that \( f^m v(t) = u(f^m t) = 0 \) (I, 1.4.1), and as \( \Gamma \) is left exact, \( f^m t = u(s) \), where \( s \in \Gamma(X, \mathcal{F}) \); as \( u \) is injective, the restriction of \( s \) to \( D(f) \)
is zero, hence (I, 1.4.1) there exists an integer $n > 0$ such that $f^ns = 0$; we finally deduce that $f^{m+n}t = u(f^ns) = 0$.

We now check (d1); let $t' \in \Gamma(D(f), \mathcal{G})$; as $\mathcal{H}$ is quasi-coherent, there exists an integer $m$ such that $f^mv(t') = v(f^mt')$ extends to a section $z \in \Gamma(X, \mathcal{H})$ (I, 1.4.1). But in virtue of Theorem (1.3.1) (or (I, 5.1.9.2)) applied to the quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the sequence $\Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H}) \to 0$ is exact, so there exists $t \in \Gamma(X, \mathcal{G})$ such that $z = v(t)$; we thus see that $v(f^mt - t'') = 0$, denoting by $t''$ the restriction of $t$ to $D(f)$; thus we have $f^{m+n}t' - f^n t'' = u(s')$, where $s' \in \Gamma(D(f), \mathcal{F})$. But as $\mathcal{F}$ is quasi-coherent, there exists an integer $n > 0$ such that $f^n s'$ extends to a section $s \in \Gamma(X, \mathcal{F})$; as $f^{m+n}t' - f^n t'' = u(f^n s')$, we see that $f^{m+n}t'$ is the restriction to $D(f)$ of a section $f^n t + u(f^n s') \in \Gamma(X, \mathcal{G})$, which finishes the proof. □

§2. Cohomological study of projective morphisms

2.1. Explicit calculations of certain cohomology groups

§3. Finiteness theorem for proper morphisms

3.1. The dévissage lemma

Definition (3.1.1). — Let $\mathcal{C}$ be an abelian category. We say that a subset $\mathcal{C}'$ of the set of objects of $\mathcal{C}$ is exact if $0 \in \mathcal{C}'$ and if, for every exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{C}$ such that two of the objects $A, A', A''$ are in $\mathcal{C}'$, then the third is also in $\mathcal{C}'$.

Theorem (3.1.2). — Let $X$ be a Noetherian prescheme; we denote by $\mathcal{C}$ the abelian category of coherent $\mathcal{O}_X$-modules. Let $\mathcal{C}'$ be an exact subset of $\mathcal{C}$, $X'$ a closed subset of the underlying space of $X$. Suppose that for every closed irreducible subset $Y$ of $X'$, with generic point $y$, there exists an $\mathcal{O}_X$-module $\mathcal{G} \in \mathcal{C}'$ such that $\mathcal{G}_y$ is a $k(y)$-vector space of dimension $1$. Then every coherent $\mathcal{O}_X$-module with support contained in $X'$ is in $\mathcal{C}'$ (and in particular, if $X' = X$, then we have $\mathcal{C}' = \mathcal{C}$).

Proof. Consider the following property $P(Y)$ of a closed subset $Y$ of $X'$: every coherent $\mathcal{O}_X$-module with support contained in $Y$ is in $\mathcal{C}'$. By virtue of the principle of Noetherian induction (0, 2.2.2), we see that we can reduce to showing that if $Y$ is a closed subset of $X'$ such that the property $P(Y')$ is true for every closed subset $Y'$ of $X$, distinct from $Y$, then $P(Y)$ is true.

Therefore, let $\mathcal{F} \in \mathcal{C}$ have support contained in $Y$, and we show that $\mathcal{F} \in \mathcal{C}'$. Denote also by $Y$ the reduced closed subscheme of $X$ having $Y$ for its underlying space (I, 5.2.1); it is defined by a coherent sheaf of ideals $\mathcal{J}$ of $\mathcal{O}_X$. We know (I, 9.3.4) that there exists an integer $n > 0$ such that $\mathcal{J}^n \mathcal{F} = 0$; for $1 \leq k \leq n$, we thus have an exact sequence

$$0 \to \mathcal{J}^{k-1} \mathcal{F} / \mathcal{J}^k \mathcal{F} \to \mathcal{F} / \mathcal{J}^k \mathcal{F} \to \mathcal{F} / \mathcal{J}^{k-1} \mathcal{F} \to 0$$

of coherent $\mathcal{O}_X$-modules ((I, 5.3.6) and (I, 5.3.3)); as $\mathcal{C}'$ is exact, we see, by induction on $k$, that it suffices to show that each of the $\mathcal{F} = \mathcal{J}^{k-1} \mathcal{F} / \mathcal{J}^k \mathcal{F}$ is in $\mathcal{C}'$. We thus reduce to proving that $\mathcal{F} \in \mathcal{C}'$ under the additional hypothesis that $\mathcal{J} \mathcal{F} = 0$; it is equivalent to say that $\mathcal{F} = j_* (j^*(\mathcal{F}))$, where $j$ is the canonical injection $Y \to X$. Let us now consider two cases:

(a) $Y$ is reducible. Let $Y = Y' \cap Y''$, where $Y'$ and $Y''$ are closed subsets of $Y$, distinct from $Y$; denote also by $Y'$ and $Y''$ the reduced closed subschemes of $X$ having $Y$ and $Y''$ for their respective underlying spaces, which are defined respectively by sheaves of ideals $\mathcal{J}'$ and $\mathcal{J}''$ of $\mathcal{O}_X$. Set $\mathcal{F}_z = \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X / \mathcal{J}')$ and $\mathcal{F}_z'' = \mathcal{F} \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y / \mathcal{J}''_z)$. The canonical homomorphisms $\mathcal{F} \to \mathcal{F}_z$ and $\mathcal{F} \to \mathcal{F}_z''$ thus define a homomorphism $u : \mathcal{F} \to \mathcal{F}_z \oplus \mathcal{F}_z''$. We show that for every $z \notin Y' \cap Y''$, the homomorphism $u_z : \mathcal{F}_z \to \mathcal{F}_z \oplus \mathcal{F}_z''$ is bijective. Indeed, we have $\mathcal{J}_z' \cap \mathcal{J}_z'' = \mathcal{J}_z$, since the question is local and the above equality follows from (I, 5.2.1) and (I, 1.1.5); if $z \notin Y''$, then we have $\mathcal{J}_z' = \mathcal{J}_z$, hence $\mathcal{F}_z' = \mathcal{F}_z$ and $\mathcal{F}_z'' = 0$, which establishes our assertion in this case; we reason similarly for $z \notin Y'$. As a result, the kernel and cokernel of $u$, which are in $\mathcal{C}$ (0, 5.3.4), have their support in $Y' \cap Y''$, and thus is in $\mathcal{C}'$ by hypothesis; for the same reason, $\mathcal{F}_z'$ and $\mathcal{F}_z''$ are in $\mathcal{C}'$, hence also $\mathcal{F}_z' \oplus \mathcal{F}_z''$, as $\mathcal{C}'$ is exact. The conclusion then follows from the consideration of the two exact sequences

$$0 \to \text{Im} u \to \mathcal{F}_z' \oplus \mathcal{F}_z'' \to \text{Coker } u \to 0,$$
Theorem (3.2.1). — If \( Y \) is irreducible, and as a result, the subscheme \( Y \) of \( X \) is integral. If \( Y \) is its generic point, then we have \((\mathcal{O}_Y)_y = k(y)\), and as \( j^*(\mathcal{F}) \) is a coherent \( \mathcal{O}_Y \)-module, \( \mathcal{F}_y = (j^*(\mathcal{F}))_y \) is a \( k(y) \)-vector space of finite dimension \( m \). By hypothesis, there is a coherent \( \mathcal{O}_X \)-module \( \mathcal{I} \in \mathcal{C}' \) (necessarily of support \( Y \)) such that \( \mathcal{I}_y \) is a \( k(y) \)-vector space of dimension 1. As a result, there is a \( k(y) \)-isomorphism \((\mathcal{I}_y)^m \simeq \mathcal{I}_y \), which is also an \( \mathcal{O}_Y \)-isomorphism, and as \( \mathcal{I}^m \) and \( \mathcal{I} \) are coherent, there exists an open neighborhood \( V \) of \( y \) in \( X \) and an isomorphism \( \mathcal{I}^m \mid V \simeq \mathcal{I} \mid V \). Let \( \mathcal{H} \) be the graph of this isomorphism, which is a coherent \( (\mathcal{O}_X \mid W) \)-submodule of \( (\mathcal{I}^m \oplus \mathcal{F}) \mid W \), canonically isomorphic to \( \mathcal{I}^m \mid W \) and to \( \mathcal{I} \mid W \); there thus exists a coherent \( \mathcal{O}_X \)-submodule \( \mathcal{H}_0 \) of \( \mathcal{I}^m \oplus \mathcal{F} \), inducing \( \mathcal{H} \) on \( W \) and \( 0 \) on \( X - Y \), since \( \mathcal{I}^m \) and \( \mathcal{I} \) have \( Y \) for their support (I, 9.4.7). The restrictions \( v : \mathcal{H}_0 \rightarrow \mathcal{I}^m \) and \( w : \mathcal{H}_0 \rightarrow \mathcal{I} \) of the canonical projections of \( \mathcal{I}^m \oplus \mathcal{F} \) are then homomorphisms of coherent \( \mathcal{O}_X \)-modules, which, on \( W \) and on \( X - Y \), reduce to isomorphisms; in other words, the kernels and cokernels of \( v \) and \( w \) have their support in the closed set \( Y - (Y \cap W) \), distinct from \( Y \). They are in \( \mathcal{C}' \); on the other hand, we have \( \mathcal{I}^m \in \mathcal{C}' \) since \( \mathcal{I} \in \mathcal{C}' \) and since \( \mathcal{C}' \) is exact. We conclude successively, by the exactness of \( \mathcal{C}' \), that \( \mathcal{H}_0 \in \mathcal{C}' \), then \( \mathcal{I} \in \mathcal{C}' \). Q.E.D.

Corollary (3.1.3). — Suppose that the exact subset \( \mathcal{C}' \) of \( \mathcal{C} \) has in addition the property that every coherent direct factor of a coherent \( \mathcal{O}_X \)-module \( \mathcal{M} \in \mathcal{C}' \) is also in \( \mathcal{C}' \). In this case, the conclusion of Theorem (3.1.2) is still valid when the condition “\( \mathcal{I}_y \) is a \( k(y) \)-vector space of dimension 1” is replaced by \( \mathcal{I}_y \neq 0 \) (this is equivalent to \( \text{Supp}(\mathcal{I}) = Y \)).

Proof. The reasoning of Theorem (3.1.2) must be modified only in the case (b); now \( \mathcal{I}_y \) is a \( k(y) \)-vector space of dimension \( q > 0 \), and as a result, we have an \( \mathcal{O}_Y \)-isomorphism \((\mathcal{I}_y)^m \simeq (\mathcal{I}_y)^n \); the end of the reasoning in Theorem (3.1.2) then proves that \( \mathcal{F}^q \in \mathcal{C}' \), and the additional hypothesis on \( \mathcal{C}' \) implies that \( \mathcal{F} \in \mathcal{C}' \).

3.2. The finiteness theorem: the case of usual schemes

Theorem (3.2.1). — Let \( Y \) be a locally Noetherian prescheme, \( f : X \rightarrow Y \) a proper morphism. For every coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \), the \( \mathcal{O}_Y \)-modules \( R^q f_* (\mathcal{F}) \) are coherent for \( q \geq 0 \).

Proof. Since the questions is local on \( Y \), we can suppose \( Y \) Noetherian, thus \( X \) Noetherian (I, 6.3.7). The coherent \( \mathcal{O}_X \)-modules \( \mathcal{F} \) for which the conclusion of Theorem (3.2.1) is true forms an exact subset \( \mathcal{C}' \) of the category \( \mathcal{C} \) of coherent \( \mathcal{O}_X \)-modules. Indeed, let \( 0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0 \) be an exact sequence of coherent \( \mathcal{O}_X \)-modules; suppose for example that \( \mathcal{F}' \) and \( \mathcal{F}'' \) belong to \( \mathcal{C}' \); we have the long exact sequence in cohomology

\[
R^{q-1} f_*(\mathcal{F}'') \rightarrow R^q f_*(\mathcal{F}') \rightarrow R^q f_*(\mathcal{F}) \rightarrow R^q f_*(\mathcal{F}'') \rightarrow R^{q+1} f_*(\mathcal{F}),
\]

in which by hypothesis the outer four terms are coherent; it is the same for the middle term \( R^q f_*(\mathcal{F}) \) by ((0, 5.3.4) and (0, 5.3.3)). We show in the same way that when \( \mathcal{F} \) and \( \mathcal{F}' \) (resp. \( \mathcal{F}'' \) and \( \mathcal{F}''' \)) are in \( \mathcal{C}' \), then so is \( \mathcal{F}'' \) (resp. \( \mathcal{F}' \)). In addition, every coherent direct factor \( \mathcal{F}' \) of an \( \mathcal{O}_X \)-module \( \mathcal{F} \in \mathcal{C}' \) belongs to \( \mathcal{C}' \); indeed, \( R^q f_*(\mathcal{F}) \) is then a direct factor of \( R^q f_*(\mathcal{F}) \) (G, II, 4.4.4), therefore it is of finite type, and as it is quasi-coherent (1.4.10), it is coherent, as \( Y \) is Noetherian. By virtue of Corollary (3.1.3), we reduce to proving that when \( X \) is irreducible with generic point \( x \), there exists one coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \) belonging to \( \mathcal{C}' \), such that \( \mathcal{F} \neq 0 \); indeed, if this point is established, then it can be applied to any irreducible closed subscheme \( Y \) of \( X \), since if \( j : Y \rightarrow X \) is the canonical injection, then \( f \circ j \) is proper (II, 5.4.2), and if \( \mathcal{F} \) is a coherent \( \mathcal{O}_Y \)-module with support \( Y \), then \( j_* (\mathcal{F}) \) is a coherent \( \mathcal{O}_X \)-module such that \( R^q (f \circ j)_* (\mathcal{F}) = R^q f_* (j_* (\mathcal{F})) \) (G, II, 4.9.1), therefore we can apply (Corollary (3.1.3).

By virtue of Chow’s lemma (II, 5.6.2), there exists an irreducible prescheme \( X' \) an a projective and surjective morphism \( g : X' \rightarrow X \) such that \( f \circ g : X' \rightarrow Y \) is projective. There exists an ample \( \mathcal{O}_X \)-module \( \mathcal{L} \) for \( g \) (II, 5.3.1); we apply the fundamental theorem of projective morphisms (2.2.1) to \( g : X' \rightarrow X \) and with \( \mathcal{L} \); there thus exists an integer \( n \) such that \( \mathcal{F} = g_* (\mathcal{O}_{X'}(n)) \) is a coherent \( \mathcal{O}_X \)-module and \( R^q g_* (\mathcal{O}_{X'}(n)) = 0 \) for all \( q > 0 \); in addition, as \( g^*(g_* (\mathcal{O}_{X'}(n))) \rightarrow \mathcal{O}_{X'}(n) \) is surjective...
for \( n \) large enough (2.2.1), we see that we can suppose, at the generic point \( x \) of \( X \), that we have \( \mathcal{F}_x \neq 0 \) (II, 3.4.7). On the other hand, as \( f \circ g \) is projective as \( Y \) is Noetherian, the \( R^i(f \circ g)_*(\mathcal{O}_X(n)) \) are coherent (2.2.1). This being so, \( R^*(f \circ g)_*(\mathcal{O}_X(n)) \) is the abutment of a Leray spectral sequence, whose \( E_2 \)-term is given by \( E_2^{pq} = R^p f_*(R^qg_*(\mathcal{O}_X(n))) \); the above shows that this spectral sequence degenerates, and we then know (0, 11.1.6) that \( E_2^{pq} = R^p f_*(\mathcal{F}) \) is isomorphic to \( R^p f_*(\mathcal{O}_X(n)) \), which finishes the proof.

**Corollary (3.2.2).** — Let \( Y \) be a locally Noetherian prescheme. For every proper morphism \( f : X \to Y \), the direct image under \( f \) of any coherent \( \mathcal{O}_X \)-module is a coherent \( \mathcal{O}_Y \)-module.

**Corollary (3.2.3).** — Let \( A \) be a Noetherian ring, \( X \) a proper scheme over \( A \); for every coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \), the \( H^p(X, \mathcal{F}) \) are \( A \)-modules of finite type, and there exists an integer \( r > 0 \) such that for every coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \) and all \( p > r \), \( H^p(X, \mathcal{F}) = 0 \).

**PROOF.** The second assertion has already been proved (1.4.12); the first follows from the finiteness theorem (3.2.1), taking into account Corollary (1.4.11). □

In particular, if \( X \) is a proper algebraic scheme over a field \( k \), then, for every coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \), the \( H^p(X, \mathcal{F}) \) are finite-dimensional \( k \)-vector spaces.

**Corollary (3.2.4).** — Let \( Y \) be a locally Noetherian prescheme, \( f : X \to Y \) a morphism of finite type. For every coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \) whose support in proper over \( Y \) (II, 5.4.10), the \( \mathcal{O}_Y \)-modules \( R^if_*(\mathcal{F}) \) are coherent.

**PROOF.** Since the questions is local on \( Y \), we can suppose \( Y \) Noetherian, and it is the same for \( X \) (I, 6.3.7). By hypothesis, every closed subscheme \( Z \) of \( X \) whose underlying space is \( \text{Supp}(\mathcal{F}) \) is proper over \( Y \), in other words, if \( j : Z \to X \) is the canonical injection, then \( f \circ j : Z \to Y \) is proper.

We can suppose that \( Z \) is such that \( \mathcal{F} = j_*(\mathcal{G}) \), where \( \mathcal{G} = j^*(\mathcal{F}) \) is a coherent \( \mathcal{O}_Z \)-module (II, 9.5.3); as we have \( R^if_*(\mathcal{F}) = R^if_*(\mathcal{G}) \) by Corollary (1.3.4), the conclusion follows immediately from Theorem (3.2.1). □

### 3.3. Generalization of the finiteness theorem (usual schemes)

**Proposition (3.3.1).** — Let \( Y \) be a Noetherian prescheme, \( \mathcal{I} \) a quasi-coherent \( \mathcal{O}_Y \)-algebra of finite type, graded in positive degrees, \( Y' = \text{Proj}(\mathcal{I}) \), and \( g : Y' \to Y \) the structure morphism. Let \( f : X \to Y \) be a proper morphism, \( \mathcal{F}' = f^*(\mathcal{I}) \), \( \mathcal{M} = \bigoplus_{k \in \mathbb{Z}} \mathcal{M}_k \) a quasi-coherent graded \( \mathcal{I}' \)-module of finite type. Then the \( R^if_*(\mathcal{M}) = \bigoplus_{k \in \mathbb{Z}} R^if_*(\mathcal{M}_k) \) are \( \mathcal{I} \)-modules of finite type for all \( p \). Suppose in addition that the \( \mathcal{I} \) are generated by \( \mathcal{I}_1 \); then, for every \( p \in \mathbb{Z} \), there exists an integer \( k_p \) such that for all \( k \geq k_p \) and all \( r > 0 \), we have

\[
R^if_*(\mathcal{M}_k) = \mathcal{I}R^if_*(\mathcal{M}_k).
\]

**PROOF.** The first assertion is identical to the statement of Theorem (2.4.1, i), where we have simply replaced “projective morphism” by “proper morphism”. In the proof of Theorem (2.4.1, i), the hypothesis on \( f \) was only used to show (with the notation of this proof) that \( R^if'_*(\mathcal{H}) \) is a coherent \( \mathcal{O}_Y \)-module. With the hypothesis of Proposition (3.3.1), \( f' \) is proper (II, 5.4.2, iii), so we can resume without change in the proof of Theorem (2.4.1, i), thanks to the finiteness theorem (3.2.1).

As for the second assertion, it suffices to remark that there is a finite affine open cover \( (U_i) \) of \( Y \) such that the restrictions to the \( U_i \) of the two sides of (3.3.1.1) are equal for all \( k \geq k_{p,i} \) (II, 2.1.6, ii); it suffices to take for \( k_{p,i} \) the largest of the \( k_{p,j} \).

**Corollary (3.3.2).** — Let \( A \) be a Noetherian ring, \( m \) an ideal of \( A \), \( X \) a proper \( A \)-scheme, and \( \mathcal{F} \) a coherent \( \mathcal{O}_X \)-module. Then, for all \( p \geq 0 \), the direct sum \( \bigoplus_{k \geq 0} H^p(X, m^k \mathcal{F}) \) is a module of finite type over the ring \( S = \bigoplus_{k \geq 0} m^k \); in particular, there exists an integer \( k_p \geq 0 \) such that for all \( k \geq k_p \) and all \( r > 0 \), we have

\[
H^p(X, m^{k+r} \mathcal{F}) = m^rH^p(X, m^k \mathcal{F}).
\]

**PROOF.** It suffices to apply Proposition (3.3.1) with \( Y = \text{Spec}(A), \mathcal{I} = \mathcal{M} = m^k \mathcal{F}, \) taking into account Corollary (1.4.11). □
It should be remembered that the S-module structure on $\bigoplus_{k \geq 0} H^p(X, m^k F)$ is obtained by considering, for every $a \in m'$, the map $H^p(X, m^k F) \to H^p(X, m^{k+r} F)$, which comes from the passage to cohomology of the multiplication map $m' F \to m^{k+r} F$ defined by $a$ (2.4.1).

### 3.4. Finiteness theorem: the case of formal schemes

The results of this section (except the definition 3.4.1)) will not be used in the rest of this chapter.

(3.4.1). Let $\mathfrak{X}$ and $\mathfrak{S}$ be two locally Noetherian formal preschemes (I, 10.4.2), $f : \mathfrak{X} \to \mathfrak{S}$ a morphism of formal preschemes. We say that $f$ is a proper morphism if it satisfies the following conditions:

1st. $f$ is a morphism of finite type (I, 10.13.3).

2nd. If $\mathcal{H}$ is a sheaf of ideals of definition for $\mathfrak{S}$ and if we set $\mathcal{J} = f^*(\mathcal{H})/\mathcal{O}_X$, $X_0 = (\mathfrak{X}, \mathcal{O}_X / \mathcal{J})$, $S_0 = (\mathfrak{S}, \mathcal{O}_S / \mathcal{H})$, then the morphism $f_0 : X_0 \to S_0$ induced by $f$ (I, 10.5.6) is proper.

It is immediate that this definition does not depend on the sheaf of ideals of definition $\mathcal{H}$ for $\mathcal{J}$ considered; indeed, if $\mathcal{H}'$ is a second sheaf of ideals of definition such that $\mathcal{H}' \subset \mathcal{H}$, and if we set $\mathcal{J}' = f^*(\mathcal{H}')/\mathcal{O}_X$, $X'_0 = (\mathfrak{X}, \mathcal{O}_X / \mathcal{J}')$, $S'_0 = (\mathfrak{S}, \mathcal{O}_S / \mathcal{H}')$, then the morphism $f'_0 : X'_0 \to S'_0$ induced by $f$ is such that the diagram

\[
\begin{array}{ccc}
X_0 & \xrightarrow{f_0} & S_0 \\
\downarrow{i} & & \downarrow{j} \\
X'_0 & \xrightarrow{f'_0} & S'_0
\end{array}
\]

is commutative, $i$ and $j$ being surjective immersions; it is equivalent to say that $f_0$ or $f'_0$ is proper, by virtue of (II, 5.4.5).

We note that, for all $n \geq 0$, if we set $X_n = (\mathfrak{X}, \mathcal{O}_X / \mathcal{J}^{n+1})$, $S_n = (\mathfrak{S}, \mathcal{O}_S / \mathcal{H}^{n+1})$, then the morphism $f_n : X_n \to S_n$ induced by $f$ (I, 10.5.6) is proper for all $n$ whenever it is for $n = 0$ (II, 5.4.6).

If $g : Y \to Z$ is a proper morphism of locally Noetherian usual preschemes, $Z'$ a closed subset of $Z$, $Y'$ a closed subset of $Y$ such that $g(Y') \subset Z'$, then the extension $\hat{g} : Y'/Y \to Z'/Z$ of $g$ to the completions (I, 10.9.1) is a proper morphism of formal preschemes, as it follows from the definition and from (II, 4.5.4).

Let $\mathfrak{X}$ and $\mathfrak{S}$ be two locally Noetherian formal preschemes, $f : \mathfrak{X} \to \mathfrak{S}$ a morphism of finite type (I, 10.13.3); the notation being the same as above, we say that a subset $Z$ of the underlying space of $\mathfrak{X}$ is proper over $\mathfrak{S}$ (or proper for $f$) if, considered as a subset of $X_0$, $Z$ is proper over $S_0$ (II, 5.4.10).

All the properties of proper subsets of usual preschemes stated in (II, 5.4.10) are still true for the proper subsets of formal preschemes, as it follows immediately from the definitions.

**Theorem (3.4.2).** — Let $\mathfrak{X}$ and $\mathfrak{Y}$ be locally Noetherian formal preschemes, $f : \mathfrak{X} \to \mathfrak{Y}$ a proper morphism. For every coherent $\mathcal{O}_X$-module $\mathcal{F}$, the $\mathcal{O}_\mathfrak{Y}$-modules $R^q f_*(\mathcal{F})$ are coherent for all $q \geq 0$.

Let $\mathcal{J}$ be a sheaf of ideals of definition for $\mathfrak{Y}$, $\mathcal{H} = f^*(\mathcal{J})/\mathcal{O}_X$, and consider the $\mathcal{O}_X$-modules

\[
\mathcal{F}_k = \mathcal{F} \otimes_{\mathcal{O}_\mathfrak{Y}} (\mathcal{O}_\mathfrak{Y} / \mathcal{J}^{k+1}) = \mathcal{F} / \mathcal{H}^{k+1} \mathcal{F} \quad (k \geq 0)
\]

which evidently form a projective system of topological $\mathcal{O}_X$-modules, such that $\mathcal{F} = \varprojlim_k \mathcal{F}_k$ (I, 10.11.3).

On the other hand, it follows from Theorem (3.4.2) that each of the $R^q f_*(\mathcal{F}_k)$, being coherent, is naturally equipped with a topological $\mathcal{O}_\mathfrak{Y}$-module structure (I, 10.11.6), and so are the $R^q f_*(\mathcal{F}_k)$. The canonical homomorphisms $\mathcal{F} \to \mathcal{F}_k = \mathcal{F} / \mathcal{H}^{k+1} \mathcal{F}$ canonically correspond to homomorphisms

\[
R^q f_*(\mathcal{F}) \to R^q f_*(\mathcal{F}_k),
\]

which are necessarily continuous for the topological $\mathcal{O}_\mathfrak{Y}$-module structures above (I, 10.11.6), and form a projective system, giving the limit a canonical functorial homomorphism

\[
R^q f_*(\mathcal{F}) \to \varprojlim_k R^q f_*(\mathcal{F}_k),
\]

which will be a continuous homomorphism of topological $\mathcal{O}_\mathfrak{Y}$-modules. We will prove along with Theorem (3.4.2) the

**Corollary (3.4.3).** — Each of the homomorphisms (3.4.2.2) is a topological isomorphism. In addition, if $\mathfrak{Y}$ is Noetherian, then the projective system $(R^q f_*(\mathcal{F} / \mathcal{H}^{k+1} \mathcal{F}))_{k \geq 0}$ satisfies the (ML)-condition (0, 13.1.1).
We will begin by establishing Theorem (3.4.2) and Corollary (3.4.3) when $Y$ is a Noetherian formal affine scheme (I, 10.4.1):

**Corollary (3.4.4).** — Under the hypotheses of Theorem (3.4.2), suppose in addition that $\mathfrak{N} = \text{Spf}(A)$, where $A$ is an adic Noetherian ring. Let $\mathfrak{J}$ be an ideal of definition for $A$, and set $\mathfrak{F}_k = \mathfrak{F} / \mathfrak{J}^{k+1}$ for $k \geq 0$. Then the $H^n(\mathfrak{X}, \mathfrak{F})$ are $A$-modules of finite type; the projective system $(H^n(\mathfrak{X}, \mathfrak{F}_k))_{k \geq 0}$ satisfies the (ML)-condition for all $n$; if we set

$$N_{h,k} = \text{Ker}(H^n(\mathfrak{X}, \mathfrak{F}) \to H^n(\mathfrak{X}, \mathfrak{F}_k))$$

(also equal to $\text{Im}(H^n(\mathfrak{X}, \mathfrak{J}^{k+1}) \to H^n(\mathfrak{X}, \mathfrak{F}))$ by the exact sequence in cohomology), then the $N_{h,k}$ define on $H^n(\mathfrak{X}, \mathfrak{F})$ a $\mathfrak{J}$-good filtration (0, 13.7.7); finally, the canonical homomorphism

$$H^n(\mathfrak{X}, \mathfrak{F}) \to \lim_{\leftarrow \mathfrak{F}_k}$$

is a topological isomorphism for all $n$ (the left hand side being equipped with the $\mathfrak{J}$-adic topology, the $H^n(\mathfrak{X}, \mathfrak{F}_k)$ with the discrete topology).

Set\[ S = \text{gr}(A) = \bigoplus_{k \geq 0} \mathfrak{J}^k / \mathfrak{J}^{k+1}, \quad \mathcal{M} = \text{gr}(\mathfrak{F}) = \bigoplus_{k \geq 0} \mathfrak{J}^k \mathfrak{F} / \mathfrak{J}^{k+1} \mathfrak{F}. \]

We know that $\mathfrak{J}^A$ is a sheaf of ideals of definition for $\mathfrak{N}$ (I, 10.3.1); let $\mathcal{X} = f^*(\mathfrak{J}^A)\mathcal{O}_X$, $X_0 = (\mathfrak{X}, \mathcal{O}_X / \mathcal{X})$, $Y_0 = (\mathfrak{N}, \mathcal{O}_N / \mathfrak{J}^A) = \text{Spec}(A_0)$, with $A_0 = A / \mathfrak{J}$. It is clear that the $\mathcal{M} = \mathfrak{J}^k \mathcal{F} / \mathfrak{J}^{k+1} \mathcal{F}$ are coherent $\mathcal{O}_{X_0}$-modules (I, 10.11.3). Consider on the other hand the quasi-coherent graded $\mathcal{O}_{X_0}$-algebra

$$\mathcal{F} = \mathcal{O}_{X_0} \otimes A_0 S = \text{gr}(\mathcal{O}_X) = \bigoplus_{k \geq 0} \mathcal{X}^k / \mathcal{X}^{k+1}.$$  

The hypothesis that $\mathcal{F}$ is a $\mathcal{O}_X$-module of finite type implies first that $\mathcal{M}$ is a graded $\mathcal{F}$-module of finite type. Indeed, the question is local on $\mathfrak{X}$, and we can thus suppose that $\mathfrak{X} = \text{Spf}(B)$, where $B$ is an adic Noetherian ring, and $\mathcal{F} = N^A$, where $N$ is a $B$-module of finite type (I, 10.10.5); we have in addition $X_0 = \text{Spec}(B_0)$, where $B_0 = B / \mathfrak{J}B$, and the quasi-coherent $\mathcal{O}_{X_0}$-modules $\mathcal{F}$ and $\mathcal{M}$ are respectively equal to $\widetilde{S}'$ and $\widetilde{M}'$, where $S' = \bigoplus_{k \geq 0}((\mathfrak{J}^k / \mathfrak{J}^{k+1}) \otimes A_0 B_0)$ and $M' = \bigoplus_{k \geq 0}((\mathfrak{J}^k / \mathfrak{J}^{k+1}) \otimes A_0 N_0)$, with $N_0 = N / \mathfrak{J}N$; we then evidently have $M' = S' \otimes B_0 N_0$, and as $N_0$ is a $B_0$-module of finite type, $M'$ is a $\mathcal{F}$-module of finite type, hence our assertion (I, 1.3.13).  

As the morphism $f_0 : X_0 \to Y_0$ is proper by hypothesis, we can apply Corollary (3.3.2) to $\mathcal{F}$, $\mathcal{M}$, and the morphism $f_0$; taking into account Corollary (1.4.11), we conclude that for all $n \geq 0$, $\bigoplus_{k \geq 0} H^n(X_0, \mathcal{M}_k)$ is a graded $S$-module of finite type. This proves that the condition (F$_n$) of (0, 13.7.7) is satisfied for all $n \geq 0$, when we consider the strictly projective system $(\mathcal{F} / \mathfrak{J}^k \mathcal{F})_{k \geq 0}$ of sheaves of abelian groups on $X_0$, each equipped with its natural “filtered $A$-module” structure. We can thus apply (0, 13.7.7), which proves that:

1st. The projective system $(H^n(\mathfrak{X}, \mathfrak{F}_k))_{k \geq 0}$ satisfies the (ML)-condition.

2nd. If $H^n = \lim_{\leftarrow k} H^n(\mathfrak{X}, \mathfrak{F}_k)$, then $H^n$ is an $A$-module of finite type.

3rd. The filtration defined on $H^n$ by the kernels of the canonical homomorphisms $H^m(\mathfrak{F}_k) \to H^n(\mathfrak{F}_k)$ is $\mathfrak{J}$-good.

Note that on the other hand, if we set $X_k = (\mathfrak{X}, \mathcal{O}_X / \mathcal{X}^{k+1})$, then $\mathfrak{F}_k$ is a coherent $\mathcal{O}_{X_k}$-module (I, 10.11.3), and if $U$ is an affine open set in $X_0$, then $U$ is also an affine open set in each of the $X_k$ (I, 5.1.9), so $H^n(U, \mathfrak{F}_k) = 0$ for all $n > 0$ and all $k \geq 1$ and $H^0(U, \mathfrak{F}_k) \to H^0(U, \mathfrak{F}_k)$ is surjective for $h \leq k$ (I, 1.3.9). We are thus in the conditions of (0, 13.3.2) and applying (0, 13.3.1) proves that $H^n$ canonically identifies with $H^n(\mathfrak{X}, \lim_{\leftarrow k} \mathfrak{F}_k) = H^n(\mathfrak{X}, \mathfrak{F})$; this finishes the proof of Corollary (3.4.4).

(3.4.5). We return to the proof of (3.4.2) and (3.4.3). We first prove the propositions for the case $\mathfrak{N} = \text{Spf}(A)$ envisaged in (3.4.4); for this, for all $g \in A$, apply (3.4.4) to the Noetherian affine formal scheme induced on the open set $\mathfrak{N}_g = \mathfrak{D}(g)$ of $\mathfrak{N}$, which is equal to $\text{Spf}(A_{(g)})$, and to the
formal prescheme induced by $\mathfrak{X}$ on $f^{-1}(\mathfrak{Y})$; note that $\mathfrak{Y}$ is also an affine open set in the prescheme
$Y_k = (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/(\mathfrak{A})^{k+1})$, and as $\mathfrak{F}_k$ is a coherent $\mathcal{O}_{\mathfrak{X}_k}$-module, we have
$$H^n(f^{-1}(\mathfrak{Y}), \mathfrak{F}_k) = \Gamma(\mathfrak{Y}, R^nf_*(\mathfrak{F}_k))$$
for all $k \geq 0$ by virtue of Corollary (1.4.11). The canonical homomorphism
$$H^n(f^{-1}(\mathfrak{Y}), \mathfrak{F}) \rightarrow \lim_{\kappa} \Gamma(\mathfrak{Y}, R^nf_*(\mathfrak{F}_k))$$
is an isomorphism; but we have $(0, 3.2.6)$
$$\lim_{\kappa} \Gamma(\mathfrak{Y}, R^nf_*(\mathfrak{F}_k)) = \Gamma(\mathfrak{Y}, \lim_{\kappa} R^nf_*(\mathfrak{F}_k)),$$
and as the sheaf $R^nf_*(\mathfrak{F})$ is the sheaf associated to the presheaf $\mathfrak{Y}_k \mapsto H^n(f^{-1}(\mathfrak{Y}), \mathfrak{F})$ on the
$\mathfrak{Y}_k (0, 3.2.1)$, we have shown that the homomorphism (3.4.2.2) is bijective. Let us now prove that
$R^nf_*(\mathfrak{F})$ is a coherent $\mathfrak{O}_{\mathfrak{Y}}$-module, and more precisely that we have
(3.4.5.1) $R^nf_*(\mathfrak{F}) = (H^n(\mathfrak{X}, \mathfrak{F}))^\Delta$.

With the above notation, we have, since $\mathfrak{F}_k$ is a coherent $\mathcal{O}_{\mathfrak{X}_k}$-module (1.4.13),
$$\Gamma(\mathfrak{Y}, R^nf_*(\mathfrak{F}_k)) = (\Gamma(\mathfrak{Y}, R^nf_*(\mathfrak{F}_k)))_g = (H^n(\mathfrak{X}, \mathfrak{F}_k))_g.$$

Now the $H^n(\mathfrak{X}, \mathfrak{F}_k)$ form a projective system satisfying (ML), and their projective limit $H^n(\mathfrak{X}, \mathfrak{F})$
is an $A$-module of finite type. We conclude (0, 13.7.8) that we have
$$\lim_{\kappa} \left( (H^n(\mathfrak{X}, \mathfrak{F}_k))_g \right) = H^n(\mathfrak{X}, \mathfrak{F}) \otimes_A A_{(g)} = \Gamma(\mathfrak{Y}, (H^n(\mathfrak{X}, \mathfrak{F}))^\Delta),$$
taking into account (I, 10.10.8) applied to $A$ and $A_{(g)}$; this proves (3.4.5.1) since $\Gamma(\mathfrak{Y}, R^nf_*(\mathfrak{F})) = \lim_{\kappa} \Gamma(\mathfrak{Y}, R^nf_*(\mathfrak{F}_k))$.

As (3.4.2.2) is then an isomorphism of coherent $\mathfrak{O}_{\mathfrak{Y}}$-modules, it is necessarily a topological
isomorphism (I, 10.11.6). Finally, it follows from the relations $R^nf_*(\mathfrak{F}_k) = (H^n(\mathfrak{X}, \mathfrak{F}_k))^\Delta$ that the
projective system $(R^nf_*(\mathfrak{F}_k))_{k \geq 0}$ satisfies (ML) (I, 10.10.2).

Once (3.4.2.2) and (3.4.3) are proved in the case where the formal prescheme $\mathfrak{Y}$ is affine Noetherian,
it is immediate to pass to the general case for (3.4.2) and the first assertion of (3.4.3), which are local
on $\mathfrak{Y}$. As for the second assertion of (3.4.3), it suffices, $\mathfrak{Y}$ being Noetherian, to cover it by a finite
number of Noetherian affine open sets $U_i$ and to note that the restrictions of the projective system
$(R^nf_*(\mathfrak{F}_k))$ to each of the $U_i$ satisfies (ML).

Along the way, we have in addition proved:

**Corollary (3.4.6).** — Under the hypotheses of Corollary (3.4.4), the canonical homomorphism
(3.4.6.1) $H^n(\mathfrak{X}, \mathfrak{F}) \rightarrow \Gamma(\mathfrak{Y}, R^nf_*(\mathfrak{F}))$
is bijective.

### §4. The Fundamental Theorem of Proper Morphisms. Applications

#### 4.1. The fundamental theorem

#### §5. An Existence Theorem for Coherent Algebraic Sheaves

5.1. Statement of the theorem

#### §6. Local and Global Tor Functors; Künneth Formula

6.1. Introduction

#### §7. Base Change for Homological Functors of Sheaves of Modules

7.1. Functors of $A$-modules
CHAPTER IV

Local study of schemes and their morphisms (EGA IV)

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SUMMARY

§1. Relative finiteness conditions. Constructible sets in preschemes.
§2. Base change and flatness.
§3. Associated prime cycles and primary decomposition.
§5. Dimension, depth, and regularity for locally Noetherian preschemes.
§6. Flat morphisms of locally Noetherian preschemes.
§7. Relations between a local Noetherian ring and its completion. Excellent rings.
§9. Constructible properties.
§11. Topological properties of finitely presented flat morphisms. Flatness criteria.
§12. Study of fibres of finitely presented flat morphisms.
§15. Study of fibres of a universally open morphism.
§17. Smooth morphisms, unramified (or net) morphisms, and étale morphisms.
§19. Regular immersions and normal flatness.
§20. Meromorphic functions and pseudo-morphisms

The subjects discussed in the chapter call for the following remarks.

(a) The common property of all the subjects discussed is that they all related to local properties of preschemes or morphisms, i.e. considered at a point, or the points of a fibre, or on a (non-specified) neighbourhood of a point or of a fibre. These properties are generally of a topological, differential, or dimensional nature (i.e. bringing the ideas of dimension and depth into play), and are linked to the properties of the local rings at the points considered. One type of problem is the relating, for a given morphisms \( f : X \to Y \) and point \( x \in X \), of the properties of \( X \) at \( x \) with those of \( Y \) at \( y = f(x) \) and those of the fibre \( X_y = f^{-1}(y) \) at \( x \). Another is the determining of the topological nature (for example, the constructibility, or the fact of being open or closed) of the set of points \( x \in X \) at which \( X \) has a certain property, or for which the fibre \( X_{f(x)} \) passing through \( x \) has a certain property at \( x \). Similarly, we are interested in the topological nature of the set of points \( y \in Y \) such that \( X \) has a certain property at all the points of the fibre \( X_y \), or those such that this fibre itself has a certain property.

\(^{1}\)The order and content of §§11–21 are given only as an indication of what the titles will be, and will possibly be modified before their publication. [Trans.] This was indeed the case: many of §§11–21 ended up having entirely different titles or content. See here.
(b) The most important idea for the following chapters is that of flat morphisms of finite presentation, as well as the particular cases of smooth morphisms and étale morphisms. Their detailed study (as well as that of connected questions) really starts in §11.

(c) Sections §§1–10 can be considered as being preliminary in nature, and as developing three types of techniques, used, not only in the other sections of the chapter, but also, of course, in the follow chapters:

(c1) Sections §§1–4 are envisaged as treating the diverse aspects of the idea of change of base, above all in relation with the conditions of finiteness or flatness; we there initiate the technique of descent, with its most elementary aspects (the questions of “effectiveness” linked to this technique will be studied in Chapter V).

(c2) Sections §§5–7 are focused on what we may call Noetherian techniques, since the preschemes considered are always locally Noetherian, whereas, on the contrary, there is generally no finiteness condition imposed on the morphisms; this is essentially due to the fact that the ideas of dimension and depth are hardly manageable except in the case of Noetherian local rings. Recall that §7 constitutes a “delicate (?)” theory of Noetherian local rings, not much used in what follows in the chapter.

(c3) Sections §§8–10 describe, amongst other things, the means of eliminating the Noetherian hypotheses on the preschemes considered, by substituting such hypotheses for suitable ones of finiteness (“finite presentation”) on the morphisms considered: the advantage of this substitution is that the latter such hypotheses (those of finiteness on the morphisms) are stable under base change, which is not the case for the Noetherian hypotheses on the preschemes. The technique permitting this substitution relies, in some part, on the use of the idea of the projective limit of preschemes, thanks to which we can reduce a question to the same question with Noetherian hypotheses; on the other hand, it relies on the systematic use of constructible sets, which have the double interest of being preserved under taking inverse images (of arbitrary morphisms) and by direct images (of morphisms of finite presentation), and having manageable topological properties in locally Noetherian preschemes. The same techniques often even allow to restrict to the case of more specific Noetherian rings, for example the \(\mathbb{Z}\)-algebras of finite type, and it is here that the properties of “excellent” rings (studied in §7) intervene in a decisive manner. Independently of the question of elimination of Noetherian hypotheses, the techniques of §§8–10, elementary in nature, find constant use in nearly all applications.

§1. RELATIVE FINITENESS CONDITIONS. CONSTRUCTIBLE SETS IN PRESCHEMES

In this section, we will resume the exposé of “finiteness conditions” for a morphism of preschemes \(f : X \to Y\) given in (I, 6.3 and 6.6). There are essentially two notions of “finiteness” of a global nature on \(X\), that of quasi-compact morphism (defined in (I, 6.6.1)) and that of quasi-separated morphism; on the other hand, there are two notions of “finiteness” of a local nature on \(X\), that of a morphism locally of finite type (defined in (I, 6.6.2)) and that of a morphism locally of finite presentation. By combining these local notions with the preceding global notions, we obtain the notion of a morphism of finite type (defined in (I, 6.3.1)) and of a morphism of finite presentation. For the convenience of the reader, we will give again in this section the properties stated in (I, 6.3 and 6.6), referring to their labels in Chapter I for their proofs.

In n° 1.8 and 1.9, we complete, in the context of preschemes, and making use of the previous notions of finiteness, the results on constructible sets given in (0, §9).
1.1. Quasi-compact morphisms

Definition (1.1.1). — We say that a morphism of preschemes \( f : X \to Y \) is quasi-compact if the continuous map \( f \) from the topological space \( X \) to the topological space \( Y \) is quasi-compact (0, 9.1.1), in other words, if the inverse image \( f^{-1}(U) \) of every quasi-compact open subset \( U \) of \( Y \) is quasi-compact (cf. (I, 6.6.1)).

If \( \mathcal{B} \) is a basis for the topology of \( Y \) consisting of affine open sets, then for \( f \) to be quasi-compact, it is necessary and sufficient that for all \( V \in \mathcal{B}, f^{-1}(V) \) is a finite union of affine open sets. For example, if \( Y \) is affine and \( X \) is quasi-compact, every morphism \( f : X \to Y \) is quasi-compact (I, 6.6.1).

If \( f : X \to Y \) is a quasi-compact morphism, then it is clear that for every open subset \( V \) of \( Y \), the restriction of \( f \) to \( f^{-1}(V) \) is a quasi-compact morphism \( f^{-1}(V) \to V \). Conversely, if \( (U_a) \) is an open cover of \( Y \) and \( f : X \to Y \) is a morphism such that the restrictions \( f^{-1}(U_a) \to U_a \) are quasi-compact, then \( f \) is quasi-compact. As a result, if \( f : X \to Y \) is an \( S \)-morphism of \( S \)-preschemes, and if there exists an open cover \( (S_\lambda) \) of \( S \) such that the restrictions \( g^{-1}(S_\lambda) \to h^{-1}(S_\lambda) \) of \( f \) (where \( g \) and \( h \) are the structure morphisms) are quasi-compact, then \( f \) is quasi-compact.

§2. Base change and flatness

§3. Associated prime cycles and primary decomposition

§4. Change of base field for algebraic preschemes

§5. Dimension, depth, and regularity in locally Noetherian preschemes

§6. Flat morphisms of locally Noetherian preschemes

§7. Relations between a local Noetherian ring and its completion. Excellent rings

§8. Projective limits of preschemes

§9. Constructible properties

§10. Jacobson preschemes

§11. Topological properties of finitely presented flat morphisms. Flatness criteria

§12. Study of fibres of finitely presented flat morphisms

§13. Equidimensional morphisms

§14. Universally open morphisms

§15. Study of fibres of a universally open morphism

§16. Differential invariants. Differentially smooth morphisms

In this paragraph we will present, in global form, some notions of differential calculus particularly useful in algebraic geometry. We will ignore many classic developments in differential geometry (connections, infinitesimal transformations associated to vector fields, jets, etc.), although these notions are translated in a particularly natural way for schemes. We will similarly ignore phenomena exclusive to characteristic \( p > 0 \) (some of which are seen, in the affine case, in (0, 21)). For certain complements to the differential formalism for preschemes the reader may consult Exposés II and VII of [eAG64] as well as subsequent chapters of this treatise.
16. Differential invariants, Differentially Smooth Morphisms

(16.1.1). Let \((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)\) be two ringed spaces and \(f = (\psi, \theta) : Y \to X\) a morphism of ringed spaces (0, 4.1.1) such that the homomorphism
\[
\theta^\#: \psi^*(\mathcal{O}_X) \longrightarrow \mathcal{O}_Y
\]
is surjective, so that \(\mathcal{O}_Y\) is identified with a sheaf of quotient rings \(\psi^*(\mathcal{O}_X)/\mathcal{I}_f\). We can then endow \(\psi^*(\mathcal{O}_X)\) with the \(\mathcal{I}_f\)-preadic filtration.

Definition (16.1.2). — The \(\mathcal{O}_Y\)-augmented sheaf of rings \(\psi^*(\mathcal{O}_X)/\mathcal{I}^n_{f}+1\) is called the \(n'\)th normal invariant of \(f\); the ringed space \((Y, \psi^*(\mathcal{O}_X)/\mathcal{I}_f^n+1)\) is called the \(n'\)th infinitesimal neighborhood of \(Y\) along \(f\) and is denoted by \(Y^n_f\) or simply \(Y^n\). The sheaf of graded rings associated to the sheaf of filtered rings \(\psi^*(\mathcal{O}_X)\)
\[
\mathcal{G}_n(f) = \bigoplus_{n \geq 0} (\mathcal{I}^n_f/\mathcal{I}_f^{n+1})
\]
is called the sheaf of graded rings associated to \(f\). The sheaf \(\mathcal{G}_1(f) = \mathcal{I}_f/\mathcal{I}_f^2\) is called the conormal sheaf of \(f\) (that will be denoted by \(\mathcal{M}_{Y/X}\) when there is no risk of confusion).

It is clear that the \(\mathcal{O}_{Y(n)} = \psi^*(\mathcal{O}_X)/\mathcal{I}_f^n+1\) (that we also denote \(\mathcal{O}_{Y(n)}\)) form a projective system of sheaves of rings on \(Y\), the transition homomorphism \(\varphi_{nm} : \mathcal{O}_{Y(m)} \to \mathcal{O}_{Y(n)}\) for \(n \leq m\) identifies \(\mathcal{O}_{Y(n)}\) with the quotient of \(\mathcal{O}_{Y(m)}\) by the power \((\mathcal{I}_f/\mathcal{I}_f^{n+1})^m\) of the augmentation ideal of \(\mathcal{O}_{Y(n)}\), kernel of \(\varphi_{0n} : \mathcal{O}_{Y(n)} \to \mathcal{O}_Y\). The \(Y^n\) therefore form a inductive system of ringed spaces, all having underlying space \(Y\), and we have canonical morphisms of ringed spaces \(h_n : Y^n \to X\) equal to \((\psi, \theta_n)\), where \(\theta_n^n\) is the canonical morphism \(\psi^*(\mathcal{O}_X) \to \psi^*(\mathcal{O}_X)/\mathcal{I}_f^n+1\). It is clear that the sheaf \(\mathcal{G}_n(f)\) is a sheaf of graded algebras over the sheaf of rings \(\mathcal{O}_Y = \mathcal{G}_0(f)\) and the \(\mathcal{G}_n(f)\) of \(\mathcal{O}_Y\)-modules.

As with every sheaf of filtered rings, we have a canonical surjective homomorphism of graded \(\mathcal{O}_Y\)-algebras
\[
S^*_{\mathcal{G}_n}(\mathcal{G}_1(f)) \longrightarrow \mathcal{G}_n(f)
\]
which coincide in degrees 0 and 1 with the identities.

Examples (16.1.3). —

(i) Suppose that \(X\) is a locally ringed space, \(Y\) is reduced to a single point \(y\) (endowed with a ring \(\mathcal{O}_y\)) and that, if \(x = \psi(y)\), \(\theta^\#: \mathcal{O}_x \to \mathcal{O}_y\) is a surjective homomorphism of rings having as kernel the maximal ideal \(m_x\) of \(\mathcal{O}_x\). So the \(\mathcal{O}_{Y(n)}\) are identified with the rings \(\mathcal{O}_x/m_x^{n+1}\) and \(\mathcal{G}_n(f)\) with the graded ring associated with the local ring \(\mathcal{O}_x\) endowed with the \(m_x\)-preadic filtration.

(ii) Suppose that \(Y\) is a closed subset of an open subspace \(U\) of \(X\) and that the \(\mathcal{O}_Y\) is induced on \(Y\) by a quotient sheaf \(\mathcal{O}_U/\mathcal{I}\), where \(\mathcal{I}\) is an ideal of \(\mathcal{O}_U\) such that \(\mathcal{I}_x = \mathcal{O}_x\) for every \(x \notin Y\); if \(X\) is a locally ringed space we also suppose that \(\mathcal{I} \neq \mathcal{O}_X\) for \(y \in Y\) so that \((Y, \mathcal{O}_Y)\) is a locally ringed space.

Let \(\psi_0 : Y \to U\) be the canonical injection and denote by \(\theta_0 : \mathcal{O}_U \to (\psi_0)_*(\mathcal{O}_Y)\) the homomorphism such that \(\theta_0^\#: (\psi_0)^*(\mathcal{O}_U) = \mathcal{O}_Y|Y \to (\mathcal{O}_U/\mathcal{I})|Y\), so that \(j_0 = (\psi_0, \theta_0) : Y \to U\) is a morphism of ringed spaces (and of locally ringed spaces if \(X\) is a locally ringed space); if \(i : U \to X\) is the canonical injection (morphism of ringed spaces), \(j = i \circ j_0\) is the morphism \((\psi, \theta) : Y \to X\) where \(\psi : Y \to X\) is the canonical injection and \(\theta : \mathcal{O}_X \to \psi_*(\mathcal{O}_Y)\) is the homomorphism such that \(\theta^\# = \theta_0^\#\). Since \(\theta^\#\) is surjective we can apply the previous definitions; \(\mathcal{O}_{Y(n)}\) is equal to \(\mathcal{O}_Y/\mathcal{I}^{n+1}\), and we have \((\psi_0)_*(\mathcal{O}_{Y(n)}) = \mathcal{O}_U/\mathcal{I}^{n+1}\), and \(\mathcal{G}_n(f) = \mathcal{G}_n(j_0) = \mathcal{G}_n(j) = \mathcal{G}_n(j_0) = \psi_0^*(\mathcal{I}_f^n/\mathcal{I}_f^{n+1}) = j_0^*(\mathcal{I}_f^n/\mathcal{I}_f^{n+1})\).

(16.1.4). The example (16.1.3, (ii)) shows that in general the \(\mathcal{O}_{Y(n)}\) are not canonically endowed with a structure of an \(\mathcal{O}_Y\)-module, or a fortiori with a structure of an \(\mathcal{O}_Y\)-algebra. The data of such structure is equivalent to the data of a homomorphism of sheaves of rings \(\lambda_n : \mathcal{O}_Y \to \mathcal{O}_{Y(n)}\), right inverse to
the augmentation morphism \( \varphi_{0n} \); it is also equivalent to the data of a morphism of ringed spaces \((1_Y, \lambda_n) : Y(n) \to Y\) left inverse to the canonical morphism \((1_Y, \varphi_{0n}) : Y \to Y(n)\).

**Proposition (16.1.5).** — Let \( f = (\varphi, \theta) : Y \to X \) be an immersion of preschemes. Then:

(i) \( \mathcal{R}_\bullet(f) \) is a quasi-coherent graded \( \mathcal{O}_Y\)-algebra.

(ii) The \( Y(n) \) are preschemes, canonically isomorphic to subpreschemes of \( X \).

(iii) Every homomorphism of sheaves of rings \( \lambda_n : \mathcal{O}_Y \to \mathcal{O}_{Y(n)} \), right inverse to the augmentation homomorphism \( \varphi_{0n} \), makes the \( \mathcal{O}_{Y(n)} \) and \( \mathcal{O}_{Y(i)} \) for \( k \leq n \) quasi-coherent \( \mathcal{O}_Y\)-algebras; the \( \mathcal{O}_Y\)-module structures induced from the above structures on the \( \mathcal{R}_k(f) \) for \( k \leq n \) coincide with the ones defined in (16.1.2).

**Proof.** (i) Since the question is local on \( X \) and \( Y \), we can reduce to the case where \( Y \) is a closed subscheme of \( X \) defined by an quasi-coherent ideal \( \mathcal{I} \) of \( \mathcal{O}_X \); since \( \mathcal{O}_Y \) is the restriction to \( Y \) of \( \mathcal{O}_X / \mathcal{I} \), the assertion (i) is evident, and \( Y(n) \) is the closed subscheme of \( X \) defined by the quasi-coherent ideal \( \mathcal{I}^{n+1} \) of \( \mathcal{O}_X \). Finally, to prove (iii) we notice that the data of \( \lambda_n \) makes the ideal \( \mathcal{I} / \mathcal{I}^n \) of the augmentation \( \varphi_{0n} \) and their quotients \( \mathcal{I} / \mathcal{I}^{k+1} (1 \leq k \leq n) \) \( \mathcal{O}_Y\)-modules, and it suffices to prove by induction on \( k \) that the \( \mathcal{I} / \mathcal{I}^{k+1} \) are quasi-coherent \( \mathcal{O}_Y\)-modules and the structure of quotient \( \mathcal{O}_Y\)-module induced on \( \mathcal{I} / \mathcal{I}^{k+1} \) is the same as defined on (16.1.2). The second assertion is immediate, \( \mathcal{I} / \mathcal{I}^{k+1} \) being killed by \( \mathcal{I} / \mathcal{I}^{n+1} \); the first result, by induction on \( k \), is trivial for \( k = 1 \) and for \( \mathcal{I} / \mathcal{I}^{k+1} \) being an extension of \( \mathcal{I} / \mathcal{I}^k \) by \( \mathcal{I} / \mathcal{I}^{k+1} \) (1.4.17).

**Corollary (16.1.6).** — Under the general hypotheses of (16.1.5), if the immersion \( f \) is locally of finite presentation then the \( \mathcal{R}_n(f) \) are quasi-coherent \( \mathcal{O}_Y\)-modules of finite type.

**Proof.** Indeed, with the notation from the proof of (16.1.5), \( \mathcal{I} \) is an ideal of finite type of \( \mathcal{O}_X \) (1.4.7), therefore the \( \mathcal{I}^n / \mathcal{I}^{n+1} \) are \( \mathcal{O}_Y\)-modules of finite type, hence the conclusion.

**Corollary (16.1.7).** — Under the general hypotheses of (16.1.5), let \( g : X \to Y \) be a morphism of preschemes, left inverse to \( f \). Therefore, for every \( n \), the composite morphism \((1, \lambda_n) : Y(n) \to X \to Y \) defines a homomorphism of sheaves of rings \( \lambda_n : \mathcal{O}_Y \to \mathcal{O}_{Y(n)} \) right inverse to the augmentation \( \varphi_{0n} \), making \( \mathcal{O}_{Y(n)} \) a quasi-coherent \( \mathcal{O}_Y\)-algebra; via these homomorphisms, the transition homomorphisms \( \varphi_{mn} : \mathcal{O}_{Y(m)} \to \mathcal{O}_{Y(n)} \) \( (n \leq m) \) are homomorphisms of \( \mathcal{O}_Y\)-algebras. Also, if \( g \) is locally of finite type, then the \( \mathcal{O}_{Y(n)} \) are quasi-coherent \( \mathcal{O}_Y\)-modules of finite type.

**Proof.** The first assertion is an immediate result from the definitions and (16.1.5). On the other hand, if \( g \) is locally of finite type, then \( f \) is locally of finite presentation (1.4.3, (v)); the \( \mathcal{R}_n(f) \) being then quasi-coherent \( \mathcal{O}_Y\)-modules of finite type by (16.1.6), the same goes for the \( \mathcal{O}_Y\)-modules \( \mathcal{I} / \mathcal{I}^{n+1} \), being extensions of a finite number of the \( \mathcal{R}_k(f) \) (III, 1.4.17).

**Proposition (16.1.8).** — Let \( X \) be a locally Noetherian prescheme, \( j : Y \to X \) an immersion; Then the \( Y(n) \) are locally Noetherian preschemes, the \( \mathcal{R}_n(j) \) are coherent \( \mathcal{O}_Y\)-modules and the \( \mathcal{R}_\bullet(j) \) is a coherent sheaf of rings over the space \( Y \).

**Proof.** Everything is local on \( X \) and \( Y \), so we reduce to the case where \( X \) is affine and \( j \) is a closed immersion and therefore all the assertions are evident except for the last, which follows from the fact that if \( A \) is a Noetherian ring and \( J \) is an ideal of \( A \), then \( \text{gr}^J_\bullet(A) \) is a Noetherian ring, taking into account the exactness of the functor \( \varphi^* \) and \((0, 5.3.7)\).

**Proposition (16.1.9).** — Let \( X \) be a prescheme, \( j : Y \to X \) an immersion locally of finite presentation, \( y \) a point of \( Y \). The following conditions are equivalent:

(a) There exists an open neighborhood \( U \) of \( y \) in \( Y \) such that \( j|U \) is a homeomorphism of \( U \) onto an open set of \( X \).

(b) There is an integer \( n > 0 \) such that the canonical homomorphism

\[
(\varphi_{n-1,n})_y : \mathcal{O}_{Y(n)}|_y \to \mathcal{O}_{Y(n-1)}|_y
\]

is bijective.

(c) There is an integer \( n > 0 \) such that \((\mathcal{R}_n(j))_y = 0\).

In addition, if the integer \( n \) satisfies (b) or (c), then there is a neighborhood \( V \) of \( y \) in \( Y \) such that \( \mathcal{R}_m(j)|_V = 0 \) for \( m \geq n \) and that \( \varphi_{nm}|_V : \mathcal{O}_{Y(m)}|_V \to \mathcal{O}_{Y(n)}|_V \) is bijective for \( m \geq n \).
PROOF. Since the questions is local on $Y$, we can restrict ourselves to the case where $j$ is a closed immersion, $Y$ being defined by a quasi-coherent ideal of finite type $\mathcal{I}$ of $\mathcal{O}_X$. The equivalence of (b) and (c), for a given $n$, is immediate; also, since $\mathcal{I}/\mathcal{I}^{n+1}$ is an $\mathcal{O}_X$-module of finite type, there is an open neighborhood $U$ of $y$ in $X$ such that $\mathcal{I}^n|U = \mathcal{I}^{n+1}|U$ (0, 5.2.2), so we also have $\mathcal{I}^n|U = \mathcal{I}^m|U$ for $m \geq n$ proving the last assertions. To prove that (a) implies (b), we can restrict ourselves to the cases where the underlying space of $Y$ is equal to the underlying space of $X$ and where $\mathcal{I}$ is generated by a finite number of sections over $X$; since $\mathcal{I}$ is contained in the nilradical $N$ of $\mathcal{O}_X$ (I, 5.1.2), it is now nilpotent which proves b). Finally, to prove that (b) implies (a), we can restrict ourselves to the case where $\mathcal{I}^n = \mathcal{I}^m$; therefore, for every $y \in Y$, since $\mathcal{I}_y \subset \mathfrak{m}_y$, maximal ideal of $\mathcal{O}_{X,y}$, we must have $\mathcal{I}_y = 0$ because of Nakayama’s lemma, since $\mathcal{I}_y$ is an ideal of finite type. The set of $x \in X$ such that $\mathcal{I}_x = 0$ is an open $U$ of $X$ contained in $Y$ (0, 5.2.2); since on the other hand $\mathcal{I}_x \neq 0$ for $x \notin Y$, we must have $U = Y$.

Corollary (16.1.10). — For a restriction of the immersion $j$ to an open neighborhood of $y$ in $Y$ to be an open immersion (in other words, for $j$ to be a local isomorphism on the point $y$), it is necessary and sufficient that $(\mathcal{I}_1(j))_y = (\mathcal{N}_{Y/X})_y = 0$.

PROOF. The condition is clearly necessary, and the previous reasoning applied to $n = 1$ proves that it is sufficient.

Remark (16.1.11). —

(i) Under the conditions of the definition (16.1.1), the projective limit of the projective system $(\mathcal{O}_{Y(n)}, \mathcal{Q}_{Ym})$ of sheaves of rings over $Y$ is called the normal invariant of infinite order of $f$, and sometimes denoted by $\mathcal{O}_{Y(\infty)}$. When $X$ is a locally noetherian prescheme, $j : Y \rightarrow X$ a closed immersion, $Y$ then is a closed subprescheme of $X$ defined by a coherent ideal $\mathcal{I}$ and $\mathcal{O}_{Y(\infty)}$ is exactly the formal completion of $\mathcal{O}_X$ along $Y$ (I, 10.8.4), and $Y(\infty) = (Y, \mathcal{O}_{Y(\infty)})$ is the formal prescheme that is the completion of $X$ along $Y$ (I, 10.8.5). In all cases, we could say that $Y(\infty)$ is the formal neighborhood of $Y$ in $X$ (via the morphism $f$). In the particular case we have just considered, it is the formal prescheme that is the inductive limit of the infinitesimal neighborhoods of order $n$.

(ii) Note that for a morphism of preschemes $f = (\psi, \theta) : Y \rightarrow X$, it can happen that the homomorphism $\theta^\# : \psi^*(\mathcal{O}_X) \rightarrow \mathcal{O}_Y$ is surjective without $f$ being a local immersion and without $f$ being injective. We have an example by taking $Y$ to be a sum of preschemes $Y_\lambda$ all isomorphic to $\text{Spec}(\mathcal{O}_X)$, where $x \in X$, ad taking $f$ to be the morphism equal to the canonical morphism in each of the $Y_\lambda$.

16.2. Functorial properties of the normal invariants of an immersion

(16.2.1). Let $f = (\psi, \theta) : Y \rightarrow X$ and $f' = (\psi', \theta') : Y' \rightarrow X'$ by two morphisms of ringed spaces such that $\theta^\#$ and $\theta'^\#$ are surjective; consider a commutative diagram of morphisms of ringed spaces

$$\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\downarrow u & & \downarrow v \\
Y' & \xrightarrow{f'} & X'
\end{array}$$

Let $u = (\rho, \lambda)$, $v = (\sigma, \mu)$. We have $\rho^*(\psi^*(\mathcal{O}_X)) = \psi'^*(\sigma^*(\mathcal{O}_X))$ and as a result a commutative diagram of homomorphisms of sheaves of rings over $Y'$

$$\begin{array}{ccc}
\rho^*(\psi^*(\mathcal{O}_X)) & = & \psi'^*(\sigma^*(\mathcal{O}_X)) \\
\downarrow \rho^*(\theta^\#) & & \downarrow \psi'^*(\theta'^\#) \\
\rho^*(\mathcal{O}_Y) & \rightarrow & \mathcal{O}_{Y'}
\end{array}$$

from which we conclude, if $\mathcal{I}$ and $\mathcal{I}'$ are the kernels of $\theta^\#$ and $\theta'^\#$, that we have $\psi'^*(y^\#)(\rho^*(\mathcal{I})) \subset \mathcal{I}'$, having in mind the exactness of the functor $\rho^*$. We deduce that, for every integer $n$, $\psi'^*(y^\#)(\rho^*(\mathcal{I}^n)) \subset \mathcal{I}'$.
\( \mathcal{I}^n \), which shows that \( \psi^* (\mu^#) \) defines, passing to the quotients, a homomorphism of sheaves of rings

\[
(16.2.1.2) \quad v_n : \rho^* (\psi^* (\mathcal{O}_X) / \mathcal{I}^{n+1}) \to \psi^* (\mathcal{O}_{X'}) / \mathcal{I}^{n+1}
\]

and therefore a morphism of ringed spaces \( w_n = (\rho, v_n) : Y^{(n)} \to Y^{(n)} \) (which, for \( n = 0 \), is none other than \( u \)). It follows immediately from this definition that the diagrams

\[
\begin{array}{ccc}
Y^{(n)} & \xrightarrow{h_{nn}} & Y^{(m)} \\
w_n & & \downarrow v \\
Y^{(n)} & \xrightarrow{h_m} & X
\end{array}
\]

(where the horizontal arrows are the canonical morphisms (16.1.2) are commutative.

By passage to the quotients via the morphisms (16.2.1.2), and taking into account the exactness of the functor \( \rho^* \), we obtain a di-homomorphism of graded algebras (relative to the morphism \( \lambda^\#: \rho^* (\mathcal{O}_Y) \to \mathcal{O}_{Y'} \))

\[
(16.2.1.3) \quad \text{gr}(u) : \rho^* (\mathcal{I}_\bullet (f)) \to \mathcal{I}_\bullet (f')
\]

(or, if you like, a \( \rho \)-morphism \( (0, 3.5.1) \) \( \mathcal{I}_\bullet (f) \to \mathcal{I}_\bullet (f') \)), and in particular a di-homomorphism of conormal sheafs

\[
\text{gr}^1(u) : \rho^* (\mathcal{I}_1 (f)) \to \mathcal{I}_1 (f')
\]

It is also immediate that these homomorphisms give rise to a commutative diagram

\[
\begin{array}{ccc}
\text{S} (\text{gr}^1(u)) & \xrightarrow{\text{gr}(u)} & \text{S} (\mathcal{I}_1 (f')) \\
\downarrow & & \downarrow \\
\text{S}^* (\mathcal{I}_1 (f')) & \xrightarrow{\mathcal{I}_\bullet (f')} & \mathcal{I}_\bullet (f')
\end{array}
\]

where the horizontal arrow are the canonical morphisms (16.1.2.2).

Finally, if we have a commutative diagram of morphisms of ringed spaces

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\downarrow u & & \downarrow v \\
Y' & \xrightarrow{f'} & X' \\
\downarrow u' & & \downarrow v' \\
Y'' & \xrightarrow{f''} & X''
\end{array}
\]

where \( f'' = (\psi'', \theta'') \) is such that \( \theta'' \rho \) is surjective, and if \( w_n' \) and \( w_n \) are defined from \( u' \), \( v' \) for one and \( u'' = u \circ u' \), \( v'' = v \circ v' \) for the other, we have \( w_n'' = w_n \circ w_n' \), which follows immediately from the definitions and from (0, 3.5.5); we have also \( \text{gr}(u'') = \text{gr}(u') \circ \rho^* (\text{gr}(u)) \) if \( u' = (\rho', \lambda') \). Therefore we can say that \( Y^{(n)} \) and \( \mathcal{I}_\bullet (f) \) depend functorially on \( f \).

**Proposition (16.2.2).** — With the notation and hypotheses of (16.2.1), suppose also that \( f, f', u, \) and \( v \) are morphisms of preschemes. We have:

(i) The morphisms \( w_n : Y^{(n)} \to Y^{(n)} \) are morphisms of preschemes.

(ii) If \( Y' = Y \times_X X' \), \( u \) and \( f' \) the canonical projections, and if \( f \) is an immersion or if \( v \) is flat, we have \( Y'^{(n)} = Y^{(n)} \times_X X' \).

(iii) If \( Y' = Y \times_X X' \) and if \( v \) is flat (resp. if \( f \) is an immersion), the homomorphism

\[
\text{Gr}(u) = \text{gr}(u) \otimes I : \mathcal{I}_\bullet (f) \otimes_{\mathcal{O}_{Y'}} \mathcal{O}_{Y'} \longrightarrow \mathcal{I}_\bullet (f')
\]

is bijective (resp. surjective).
Proof.

(i) The hypotheses immediately imply that, for every \( y' \in Y', \rho^*_\psi (\theta^\psi (y')) \) is a local homomorphism (I, 1.6.2), so \( w_n \) is a morphism of preschemes (I, 2.2.1).

(ii) and (iii) If \( f \) is an immersion, we can restrict ourselves to the case where \( f \) is a closed immersion, \( Y \) being defined by a quasi-coherent ideal \( \mathcal{I} \) of \( \mathcal{O}_X \) and \( Y(n) \) by the ideal \( \mathcal{I}^{n+1} \); the assertions follows from (I, 4.4.5).

Second, suppose that \( \nu \) is flat; we can restrict ourselves to the case where \( X = \text{Spec}(A) \), \( Y = \text{Spec}(B) \), \( X' = \text{Spec}(A') \) are affines, \( A' \) being a flat \( A \)-module; so \( Y' = \text{Spec}(B') \) where \( B' = B \otimes_A A' \); in addition, if \( J \) is the kernel of the homomorphism \( A \to B \), the kernel \( \mathcal{J}' \) of \( A' \to B' \) is identified with \( J \otimes_A A' \) by flatness, and \( \mathcal{I}^n / \mathcal{I}^{n+1} \) is equal to

\[
\psi^*(\mathcal{I}^n / \mathcal{I}^{n+1}) = \mathcal{I}^n / \mathcal{I}^{n+1} \]

and in particular for \( n = 0 \), we have

\[
\theta_{Y'} = \rho^*(\theta_Y) \otimes \rho^*(\mathcal{O}_X) \psi^*(\mathcal{O}_{X'})
\]

from which we have canonical isomorphism of \( \mathcal{I}^n / \mathcal{I}^{n+1} \) with

\[
\rho^*(\mathcal{I}^n / \mathcal{I}^{n+1}) \otimes \rho^*(\mathcal{O}_Y) \theta_{Y'} = (\mathcal{I}^n / \mathcal{I}^{n+1}) \otimes \mathcal{O}_{Y'}
\]

which proves (iii). Let now \( C_n = \Gamma(Y, \mathcal{O}_{Y(n)}) \), \( C'_n = \Gamma(Y', \mathcal{O}_{Y'(n)}) \). As \( Y(n) \) and \( Y'(n) \) are affine schemes (16.1.5), the kernel \( \mathfrak{a}_n \) (resp. \( \mathfrak{a}'_n \)) of the homomorphism \( C_n \to C_{n-1} \) (resp. \( C'_n \to C'_{n-1} \)) is \( \Gamma(Y, \mathcal{I}^n / \mathcal{I}^{n+1}) \) (resp. \( \Gamma(Y', \mathcal{I}^n / \mathcal{I}^{n+1}) \)); therefore we can deduce from the above results that \( \mathfrak{a}_n = \mathfrak{a}_n \otimes_A A' \). Now, we have a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathfrak{a}_n \otimes_A A' & \rightarrow & C_n \otimes_A A' & \rightarrow & C_{n-1} \otimes_A A' & \rightarrow & 0 \\
& & \uparrow r & & \uparrow s & & \uparrow s_{n-1} & & \\
0 & \rightarrow & \mathfrak{a}'_n & \rightarrow & C'_n & \rightarrow & C'_{n-1} & \rightarrow & 0
\end{array}
\]

where the vertical arrow of the left is bijective and the two lines are exact (\( A' \) being a flat \( A \)-module). We deduce by induction that \( s_n \) is bijective for every \( n \), because it is true by hypothesis for \( n = 0 \), and is deduced by application of the five lemma for all \( n \). That proves the second assertion of (ii).

\( \square \)

**Corollary (16.2.3).** — Let \( g : X \to Y, u : Y' \to Y \) be two morphisms of preschemes, \( X' = X \times_Y Y' \), \( g' : X' \to Y' \) and \( v : X' \to X \) by the canonical projections. Let \( f : Y \to X \) by a \( Y \)-section of \( X \) (and therefore a morphism), \( f' = f(g') : Y' \to X' \) the \( Y' \)-section of \( X' \) deduced from \( f \) by the base change \( u \). We have:

(i) The morphism \( w_n : Y'_f^{(n)} \to Y_f^{(n)} \) corresponding to \( f, f', u, v \) (16.2.1) and the canonical morphism \( h_n : Y'_f^{(n)} \to X' \) identifies \( Y'_f^{(n)} \) with the product \( Y_f^{(n)} \times_X X' \).

(ii) If we endow \( \mathcal{O}_{Y'_f} \) (resp. \( \mathcal{O}_{Y_f} \)) with the structure of an \( \mathcal{O}_Y \)-algebra defined by \( g \) (resp. with the structure of an \( \mathcal{O}_{Y'} \)-algebra defined by \( g' \)) (16.1.5, (iii)), then the homomorphism of \( \mathcal{O}_{Y'} \)-algebras induced by the homomorphism \( v_n \) (16.2.1.2) is bijective. Also, the homomorphism of \( \mathcal{O}_Y \)-modules

\[
\text{Gr}_1(u) : \mathcal{R}_1(f) \otimes \mathcal{O}_Y \mathcal{O}_Y' \rightarrow \mathcal{R}_1(f')
\]

is bijective.

Proof.

(i) Let us first note that \( f' : Y' \to X' \) and \( u : Y' \to Y \) identifies \( Y' \) with the product \( Y \times_X X' \) (via the structure morphisms \( f : Y \to X \) and \( v : X' \to X \) (14.5.12.1). The conclusion of (i) now follows from (16.2.2, (ii)), the morphism \( g \) being an immersion.
(ii) The commutative diagram

\[
\begin{array}{ccc}
Y_f(n) & \xleftarrow{v_n} & Y'_f(n) \\
\downarrow h_n & & \downarrow h'_n \\
X & \xleftarrow{v} & X' \\
\downarrow g & & \downarrow g' \\
Y & \xleftarrow{u} & Y'
\end{array}
\]

identifies \(Y'_f(n)\) with the product \(Y_f(n) \times_X X'\), so \((I, 3.3.9)\) it identifies (via the morphisms \(g' \circ h'_n\) and \(w_n\)) \(Y'_f(n)\) to the product \(Y_f(n) \times_Y Y'\). Since \(Y_f(n)\) (resp. \(Y'_f(n)\)) is the affine prescheme over \(Y\) (resp. over \(Y'\)) associated with the \(\mathcal{O}_Y\)-module \(\mathcal{O}_{Y_f(n)}\) (resp. to the \(\mathcal{O}_Y\)-module \(\mathcal{O}_{Y'_f(n)}\)), the fact that the canonical homomorphism \((16.2.3.1)\) is bijective follows from \((II, 1.5.2)\). Finally, the canonical homomorphism \((16.2.3.1)\) is compatible with the augmentations \(\mathcal{O}_{Y_f(n)} \rightarrow \mathcal{O}_Y\) and \(\mathcal{O}_{Y'_f(n)} \rightarrow \mathcal{O}_{Y'}\); since \(\mathcal{O}_{Y_f(n)}\) is a direct sum (as an \(\mathcal{O}_Y\)-module) of \(\mathcal{O}_Y\) and the augmentation ideal \(\mathcal{I}/\mathcal{I}^{n+1}\), we can therefore see that the canonical homomorphism \((16.2.3.1)\), restricted to \(\mathcal{I}/\mathcal{I}^{n+1} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'}\), is a bijection of the latter onto \(\mathcal{I}'/\mathcal{I}'^{n+1}\). For \(n = 1\) this shows that \(\text{Gr}_1(u)\) is bijective.

We note that, under the hypotheses of \((16.2.3)\), the homomorphisms \(\text{Gr}_n(u)\) are surjective in view of the above, but are not bijective in general for \(n \geq 2\). However:

**Corollary (16.2.4).** — Under the hypotheses of \((16.2.3)\), suppose that \(u : Y' \rightarrow Y\) is a flat morphism (resp. that the \(\mathcal{R}_n(f)\) are flat \(\mathcal{O}_Y\)-modules for \(n \leq m\)). Then the homomorphism

\[
\text{Gr}_n(u) : \mathcal{R}_n(f) \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'} \rightarrow \mathcal{R}_n(f')
\]

is bijective for all \(n\) (resp. for \(n \leq m\)).

**Proof.** If \(u\) is flat, then we deduce by base change that the same is true for \(v : X' \rightarrow X\), and we already know in this case that \(\text{Gr}(u)\) is bijective \((16.2.2, (iii))\). If the \(\mathcal{R}_n(f)\) are flat for \(n \leq m\), then we first see by induction on \(n\) that the same holds for \(\mathcal{I}/\mathcal{I}^{n+1}\) for \(n \leq m\), because of the exact sequences

\[
0 \rightarrow \mathcal{I} / \mathcal{I}^{n+1} \rightarrow \mathcal{I} / \mathcal{I}^{n} \rightarrow 0
\]

\((0, 6.1.2)\); in addition, we have the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & (\mathcal{I} / \mathcal{I}^{n+1}) \otimes_{\mathcal{O}_{Y'}} \mathcal{O}_{Y'} & \rightarrow & (\mathcal{I} / \mathcal{I}^{n+1}) \otimes_{\mathcal{O}_{Y}} \mathcal{O}_{Y'} & \rightarrow & (\mathcal{I} / \mathcal{I}^{n}) \otimes_{\mathcal{O}_{Y}} \mathcal{O}_{Y'} & \rightarrow & 0 \\
0 & \rightarrow & \mathcal{J} / \mathcal{J}^{n+1} & \rightarrow & \mathcal{J} / \mathcal{J}^{n} & \rightarrow & 0
\end{array}
\]

in which the lines are exact (the first by flatness \((0, 6.1.2)\)) and the two last vertical arrows are bijective by virtue of \((16.2.2, (ii))\); hence the conclusion.

**Remarks (16.2.5).** —

(i) The reasoning of \((16.2.2, (i))\) still applies to \((16.2.1.1)\) when these are morphisms of \(\text{locally ringed spaces} (I, 1.8.2)\).

(ii) In \((16.2.2, (ii))\), the conclusion is no longer necessarily valid if we only suppose that \(v\) and \(f\) are morphisms of preschemes \((f\) satisfying the condition of \((16.1.1))\). For example (with the notation of the proof of \((16.2.2, (ii))\)), it can happen that \(\mathcal{I} = 0\) but the kernel \(\mathcal{J}'\) of \(\mathcal{A}' \rightarrow \mathcal{B}' = B \otimes_A A'\) is not zero and that \(\mathcal{B}' \neq 0\), in which case we have \(Y^{(n)} = Y\) for all \(n\), but \(Y^{(n)}(n) \neq Y'\). We have an example of this by taking \(A = \mathbb{Z}, B = \mathbb{Q}, A' = \prod_{k=1}^{\infty} (\mathbb{Z}/m^k\mathbb{Z})\) where \(m > 1\).
Consider the particular case of the diagram (16.2.1.1) where $X' = X$, $v$ is the identity, $X$ a prescheme, $Y$ a subscheme of $X$, $Y'$ a subscheme of $Y$, $f$, $u$, and $f' = f \circ u$ the canonical injections; the di-homomorphism (16.2.1.3) gives us, by tensoring with $\mathcal{O}_{Y'}$ over $\mathcal{O}_Y$ of graded $\mathcal{O}_{Y'}$-algebras

\begin{equation}
\mathcal{O}_Y \otimes_{\mathcal{O}_{Y'}} (\mathcal{O}_{Y'}(f')) \to \mathcal{O}_Y(\mathcal{O}_{Y'}). \tag{16.2.6.1}
\end{equation}

On the other hand, we identify $\mathcal{O}_{Y'}$ to $\mathcal{O}_Y$ and $\mathcal{O}_{Y'}$ to $\mathcal{O}_Y$; since $\mathcal{O}_Y$ is an exact functor, we have $\mathcal{O}_Y(\mathcal{O}_{Y'}) = \mathcal{O}_Y(\mathcal{O}_Y) = \mathcal{O}_Y(\mathcal{O}_Y) = \mathcal{O}_Y(\mathcal{O}_Y)$, and since $\mathcal{O}_{Y'}$ is moreover identified with $\mathcal{O}_Y$ of $\mathcal{O}_Y$, we see that $\mathcal{O}_Y = \mathcal{O}_Y(\mathcal{O}_Y)$. We deduce that for every integer $n$ there is a canonical homomorphism $\mathcal{O}_Y^n \to \mathcal{O}_Y^n$, from which we have a canonical morphism of graded $\mathcal{O}_{Y'}$-algebras

\begin{equation}
\mathcal{O}_Y^n(\mathcal{O}_{Y'}(f')) \to \mathcal{O}_Y^n(\mathcal{O}_Y). \tag{16.2.6.2}
\end{equation}

**Proposition (16.2.7).** — Let $X$ be a prescheme, $Y$ a subscheme of $X$, $Y'$ a subscheme of $Y$, $j : Y' \to Y$ the canonical injection. We then have an exact sequence of conormal sheaves ($\mathcal{O}_{Y'}$-modules)

\begin{equation}
\mathcal{J}(\mathcal{N}_{Y/X}) \to \mathcal{N}_{Y'/X} \to \mathcal{N}_{Y'/Y} \to 0 \tag{16.2.7.1}
\end{equation}

where the arrows are the degree 1 components of the canonical homomorphisms (16.2.6.1) and (16.2.6.2).

**Proof.** The problem being local, we can restrict to the case where $X = \text{Spec}(A)$, $Y = \text{Spec}(A/I)$, and $Y' = \text{Spec}(A/I)$ and $I$ and $\mathcal{R}$ being ideals of $A$ such that $I \subset \mathcal{R}$; everything reduces to seeing that the sequence of canonical morphisms $\mathcal{J}/\mathcal{R}\mathcal{O} \to \mathcal{R}/\mathcal{R}^2 \to (\mathcal{J}/\mathcal{R})/(\mathcal{R}/\mathcal{R})^2 \to 0$ is exact, which is immediate given that the image of $\mathcal{J}/\mathcal{R}\mathcal{O}$ in $\mathcal{R}/\mathcal{R}^2$ is $\mathcal{J}/\mathcal{R}^2$ and that $\mathcal{R}/\mathcal{R}^2$ is identified with $\mathcal{R}/(\mathcal{J} + \mathcal{R}^2)$. \( \Box \)

It is easy to give examples where the sequence (16.2.7.1) extended on the left by 0 is not exact; with the above notation, it suffices to take $A = k[T], I = AT^2, \mathcal{R} = AT$, because then $(\mathcal{J} + \mathcal{R}^2)/\mathcal{R}^2 = 0$ and $\mathcal{J}/\mathcal{R}\mathcal{O} \neq 0$. See however (16.9.13) and (19.1.5) for some cases where the extended sequence is indeed exact.

### 16.3. Fundamental differential invariants of morphisms of preschemes

**Definition (16.3.1).** — Let $f : X \to S$ be a morphism of preschemes, $\Delta_f : X \to X \times_S X$ the corresponding diagonal morphism, which is an immersion (I, 5.3.9). We denote by $P_n^f$ or $P_n^{X/S}$ and call the **sheaf of principal parts of order $n$ of the $S$-prescheme $X$**, the $\mathcal{O}_X$-augmented sheaf of rings, $n$-th normal invariant of $\Delta_f$ (16.1.2). We will also write $P_n^0 = \lim_n P_n^{X/S}$, $P_n(\mathcal{O}_f) = P_n(\mathcal{O}_f/S)$, $P_n(\mathcal{O}_f)$, augmentation sheaf of ideals of $\mathcal{O}_f$, is denoted by $\mathcal{O}_f^1$ or $\mathcal{O}_f^{1/S}$, and is called the $\mathcal{O}_X$-module of 1-differentials of $f$, or of $X$ with respect to $S$, or of the $S$-prescheme $X$.

It follows from this definition that $P_n^{X/S}$ is canonically identified with $\mathcal{O}_X$ (16.1.2).

We have (16.1.2.2) a canonical surjective morphism of graded $\mathcal{O}_X$-algebras

\begin{equation}
S^{\bullet}(\mathcal{O}_X^1) \to P_\bullet(\mathcal{O}_X/S). \tag{16.3.1.1}
\end{equation}

And it follows from Definition (16.3.1) that for every open $U$ of $X$ we have $P_n(U) = P_n|U, P_n^{\bullet}|U = P_n^{\bullet}|U, P_n(\mathcal{O}_f(U)) = P_n(\mathcal{O}_f(U))|U, \mathcal{O}_f^1(U) = \mathcal{O}_f^1|U$ (in other words, the notions introduced are local on $X$).

**Proposition (16.3.2).** Denote by $p_1, p_2$ the two canonical projections of the product $X \times_S X$; since $\Delta_f$ is an $X$-section of $X \times_S X$ for both $p_1$ and $p_2$, each of these morphisms define, for all $n$, a homomorphism of sheaves of rings $\mathcal{O}_X \to P_n^{X/S}$, right inverse of the augmentation $P_n^{X/S} \to \mathcal{O}_X$ (16.1.7); we can also say that we thus define on $P_n^{X/S}$ two quasi-coherent augmented $\mathcal{O}_X$-algebra structures; the corresponding $\mathcal{O}_X$-module structures on $P_n(\mathcal{O}_f)$ are the same. We also have, by passing to the limit, two $\mathcal{O}_X$-algebra structures on $P_n^{\bullet}$. 

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The morphism \( s = (p_2, p_1)_S : X \times_S X \to X \times_S X \) is an involutive automorphism of \( X \times_S X \), called the canonical symmetry, such that

\[(16.3.3.1) \quad p_1 \circ s = p_2, \quad p_2 \circ s = p_1, \quad s \circ \Delta_f = \Delta_f.\]

If we put \( s = (\rho, \lambda), p_i = (\pi_i, \mu_i) \) (\( i = 1, 2 \)), \( \Delta_f = (\delta, v) \), \( \lambda^\# \) is then an isomorphism of \( \rho^i(\pi_1^* (\mathcal{O}_X)) \) onto \( \pi_2^* (\mathcal{O}_X) \), and \( \delta^i(\mathcal{O}_{X \times_S X}) \) the kernel \( \mathcal{I} \) of the homomorphism \( \iota^\#: \delta^i(\mathcal{O}_{X \times_S X}) \to \mathcal{O}_X \). Therefore:

**Proposition (16.3.4).** — The homomorphism \( \sigma = \delta^i(\lambda^\#) \) induced from \( s \) (and also called the canonical symmetry) is an involutive automorphism of the projective system \( (\mathcal{P}^n_{X/S}) \) of \( \mathcal{O}_X \)-augmented sheaves of rings, and as a result also of the projective limit \( \mathcal{P}^\infty_{X/S} \). This automorphism permutes the \( \mathcal{O}_X \)-algebra structure on \( \mathcal{P}^n_{X/S} \) and on \( \mathcal{P}^\infty_{X/S} \).

(16.3.5). In what follows, the two \( \mathcal{O}_X \)-algebra structures defined on \( \mathcal{P}^n_{X/S} \) and on \( \mathcal{P}^\infty_{X/S} \) will play very different roles: we will now agree, unless said otherwise, that when \( \mathcal{P}^n_{X/S} \) or \( \mathcal{P}^\infty_{X/S} \) is considered as an \( \mathcal{O}_X \)-algebra, it is the algebra structure induced by \( p_1 \).

For every open \( U \) of \( X \) and every section \( t \in \Gamma(U, \mathcal{O}_X) \), we will simply denote by \( t.1 \) or even \( t \) the image of \( t \) under the structure morphism \( \Gamma(U, \mathcal{O}_X) \to \Gamma(U, \mathcal{P}^n_{X/S}) \) (resp. \( \Gamma(U, \mathcal{O}_X) \to \Gamma(U, \mathcal{P}^\infty_{X/S}) \)) (that is to say, the automorphism corresponding to \( p_1 \)).

**Definition (16.3.6).** — We denote by \( d^n, d^n_X \) (resp. \( d^n_f, d^n_{X/S} \)), or simply \( d^n \) (resp. \( d^n_{\mathcal{O}} \)), the homomorphism of sheaves of rings \( \mathcal{O}_X \to \mathcal{P}_f^n = \mathcal{P}^n_{X/S} \) (resp. \( \mathcal{O}_X \to \mathcal{P}^\infty_{X/S} \)) induced by \( p_2 \). For every open \( U \) of \( X \), and every \( t \in \Gamma(U, \mathcal{O}_X) \), \( d^n(t) \) (resp. \( d^n_{\mathcal{O}}(t) \)) is called the principal part of order \( n \) (resp. principal part of infinite order) of \( t \). We set \( dt = d^1 t - t \), and we say that \( dt \) is the differential of \( t \), or simply \( \mathcal{O}_X \)-algebra (an element of \( \Gamma(U, \mathcal{O}^1_{X/S}) \), also denoted \( d_{X/S}(t) \)).

It follows immediately \(^2\) from this definition that we have

\[(16.3.6.1) \quad d(t_1 t_2) = t_1 dt_2 + t_2 dt_1 \]

for every \( t_1, t_2 \in \Gamma(U, \mathcal{O}_X) \), that is, \( d \) is a derivation of the ring \( \Gamma(U, \mathcal{O}_X) \) in the \( \Gamma(U, \mathcal{O}_X) \)-module \( \Gamma(U, \mathcal{O}^1_{X/S}) \).

In all notation introduced in (16.3.1) and (16.3.6), we will sometimes replace \( S \) by \( A \) when \( S = \text{Spec}(A) \).

(16.3.7). Suppose in particular that \( S = \text{Spec}(A) \) and \( X = \text{Spec}(B) \) are affine schemes, \( B \) then being an \( A \)-algebra. Then \( \Delta_f \) corresponds to the canonical surjective homomorphism \( \pi: B \otimes_A B \to B \) such that \( \pi(b \otimes b') = bb' \), with kernel \( \mathcal{J} = J_{B/A}(0, \mathcal{O}_X) \); \( \mathcal{P}^n_f \) is the structure sheaf of the presheve \( \text{Spec}(P^n_{B/A}) \), where

\[ P^n_{B/A} = (B \otimes_A B)/\mathcal{J}^{n+1}; \]

\( \mathcal{G}^\bullet_\mathcal{I}(\mathcal{P}_f) \) is the quasi-coherent \( \mathcal{O}_X \)-module corresponding to the graded \( B \)-module

\[ \operatorname{gr}^\bullet_\mathcal{I}(B \otimes_A B) = \bigoplus_{n \geq 0} (\mathcal{J}^n/\mathcal{J}^{n+1}); \]

in particular \( \mathcal{O}^1_f = \mathcal{O}^1_{X/S} \) is the quasi-coherent \( \mathcal{O}_X \)-module corresponding to the \( B \)-module of 1-differentials of \( B \) over \( A \), \( \mathcal{O}^1_{B/A} \) (0, 20.4.3). The projection morphisms \( p_1 : X \times_S X \to X, p_2 : X \times_S X \to X \) corresponding to the two homomorphisms of rings \( j_1 : B \to B \otimes_A B, j_2 : B \to B \otimes_A B \) such that \( j_1(b) = b \otimes 1, j_2(b) = 1 \otimes b \), so that (by the convention of (16.3.5)), \( P^n_{B/A} \) is always considered as a \( B \)-algebra via the composite homomorphism \( B \xrightarrow{j_1} B \otimes_A B \to P^n_{B/A} \); the ring homomorphism \( B \xrightarrow{j_2} B \otimes_A B \to P^n_{B/A} \) is denoted by \( d^n_{B/A} \) and corresponds to \( d^n_{X/S} \) acting on \( \Gamma(X, \mathcal{O}_X) \) for every \( t \in B \), \( dt \) is equal to \( d_{B/A} t \), defined in (0, 20.4.6).

If \( \pi_n : B \otimes_A B \to P^n_{B/A} \) is the canonical homomorphism, so we have, in light of the preceding definitions,

\[(16.3.7.1) \quad \pi_n(b \otimes b') = b \cdot \pi_n(1 \otimes b') = b \cdot d^n_{B/A}(b') \quad \text{for } b \in B, b' \in B.\]

\(^2\) [Trans.] This is, locally we have (0, 20.1.1).
Proposition (16.3.8). — The image of the canonical homomorphism \( d^n_{X/S} : \mathcal{O}_X \to \mathcal{P}^n_{X/S} \) generates the \( \mathcal{O}_X \)-module \( \mathcal{P}^n_{X/S} \).

**Proof.** We immediately reduce to the case where \( X = \text{Spec}(B) \) and \( S = \text{Spec}(A) \) are affine and the proposition follows from (16.3.7.1) since \( \pi_n \) is surjective. We note that in general \( d^n_{X/S} \) is not surjective (even for \( n = 1 \)). \( \square \)

Proposition (16.3.9). — Suppose that \( f : X \to S \) is a morphism locally of finite type. Then the \( \mathcal{P}^n_f \) and the \( \mathcal{G}^n_f(\mathcal{P}_f) \) are quasi-coherent \( \mathcal{O}_X \)-modules of finite type.

**Proof.** This follows from (16.1.6) and from the fact that \( \Delta_f \) is locally of finite presentation (1, 4.3.1). \( \square \)

16.4. Functorial properties of differential invariants

(16.4.1). Consider a commutative diagram of morphisms of preschemes

\[
\begin{array}{ccc}
X & \xleftarrow{u} & X' \\
\downarrow f & & \downarrow f' \\
S & \xleftarrow{w} & S'
\end{array}
\]

We deduce a commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{u} & X' \\
\downarrow \Delta_f & & \downarrow \Delta_{f'} \\
X \times_S X & \xleftarrow{v} & X' \times_{S'} X'
\end{array}
\]

where \( v \) is the composite homomorphism (I, 5.3.5) and (I, 5.3.15).

(16.4.1.2)

\[
X' \times_{S'} X' \xrightarrow{(\rho_1^*, \rho_2^*)} X' \times_S X' \xrightarrow{u \times s} X \times_S X.
\]

So we induce from \( u \) and \( v \), as explained in (16.2.1), homomorphisms of augmented sheaves of rings

(16.4.1.3)

\[
\nu_n : \rho^*(\mathcal{P}^n_{X/S}) \to \mathcal{P}^n_{X'/S'}
\]

(where we put \( u = (\rho, \lambda) \)); these homomorphisms form a projective system, and therefore give at the limit a homomorphism of sheaves of graded rings

(16.4.1.4)

\[
\nu_\infty : \rho^*(\mathcal{G}^n_{X/S}) \to \mathcal{G}^n_{X'/S'};
\]

on the other hand, by passing to the quotient, the homomorphisms \( \nu_n \) give rise to a di-homomorphism of graded algebras (relative to \( \lambda^n \)):

(16.4.1.5)

\[
\text{gr}(u) : \rho^*(\mathcal{G}_X(\mathcal{P}_X)) \to \mathcal{G}_X(\mathcal{P}_{X'/S'}).
\]

(16.4.2). If we have a commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{u} & X' & \xleftarrow{u'} & X'' \\
\downarrow f & & \downarrow f' & & \downarrow f'' \\
S & \xleftarrow{w} & S' & \xleftarrow{w'} & S''
\end{array}
\]

we deduce a commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{u} & X' & \xleftarrow{u'} & X'' \\
\downarrow \Delta_f & & \downarrow \Delta_{f'} & & \downarrow \Delta_{f''} \\
X \times_S X & \xleftarrow{v} & X' \times_{S'} X' & \xleftarrow{v'} & X'' \times_{S''} X''
\end{array}
\]
where \( \nu' \) is defined from \( u', w', f', f'' \) as \( v \) is from \( u, w, f, f' \). We verify immediately that if \( u'' = u \circ u' \), \( w'' = w \circ w' \), then the composite homomorphism \( \nu = \nu' \circ \nu'' \) is equal to the homomorphism \( \nu'' \) deduced from \( u', w', f', f'' \) as \( v \) is from \( u, w, f, f' \). If we put \( u' = (\rho', \lambda') \), \( u'' = (\rho'', \lambda'') \) it follows (16.2.1) that the homomorphism \( \nu'' : \rho''^* (\mathcal{P}_n^{X/S}) \rightarrow \mathcal{P}_n^{X'/S'} \) is equal to the composite

\[
\rho''^*(\rho^* (\mathcal{P}_n^{X/S})) \xrightarrow{\rho''^*(\nu_n)} \rho''^* (\mathcal{P}_n^{X'/S'}) \xrightarrow{\nu'_n} \mathcal{P}_n^{X'/S'},
\]

and we have analogous transitivity properties for the homomorphisms (16.4.1.4) and (16.4.1.5), which lets us say that the \( \mathcal{P}_n^{X/S} \), \( \mathcal{P}_n^{X'/S'} \) and \( \mathcal{R}_n^* (\mathcal{P}_X/S) \) depend functorially on \( f \).

(16.4.3). We verify immediately (for example, by restricting ourselves to the affine case with help of (16.3.7)) that with the notation of (16.4.1), the diagram

\[
\begin{array}{ccc}
\rho^* (\mathcal{O}_X) & \xrightarrow{\lambda^*} & \mathcal{O}_{X'} \\
& \downarrow & \\
\rho^* (\mathcal{P}_n^{X/S}) & \xrightarrow{\nu_n} & \mathcal{P}_n^{X'/S'}
\end{array}
\]

where the vertical arrows are the ones defining the algebra structure chosen in (16.3.5) (that is to say, the ones coming from the first projections) is commutative; the same goes for the diagram

\[
\begin{array}{ccc}
\rho^* (\mathcal{O}_X) & \xrightarrow{\lambda^*} & \mathcal{O}_{X'} \\
& \downarrow_{\rho^*(d^e_{X/S})} & \downarrow_{\rho'_n(\mathcal{P}_n^{X/S})} \\
\rho^* (\mathcal{P}_n^{X/S}) & \xrightarrow{\nu_n} & \mathcal{P}_n^{X'/S'}
\end{array}
\]

the vertical arrows defining here the algebra structure from the second projection; besides, if \( \sigma \) and \( \sigma' \) are the canonical symmetries corresponding to \( f \) and \( f' \) (16.3.4), we have

\[
v_n \circ \rho^*(\sigma) = \sigma' \circ v_n
\]

which switches one diagram with the other. We deduce from (16.4.3.1) a canonical homomorphism of augmented \( \mathcal{O}_{X'} \)-algebras

\[
P_n^* (u) : u^* (\mathcal{P}_n^{X/S}) = \mathcal{P}_n^{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'} \rightarrow \mathcal{P}_n^{X'/S'}
\]

and it follows from (16.4.3.2) that the diagram

\[
\begin{array}{ccc}
\mathcal{O}_{X'} & \xrightarrow{\text{id}} & \mathcal{O}_{X'} \\
& \downarrow_{u^*(d^e_{X/S})} & \downarrow_{d^e_{X'/S'}} \\
u^* (\mathcal{P}_n^{X/S}) & \xrightarrow{p_n(u)} & \mathcal{P}_n^{X'/S'}
\end{array}
\]

is commutative. We deduce a homomorphism of graded \( \mathcal{O}_{X'} \)-algebras

\[
\text{Gr}_n^* (u) : u^* (\mathcal{R}_n^* (\mathcal{P}_X/S)) \rightarrow \mathcal{R}_n^* (\mathcal{P}_{X'/S'})
\]

and in particular a homomorphism of \( \mathcal{O}_{X'} \)-modules

\[
\text{Gr}_1 (u) : \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'} \rightarrow \Omega^1_{X'/S'}
\]

giving rise to a commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_{X'} & \xrightarrow{\text{id}} & \mathcal{O}_{X'} \\
& \downarrow_{d^e_{X/S} \otimes 1} & \downarrow_{d^e_{X'/S'}} \\
\Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'} & \xrightarrow{} & \Omega^1_{X'/S'}
\end{array}
\]
(16.4.4). When $S = \text{Spec}(A), S' = \text{Spec}(A'), X = \text{Spec}(B), X' = \text{Spec}(B')$ are affine, so that we have a commutative diagram of ring homomorphisms

$$
\begin{array}{ccc}
B & \longrightarrow & B' \\
\uparrow & & \uparrow \\
A & \longrightarrow & A'
\end{array}
$$

the image of $\mathfrak{I}_{B/A}$ in $B' \otimes_A B'$ is contained in $\mathfrak{I}_{B'/A'}$, and the homomorphism $\nu_n$ corresponds to the homomorphism of rings $P^n_{B/A} \to P^n_{B'/A'}$ induced from the homomorphism $B \otimes_A B \to B' \otimes_A B'$ by passing to quotients. The homomorphism (16.4.3.6) corresponds to the homomorphism defined in (0, 20.5.4.1), and the commutative diagram (16.4.3.7) to the diagram (0, 20.5.4.2).

**Proposition (16.4.5).** — Suppose that $X' = X \times_S S'$, $f'$ and $u$ the canonical projections. Then the canonical homomorphisms $P^n(u)$ (16.4.3.3) and $\text{Gr}_1(u)$ (16.4.3.6) are bijective.

**Proof.** We have $X' \times_S X' = (X \times_S X) \times_S S'$, and it suffices to apply (16.2.3, (ii)) replacing $g$ by the first $p_1: X \times_S X \to X$ and $f$ by the diagonal $\Delta_f$.

We note that under the hypotheses of (16.4.5) the homomorphism $\text{Gr}_n(u)$ (16.4.3.5) is surjective, but not bijective in general. However (16.2.4):

**Corollary (16.4.6).** — Under the hypotheses of (16.4.5), suppose in addition that $w: S \to S'$ is flat (resp. that $\mathfrak{F}_n(\mathcal{P}^n_{X/S})$ are flat $\mathcal{O}_X$-modules for $n \leq m$); then the homomorphism

$$
\text{Gr}_n(u): u^*(\mathfrak{F}(\mathcal{P}^n_{X/S})) \longrightarrow \mathfrak{F}(\mathcal{P}^n_{X'/S'})
$$

is bijective for each $n$ (resp. for $n \leq m$).

**Proof.** Indeed, if $w$ is flat, then so is $v: X' \times_S X' \to X \times_S X$, so the conclusion follows from (16.2.4). □

(16.4.7). Let $S$ be a prescheme, $\mathcal{E}$ a quasi-coherent $\mathcal{O}_S$-Module, and set $X = V(\mathcal{E})$ (II, 1.7.8), the vector bundle associated to $\mathcal{E}$. Let $f: X \to S$ be the structure morphism. For every open $U$ of $S$ and every section $t \in \Gamma(U, \mathcal{E})$, $t$ is identified with a section of $\mathcal{S}_{\mathcal{O}_S}(\mathcal{E})$ over $U$; let $t'$ be its image in $\Gamma(f^{-1}(U), \mathcal{O}_X) = \Gamma(U, f_*(\mathcal{O}_X)) = \Gamma(U, \mathcal{S}_{\mathcal{O}_S}(\mathcal{E}))$, and set

$$
\delta(t) = \mathcal{P}_U^n(X)/\mathcal{X}^{n+1}.
$$

It is clear that $\delta$ is a di-homomorphism of modules (corresponding to the homomorphism of rings $\Gamma(U, \mathcal{O}_S) \to \Gamma(f^{-1}(U), \mathcal{O}_X)$) into $\Gamma(f^{-1}(U), \mathcal{P}_U^n(X)/\mathcal{X}^{n+1})$, and therefore the image belongs to the augmentation ideal of $\Gamma(f^{-1}(U), \mathcal{P}_U^n(X)/\mathcal{X}^{n+1})$. We deduce (by varying $U$) a canonical homomorphism of $\mathcal{O}_X$-algebras

$$
\delta_n: f^*(\mathcal{S}_{\mathcal{O}_S}(\mathcal{E}))/\mathcal{X}^{n+1} \longrightarrow \mathcal{P}^n_{X/S}.
$$

**Proposition (16.4.8).** — Under the conditions of (16.4.7), the homomorphisms $\delta_n$ are bijective and form a projective system of isomorphisms; we deduce an isomorphism of graded $\mathcal{O}_S$-algebras

$$
\delta_n: f^*(\mathcal{S}_\mathcal{O}_S(\mathcal{E})) \longrightarrow \mathfrak{F}_n(\mathcal{P}^n_{X/S})
$$

**Proof.** The fact that homomorphisms (16.4.7.3) form a projective system follows immediately from their definition. To prove they are isomorphisms, it suffices to prove that (16.4.8.1) is an isomorphism, since both filtrations involved in (16.4.7.3) are finite (Bourbaki, Alg. comm., chap. III, §2, no 8, cor. 3 of th. 1). To do this, consider the split exact sequence of $\mathcal{O}_S$-modules

$$
0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow 0
$$
where, for every pair of sections \( s, t \) of \( \mathcal{E} \) over an open \( U \) of \( S \), we take \( u(s) = (-s, s) \) and \( v(s, t) = s + t \). We have

\[
X \times_S X = \text{Spec}(S \otimes \mathcal{E}, S \otimes \mathcal{E}) = \text{Spec}(S \otimes \mathcal{E}, S \otimes \mathcal{E})
\]

((II, 14.6) and (II, 1.7.11)), and the diagonal morphism \( X \to X \times_S X \) corresponds to the diagonal homomorphism of \( \mathcal{O}_X \)-algebras \( S(v): S \otimes \mathcal{E} \to S \otimes \mathcal{E} \) (II, 1.7.4), such that if \( \mathcal{I} \) is the kernel of this homomorphism, then we have

\[
\mathcal{R}_{X/S}^n = f^*(S \otimes \mathcal{E}) / \mathcal{I}^{n+1}.
\]

The proposition now will be a consequence of the following lemma:

**Lemma (16.4.8.3).** Let \( Y \) be a ringed space, \( 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \) an exact sequence of \( \mathcal{O}_Y \)-modules such that each point \( y \in Y \) has an open neighborhood \( V \) such that the sequence \( 0 \to \mathcal{F}'|V \to \mathcal{F}|V \to \mathcal{F}''|V \to 0 \) is split. Let \( \mathcal{I} \) be the kernel ideal of \( S(v) \):

\[
S_{\mathcal{O}_Y}(\mathcal{F}) \to S_{\mathcal{O}_Y}(\mathcal{F}''),
\]

and let \( \text{gr}_{\mathcal{I}}(S_{\mathcal{O}_Y}(\mathcal{F})) \) be the graded \( \mathcal{O}_Y \)-algebra associated to the \( \mathcal{O}_Y \)-algebra \( S_{\mathcal{O}_Y}(\mathcal{F}) \) endowed with the \( \mathcal{I} \)-preadic filtration. Then the homomorphism of graded \( \mathcal{O}_Y \)-algebras

\[
(S_{\mathcal{O}_Y}(\mathcal{F}')) \otimes_{\mathcal{O}_Y} S_{\mathcal{O}_Y}(\mathcal{F}'') \to \text{gr}_{\mathcal{I}}(S_{\mathcal{O}_Y}(\mathcal{F}))
\]

(16.4.8.4)

is bijective.

**Proof.** The injection \( \mathcal{F}' \to \mathcal{I} \) indeed canonically gives a homomorphism of graded \( \mathcal{O}_Y \)-algebras \( \text{gr}_{\mathcal{I}}(S_{\mathcal{O}_Y}(\mathcal{F}')) \to \text{gr}_{\mathcal{I}}(S_{\mathcal{O}_Y}(\mathcal{F})) \), and since the second member is by definition a graded \( \mathcal{O}_Y \)-algebra, we induce the canonical homomorphism (16.4.8.4) by tensoring the above with \( S_{\mathcal{O}_Y}(\mathcal{F}'') \). To prove the lemma we can, being a local problem, restrict to the case where \( \mathcal{F} = \mathcal{F}' + \mathcal{F}'' \) and \( \mathcal{I} = \mathcal{I}' + \mathcal{I}'' \), and prove that the canonical homomorphisms. Then the graded algebra \( S_{\mathcal{O}_Y}(\mathcal{F}) \) is canonically identified with the graded tensor product \( S_{\mathcal{O}_Y}(\mathcal{F}') \otimes_{\mathcal{O}_Y} S_{\mathcal{O}_Y}(\mathcal{F}'') \) (II, 1.7.4), and it is immediate that \( \mathcal{I} \) is therefore the ideal \( \mathcal{I}' \otimes_{\mathcal{O}_Y} S_{\mathcal{O}_Y}(\mathcal{F}'') \), where \( \mathcal{F}' \) is the augmentation ideal of \( S_{\mathcal{O}_Y}(\mathcal{F}') \), that is to say the (direct) sum of the \( S_{\mathcal{O}_Y}(\mathcal{F}') \) for \( m \geq 1 \). We conclude that \( \mathcal{I} = \mathcal{I}' \otimes_{\mathcal{O}_Y} S_{\mathcal{O}_Y}(\mathcal{F}'') \), where this time \( \mathcal{I}' \) is the direct sum of the \( S_{\mathcal{O}_Y}(\mathcal{F}') \) for \( m \geq n \); we have therefore \( \mathcal{I} \mathcal{I}' = S_{\mathcal{O}_Y}(\mathcal{F}') \mathcal{I}' \), which proves that (16.4.8.4) is bijective.

Having proved the lemma, it remains to see that the homomorphism (16.4.8.1) is the image by \( f^* \) of the homomorphism (16.4.8.4) corresponding to the exact sequence (16.4.8.2); we can easily see that follows from the definition of \( f^* \) (16.4.8.2) and of \( \delta \) (16.4.7.1), given the definition of the \( \mathcal{O}_X \)-algebra structures of \( \mathcal{P}_{X/S}^n \) and of the \( d_{X/S}^n \) (16.3.5) and (16.4.3.6).

In particular:

**Corollary (16.4.9).** Under the conditions of (16.4.7), we have a canonical isomorphism

(16.4.9.1)

\[
\text{gr}_{1}(\delta) : f^*(\mathcal{E}) \simeq \Omega_{X/S}^1.
\]

**Corollary (16.4.10).** If \( S = \text{Spec}(A) \), \( \mathcal{E} = \mathcal{O}_S^m \), so that

\[
X = \text{Spec}(A[T_1, \ldots, T_m]),
\]

then \( \mathcal{P}_{X/S}^n \) is canonically identified with the \( \mathcal{O}_X \)-algebra corresponding to the quotient \( A[T_1, \ldots, T_m] / \mathcal{R}^{n+1} \), where the \( U_i \) (1 \( \leq i \leq m \)) are \( m \) new indeterminates and \( \mathcal{R} \) is the ideal generated by \( U_i \), \( 1 \leq i \leq m \).

We thus recover in particular the structure of \( \Omega_{X/S}^1 \) in this case (0, 20.5.13).

In addition, note that the \( d_{X/S}^n \) then corresponds to a polynomial \( F(T_1, \ldots, T_m) \), class modulo \( \mathcal{R}^{n+1} \) of \( F(T_1 + U_1, \ldots, T_m + U_m) \), which follows from the definition (16.4.7.1).
\textbf{Proposition (16.4.11).} — Let $f : X \to S$ be a morphism, $g : S \to X$ a $S$-section of $X$, $S^{(n)}$ the $n$-th infinitesimal neighborhood of $S$ by the immersion $g$ (16.1.2). Then there exists a unique isomorphism of $\mathcal{O}_S$-algebras

$$\omega_n : g^*(\mathcal{P}^n_{X/S}) \to \mathcal{O}_{S^{(n)}}$$

(via the $\mathcal{O}_S$-algebra structure on $\mathcal{O}_{S^{(n)}}$ defined by $f$ (16.1.7)), making the diagram

$$\begin{array}{ccc}
\mathcal{O}_S &=& g^*(\mathcal{O}_X) \\
&\xrightarrow{\omega_n}& \mathcal{O}_{S^{(n)}} \\
\downarrow_{\lambda_n} & & \downarrow_{g^*(\Delta^n_{X/S})} \\
\downarrow_{\omega_n} & & \downarrow_{g^*(\mathcal{P}^n_{X/S})} \\
\end{array}$$

commutative (where $\lambda_n$ is the structure morphism).

\textbf{Proof.} In light of (I, 5.3.7), where we replace $X, Y, S$ by $X, S, S$ respectively and $f$ by $g$, the diagrams

$$\begin{array}{ccc}
S &=& X \\
\downarrow_{g} & \searrow_{\Delta_f} & \searrow_{\Delta_f} \\
X &\to& X \times_S X \\
\downarrow_{(g \circ f)_{S}} & & \downarrow_{(1_X, g \circ f)_{S}} \\
\end{array}$$

identifies $S$ with the product of the $(X \times_S X)$-preschemes $X$ and $X$ by the morphisms $\Delta_f$ and $(g \circ f, 1_X)_S$ (resp. $(1_X, g \circ f)_S$). On the other hand, the diagrams

$$\begin{array}{ccc}
X &\to& X \\
\downarrow_{f} & \searrow_{p_1} & \searrow_{p_2} \\
S &\to& S \\
\downarrow_{g} & & \downarrow_{g} \\
\end{array}$$

identify $X$ to the product of $X$-preschemes $S$ and $X \times_S X$ via the morphisms $g$ and $p_1$ (resp. $p_2$) (particular case of the associativity formula (I, 3.3.9.1)). We can say that $\Delta_f$, considered as an $X$-section of $X \times_S X$ (relative to $p_1$ or $p_2$) plays the role of a universal section for the $S$-sections of $X$: each of these sections $g$ in fact are deduced by base change $(g \circ f, 1_X)_S : X \to X \times_S X$. The definition of the homomorphism $\omega_n$ and the fact that it is bijective follows from the remarks of (16.2.3, (ii)) applied to the first diagram (16.4.11.4). The commutativity of the first diagram (16.4.11.4) follows also from (16.2.3, (iii)) this time applied to the second diagram (16.11.4). To explain $\omega_n$, we can restrict ourselves to the case where $g$ is a closed immersion: Indeed, for every $s \in S$, there is an open neighborhood $W$ of $s$ in $S$ such that $g(W)$ is closed in an open set $U$ of $X$, and it is clear that $g|W$ is a $W$-section of the morphism $U \cap f^{-1}(W)$. We can then suppose that $S$ is a closed subscheme of $X$ defined by a quasi-coherent ideal $\mathcal{I}$. Then the preceding definitions show that if $W$ is an open of $S$, $t$ is a section of $\mathcal{O}_X$ over $f^{-1}(W)$, $\omega_n(d^n t|W)$ is equal to the canonical image of $t$ in $\Gamma(W, (\mathcal{O}_X/\mathcal{I}^{n+1})|W)$. The uniqueness of $\omega_n$, then follows since the image of $\mathcal{O}_X$ under $d^n|W$ generates the $\mathcal{O}_X$-module $\mathcal{P}^n_{X/S}$.

\textbf{Corollary (16.4.12).} — Let $k$ be a field, $X$ a $k$-prescheme, $x$ a point of $X$ rational over $k$. Then $(\mathcal{P}^n_{X/S})_x \otimes_{\mathcal{O}_X} k(x)$ is canonically isomorphic (as an augmented $k(x)$-algebra) to $\mathcal{O}_x / m_x^{n+1}$.

\textbf{Proof.} It suffices to consider the unique $k$-section $g$ of $X$ such that $g(\text{Spec}(k)) = \{x\}$.

\textbf{Corollary (16.4.13).} — Let $f : X \to S$ be a morphism, $s$ a point of $S$, $X_s = X \times_S \text{Spec}(k(s))$ the fibre of $f$ in $s$. If $x \in X_s$ is rational over $k(s)$, $(\mathcal{P}^n_{X/S})_x \otimes_{\mathcal{O}_S} k(s)$ is canonically isomorphic to $\mathcal{O}_{X,x}/m_x^{n+1}$, is the maximum ideal of $\mathcal{O}_{X,x}$: more precisely, this isomorphism sends $(d^n t)_x \otimes 1$ (where $t$ is a section of $\mathcal{O}_X$ over an open neighborhood of $x$ in $X$) to the class of $t_x \otimes 1$ modulo $m_x^{n+1}$. 

The preceding corollaries justify the terminology “sheaf of principal parts of order $n$”.

**Proposition (16.4.14).** — Let $\rho : A \to B$ be a morphism of rings, $S$ a multiplicative subset of $B$. Then the canonical homomorphisms

\[
S^{-1}P^n_{B/A} \to P^n_{S^{-1}B/A}
\]

deduced from the canonical homomorphisms $P^n_{B/A} \to P^n_{S^{-1}B/A}$ (16.4.4), form a projective system and are bijective.

**Proof.** It suffices to remark that $S^{-1}((B \otimes_A B)/\mathfrak{I}^{n+1}) = S^{-1}(B \otimes_A B)/(S^{-1}\mathfrak{I})^{n+1}$ by flatness, and that $S^{-1}(B \otimes_A B) = (S^{-1}B) \otimes_A (S^{-1}B)$ (I, 1.3.4).

**Corollary (16.4.15).** — The notation being that of (16.4.14), let $R$ be a multiplicative subset of $A$ such that $\rho(R) \subset S$. Then we have canonical isomorphisms

\[
S^{-1}P^n_{B/A} \simeq P^n_{S^{-1}B/R^{-1}A}
\]

forming a projective system.

**Proof.** It evidently suffices to define canonical isomorphisms

\[
P^n_{S^{-1}B/A} \simeq P^n_{S^{-1}B/R^{-1}A}
\]

that is to say that we reduce to the case there $\rho(R)$ is consists of invertible elements of $B$. But then the isomorphism (16.4.15.2) is simply induced by the canonical isomorphism $B \otimes_A B \to B \otimes_{R^{-1}A} B$ by passing to quotients (I, 1.5.3).

**Corollary (16.4.16).** — Let $f : X \to S$ be a morphism of preschemes, $x$ a point of $X$, $s = f(x)$. Then we have canonical isomorphisms

\[
(\mathcal{O}^n_{X/S})_x \simeq P^n_{\mathcal{O}_X/\mathcal{O}_x}
\]

forming a projective system.

We deduce from these isomorphisms of the associated graded rings, and in particular a canonical isomorphism

\[
(\Omega^1_{X/S})_x \simeq \Omega^1_{\mathcal{O}_X/\mathcal{O}_x}.
\]

**Corollary (16.4.17).** — Let $k$ be a field, $K$ the field of rational functions $k(T_1, \ldots, T_r)$. Then, for every integer $n$, the homomorphism of $K[U_1, \ldots, U_r]$ (with indeterminates) into $P^n_{K/k}$ which sends $U_i$ to $dT_i - T_i$ is surjective and defines an isomorphism from the quotient $K[U_1, \ldots, U_r]/m^{n+1}$ (where $m$ is the ideal generated by the $U_i$) to $P^n_{K/k}$.

**Proof.** This follows from (16.4.8), (16.4.10) and (16.4.14), where we take $A = k$, $B = k[T_1, \ldots, T_r]$ and $S = B - \{0\}$.

We thus recover the fact that the $dT_i$ form a basis of the $K$-vector space $\Omega^1_{K/k}(0, 20.5.10)$.

**16.18.** Let $f : X \to Y$, $g : Y \to Z$ be two morphisms of preschemes, and consider the canonical homomorphism of augmented $\mathcal{O}_X$-algebras (16.4.3.3)

\[
g_{X/Y/Z} : \mathcal{P}^n_{X/Z} \to \mathcal{P}^n_{X/Y}
\]

\[
f_{X/Y/Z} : f^*(\mathcal{P}^n_{Y/Z}) \to \mathcal{P}^n_{X/Z}.
\]

Then $g_{X/Y/Z}$ is surjective, and its kernel is the sheaf of ideals generated by the image under $f_{X/Y/Z}$ of the augmentation ideal of $f^*(\mathcal{P}^n_{X/Z})$. 

**Proof.** This follows from (16.4.5) and (16.4.12). 

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Proof. First note that $g_{X/Y/Z}$ corresponds to the case in (16.4.3.3) where $X' = X$, $S' = Y$ and $S = Z$, $u = 1_X$, and $f_{X/Y/Z}$ to the case where we replace $X'$, $X, S, S'$ by $X, Y, Z, Z$ respectively and $u, f$ by $g, f$ respectively.

We have a commutative diagram (16.4.18.3)

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta} & X \times_Y X \\
\downarrow f & & \downarrow j \\
Y & \xrightarrow{\Delta_g} & Y \times_Z Y
\end{array}
\]

where $j = (1_X, 1_X)_Z$ is an immersion, $j \circ \Delta_f = \Delta_{g \circ f}$, and $p$ is the structure morphism. Since we can restrict ourselves to the case where $X, Y$ and $Z$ are affine, we can suppose that the immersions $\Delta_f, \Delta_g$ and $j$ are closed, so that $\mathcal{O}_X$ and $\mathcal{O}_{X \times_Y X}$ are identified respectively with $\mathcal{O}_{X \times_Z X}/\mathcal{I}$ and $\mathcal{O}_{X \times Z X}/\mathcal{J}$, where $\mathcal{L} \supset \mathcal{I}$ are two quasi-coherent ideals corresponding respectively to the immersions $\Delta_{g \circ f}$ and $j$. The $\mathcal{O}$-algebra $\mathcal{P}_{X/Z}$ is identified with $\mathcal{O}_{X \times Z X}/(\mathcal{I} \mathcal{L})^{n+1}$, and $\mathcal{P}_{X/Y}$ is identified with $\mathcal{O}_{X \times Z X}/(\mathcal{J} \mathcal{L})^{n+1}$, which is to say with $\mathcal{O}_{X \times Z X}/(\mathcal{I} \mathcal{L})^{n+1}$, and therefore with the quotient of $\mathcal{P}_{X/Z}$ by $(\mathcal{I} \mathcal{L})^{n+1}$. But we know (loc. cit) that if $p$ and $j$ make $X \times_Y X$ the product of the $(Y \times Z)$-preschemes $Y$ and $X \times Z X$, so if $\mathcal{O}_Y$ is identified to $\mathcal{O}_{X \times Z X}/\mathcal{J}$, where $\mathcal{J}$ is the ideal corresponding to $\Delta_g, \mathcal{L}$ is equal to $(f \times Z f)^*(\mathcal{J}) \mathcal{O}_{X \times Z X} (I, 4.4.5)$. Since $(\mathcal{I} \mathcal{L})^{n+1}$ is the ideal of $\mathcal{P}_{X/Z}$ generated by the image of $\mathcal{L}$, we deduce the proposition. \(\square\)

**Corollary (16.4.19).** — With the notation of (16.4.18), we have an exact sequence of quasi-coherent $\mathcal{O}_X$-modules

\[
(16.4.19.1) \quad \xymatrix{ f^*(\Omega^1_{Y/Z}) \ar[r]^{f_{X/Y/Z}^*} & \Omega^1_{X/Z} \ar[r]^{g_{X/Y/Z}} & \Omega^1_{X/Y} \ar[r] & 0. }
\]

When $X, Y, Z$ are affine, we recover the exact sequence (0, 5.7.1).

**Proposition (16.4.20).** — Let $f : Y \to Z$ be a morphism, $j : X \to Y$ a closed immersion, $\mathcal{K}$ the quasi-coherent sheaf of ideals of $\mathcal{O}_Y$ corresponding to $j$. It follows that $\mathcal{P}^n_{Y/Z} = \mathcal{O}_X = \mathcal{O}_Y/\mathcal{K}$, the canonical homomorphism $j^*(\mathcal{P}^n_{Y/Z}) \to \mathcal{P}^n_{X/Z}$ is surjective, and its kernel is the ideal of $\mathcal{O}^n_{X/Z}$ generated by $j^*(\mathcal{O}_Y/\mathcal{K})$.

**Proof.** We know (I, 5.3.8) that the diagonal $\Delta : X \to X \times_Y X$ is an isomorphism, from which the first assertion follows. If $\omega_1$ and $\omega_2$ are the two canonical homomorphisms of algebras $\mathcal{O}_Y \to \mathcal{P}^n_{Y/Z}$ corresponding respectively to the two canonical projections $p_1, p_2$ of $X \times Z Y \to Y$, recall that by definition ((16.3.5) and (16.3.6)) $\omega_1$ is the structure homomorphism of the $\mathcal{O}_Y$-algebra $\mathcal{P}^n_{Y/Z}$ and $\omega_2 = d^n_{Y/Z}$. The $\mathcal{O}_X$-algebra $j^*(\mathcal{P}^n_{Y/Z})$ is therefore identified with $\mathcal{P}^n_{Y/Z}/\omega_1(\mathcal{K}) \mathcal{P}^n_{Y/Z}$ and its quotient by the ideal generated by $j^*(\mathcal{O}_Y/\mathcal{K})/\mathcal{P}^n_{Y/Z}$.

Now note that we have a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{j} & X \\
\downarrow \Delta & & \downarrow \Delta_{f_1} \\
Y \times_Z Y & \xrightarrow{j \times_Z j} & X \times_Z X
\end{array}
\]

identifying $X$ with the product of the $(Y \times Z)$-preschemes $Y$ and $X \times_Z X$ (I, 5.3.7). Since $j \times_Z j$ is an immersion, we therefore deduce from this remark and from (16.2.2) that if $\Delta^n_{Y/Z}$ and $\Delta^n_{X/Z}$ denote the infinitesimal neighborhoods of order $n$ of $Y$ and $X$ by the canonical immersions $\Delta_f$ and $\Delta_{f_1}$ respectively, then we have a diagram

\[
\begin{array}{ccc}
\Delta^n_{Y/Z} & \xleftarrow{\Delta^n_{X/Z}} & \Delta^n_{X/Z} \\
\downarrow j \times_Z j & & \downarrow j \times_Z j \\
Y \times_Z Y & \xrightarrow{j \times_Z j} & X \times_Z X
\end{array}
\]
making $\Delta^n_{X/Z}$ the product of the $(Y \times Z - Y)$-preschemes $\Delta^n_{Y/Z}$ and $X \times_Z X$. We can also say that $\mathcal{P}^n_{X/Z}$ is identified with the sheaf of rings $\mathcal{P}^n_{Y/Z} \otimes_{\mathcal{O}_{Y \times_Z Y}} \mathcal{O}_{X \times_Z X}$. But we see immediately that (for example, by restricting to the affine case) that $\mathcal{O}_{X \times_Z X} = \mathcal{O}_{Y \times_Z Y}/(p_1^*(\mathcal{X}) + p_2^*(\mathcal{X}))\mathcal{O}_{Y \times_Z Y}$. Therefore $\mathcal{P}^n_{X/Z}$ is identified with the quotient of $\mathcal{P}^n_{Y/Z}$ by the ideal generated by the image in $\mathcal{P}^n_{Y/Z}$ of $p_1^*(\mathcal{X}) + p_2^*(\mathcal{X})$. But by definition this ideal is generated by $\omega_1(\mathcal{X}) + \omega_2(\mathcal{X})$. □

**Corollary (16.4.21).** — Let $f : Y \to Z$ be a morphism, $j : X \to Y$ an immersion. We have an exact sequence of quasi-coherent $\mathcal{O}_X$-modules

\[ \mathcal{N}_{X/Y} \to j^*(\Omega^1_{Y/Z}) \to \Omega^1_{X/Z} \to 0. \]

When $X, Y, Z$ are affine, we recover the exact sequence (0, 20.5.12.1).

**Corollary (16.4.22).** — If $f : X \to S$ is a morphism locally of finite presentation, $\mathcal{P}^n_{X/S}$ and $\Omega^1_{X/S}$ are quasi-coherent $\mathcal{O}_X$-modules of finite presentation.

**Proof.** We immediately reduce to the case where $S = \text{Spec}(A)$ is affine, $X = \text{Spec}(B)$, where $B = A[T_1, \ldots, T_r]/\mathfrak{a}$, $\mathfrak{a}$ being an ideal of finite type of $C = A[T_1, \ldots, T_r]$. Applying (16.4.20) where $Z = S$, $Y = \text{Spec}(C)$ and $\mathcal{X} = \mathfrak{a}$. Then $j^*(\mathcal{P}^n_{Y/Z})$ is a free $\mathcal{O}_X$-module of finite rank (16.4.10) and the hypothesis on $\mathfrak{a}$ implies that $j^*(\mathcal{O}_X, \mathcal{P}^n_{Y/Z}(\mathcal{X}))$ generates a quasi-coherent $\mathcal{O}_X$-module of finite type; hence the conclusion. □

**Proposition (16.4.23).** — Let $X, Y$ be two $S$-preschemes, $Z = X \times_S Y$ their product, $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ the canonical projections. Then the canonical homomorphism

\[ p^*_{Z/X/S} \oplus q^*_{Z/Y/S} : p^*(\Omega^1_{X/S}) \oplus q^*(\Omega^1_{Y/S}) \to \Omega^1_{X \times_S Y/S} \]

is bijective.

**Proof.** The commutative diagram

\[ \begin{array}{ccc} Y & \xleftarrow{q} & X \times_S Y \xrightarrow{id} X \times_S Y \\ \downarrow s & & \downarrow h \\ S & \xleftarrow{\text{id}} & S \xrightarrow{f} X \end{array} \]

gives us a factorization of the canonical isomorphism $P^m(p)$ (16.4.5)

\[ p^*_{Z/X/S} \to \mathcal{P}^n_{Z/S} \to \mathcal{P}^n_{Z/Y} \]

and similarly, switching $X$ with $Y$, we have a factorization of the isomorphism $P^m(q)$

\[ q^*_{P^m_{Z/Y/S}} \to \mathcal{P}^n_{Z/S} \to \mathcal{P}^n_{Z/Y/S}. \]

This proves that the canonical homomorphism (16.4.18.1)

\[ p_{Z/X/S}^* : P^m(X) \to \mathcal{P}^n_{Z/S} \quad (\text{resp. } q_{Z/X/S}^* : P^m(X) \to \mathcal{P}^n_{Z/S}) \]

is injective, and that the kernel of the canonical surjective homomorphism (16.4.18.2)

\[ \mathcal{P}^n_{Z/S} \to \mathcal{P}^n_{Z/Y} \quad (\text{resp. } \mathcal{P}^n_{Z/S} \to \mathcal{P}^n_{Z/X}) \]

is direct summand of the image $p_{Z/X/S}$ (resp. $q_{Z/X/S}$). On the other hand, this kernel is, by virtue of (16.4.18), generated by the image by $q_{Z/Y/S}$ (resp. $p_{Z/X/S}$) of the augmentation ideal of $q^*_{Z/Y/S}$ (resp. $p^*_{Z/X/S}$). We conclude the proposition by considering the case $n = 1$. □

We immediately generalize (16.4.23) to the case of a product of any finite number of $S$-preschemes.

**Remarks (16.4.24).** —

(i) We will see (17.2.3) that when the morphism $f : X \to Y$ in (16.4.18) is smooth, the homomorphism $f_{Y/Z}$ in (16.4.19.1) is locally left invertible and in particular injective. Similarly, when the morphism $f \circ j : X \to Z$ of (16.4.20) is smooth, the homomorphism on the left in (16.4.21.1) is locally left invertible and a fortiori injective (17.2.5). In Chapter V, we will also give a variant, in the case of modules over a prescheme, of the “imperfection modules” studied in (0, 20.6), and the exact sequences where they occur.
(ii) Let $X$ be a topological space, $\mathcal{A}$ a sheaf of rings over $X$ and $\mathcal{B}$ a $\mathcal{A}$-algebra over $X$. Then it is clear that

$$U \mapsto p^n_{U, \mathcal{B}}(U, \mathcal{A})$$

is a presheaf of augmented $\Gamma(U, \mathcal{B})$-algebras, and therefore the associated sheaf $p^n_{\mathcal{B}}|_{\mathcal{A}^0}$ is an augmented $\mathcal{B}$-algebra. In the particular case where $X$ is a prescheme, $f = (\psi, \theta) : X \to S$ a morphism of preschemes, it follows easily from (16.4.16) and from the exactness of the functor $\lim$ that $p^n_{\mathcal{B}/S}$ is canonically isomorphic to $p^n_{\mathcal{B}/f^*S}$. It follows that the formalism developed in the present paragraph could be considered as a particular case of a differential formalism for ringed spaces endowed with a sheaf of algebras over the structure sheaf. However, we did not start with this point of view, which is less intuitive and less convenient for applications. It also seems that, for various kinds of “varieties”, the “global” constructions of the $\mathcal{P}^n$ analogous to those we have used here are also better suited for applications.

16.5. Relative tangent sheaves and bundles; derivations.

(16.5.1). Let $f = (\psi, \theta) : X \to S$ be a morphism of ringed spaces. For every $\mathcal{O}_X$-module $\mathcal{F}$, we say $\mathcal{S}$-derivation (or $(X/S)$-derivation, or $f$-derivation) of $\mathcal{O}_X$ to $\mathcal{F}$ for every homomorphism of sheaves of additive groups $D : \mathcal{O}_X \to \mathcal{F}$ satisfying the following conditions:

(a) for every open $V$ of $X$, and all pair of sections $(t_1, t_2)$ of $\mathcal{O}_X$ over $V$, we have

$$D(t_1 t_2) = t_1 D(t_2) + D(t_1) t_2;$$

(b) for every open $V$ of $X$, every section $t$ of $\mathcal{O}_X$ over $V$, and every section $s$ of $\mathcal{O}_S$ over an open $U$ of $S$ such that $V \subset f^{-1}(U)$, we have

$$D((s|V)t) = (s|V)D(t).$$

It is clear that this amounts to saying that, for all $x \in X$, the homomorphism of additive groups $D_x : \mathcal{O}_X \to \mathcal{F}_x$ is an $\mathcal{O}_x$-derivation.

Another interpretation consists of considering the $\mathcal{O}_X$-algebra $\mathcal{D}_{\mathcal{O}_X}(\mathcal{F})$ as equal to $\mathcal{O}_X \oplus \mathcal{F}$, the algebra structure being defined by the condition that for every open $V$ of $X$, the product of two sections of $\mathcal{O}_X$ (resp. of a section of $\mathcal{O}_X$ and a section of $\mathcal{F}$) over $V$ is defined by the ring structure of $\Gamma(V, \mathcal{O}_X)$ (resp. the $\Gamma(V, \mathcal{O}_X)$-module structure on $\Gamma(V, \mathcal{F})$), and the product of two sections of $\mathcal{F}$ over $V$ is chosen to be zero; then $\mathcal{F}$ is an ideal of $\mathcal{D}_{\mathcal{O}_X}(\mathcal{F})$, the kernel of the canonical augmentation $\mathcal{D}_{\mathcal{O}_X}(\mathcal{F}) \to \mathcal{O}_X$, and to say that $D$ is an $\mathcal{S}$-derivation of $\mathcal{O}_X$ to $\mathcal{F}$ means that $1 + D$ is an $\mathcal{S}$-homomorphism of algebras from $\mathcal{O}_X$ to $\mathcal{D}_{\mathcal{O}_X}(\mathcal{F})$, which, composed with the augmentation, gives $1_{\mathcal{O}_X}$.

The $\mathcal{S}$-derivations of $\mathcal{O}_X$ to $\mathcal{F}$ clearly form a $\Gamma(X, \mathcal{O}_X)$-module $\text{Der}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F})$.

When $\mathcal{F} = \mathcal{O}_X$, an $\mathcal{S}$-derivation of $\mathcal{O}_X$ to itself is simply called an $\mathcal{S}$-derivation of $\mathcal{O}_X$.

Proposition (16.5.2). — Let $A$ be a ring, $B$ an $A$-algebra, $L$ a $B$-module; let $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, $\mathcal{F} = L$. Then the map $D \mapsto \Gamma(D)$ which sends every $\mathcal{S}$-derivation $D$ of $\mathcal{O}_X$ to $\mathcal{F}$ to the map $\Gamma(D) : t \mapsto D(t)$ of $B$ to $L$, is an isomorphism of $B$-modules from $\text{Der}_S(\mathcal{O}_X, \mathcal{F})$ to $\text{Der}_A(B, L)$ (cf. (0, 20.1.2)).

Proof. This follows immediately from the given interpretation of $\mathcal{S}$-derivations in terms of homomorphisms of algebras, analogous to the interpretation given in (0, 20.1.6), and from the canonical correspondence between homomorphisms of $\mathcal{O}_X$-algebras and homomorphisms of $B$-algebras (I, 1.3.13) and (I, 1.3.8)).

Proposition (16.5.3). — Let $f = (\psi, \theta) : X \to S$ be a morphism of preschemes.

(i) The differential $d_{X/S} : \mathcal{O}_X \to \Omega^1_{X/S}$ (16.3.6) is an $\mathcal{S}$-derivation.

(ii) For every $\mathcal{O}_X$-module $\mathcal{F}$, the map $u \mapsto u \circ d_{X/S}$ is an isomorphism of $\Gamma(X, \mathcal{O}_X)$-modules

$$\text{Hom}_{\mathcal{O}_X}^{\Gamma}(\Omega^1_{X/S}, \mathcal{F}) \simeq \text{Der}_S(\mathcal{O}_X, \mathcal{F}).$$

Proof. The assertion (i) has already been written (16.3.6). On the other hand, it is immediate (in light of (0, 20.8)) that $u \mapsto u \circ d_{X/S}$ is injective, considering the restrictions to a fibre $\mathcal{O}_x$ of the two sides and using (16.4.16.2). To see that the homomorphism (16.5.3.1) is surjective, consider an $\mathcal{S}$-derivation $D : \mathcal{O}_X \to \mathcal{F}$; for every affine open $V = \text{Spec}(B)$ of $X$, such that $f(V)$ is contained in
an affine open \( U = \text{Spec}(A) \) of \( S \), \( D_U : B \to \Gamma(V, \mathcal{F}) \) is an \( A \)-derivation, and therefore there exists a unique \( B \)-homomorphism \( u_V : \Omega^1_{B/A} \to \Gamma(V, \mathcal{F}) \) such that \( D_V = u_V \circ d_{B/A} \) \((0, 20.4.8)\); in addition, the uniqueness of \( u_V \) shows immediately that for an affine open \( W \subset V \) we have \( u_W = u_V|_W \), and therefore the \( u_V \) define a homomorphism of \( \mathcal{O}_X \)-modules \( u : \mathcal{O}_X \to \mathcal{F} \) answering the question. \( \square \)

(16.5.4). With the notation of (16.5.1), for every open \( U \) of \( X \), \( \text{Der}_S(\mathcal{O}_U, \mathcal{F}|U) \) is a \( \Gamma(U, \mathcal{O}_X) \)-module and it is clear that the map \( U \to \text{Der}_S(\mathcal{O}_U, \mathcal{F}|U) \) is a presheaf; in fact, it is even a sheaf (and therefore \( \mathcal{O}_X \)-module), in light of the pointwise characterization of \( S \)-derivations, seen in (16.5.1). This \( \mathcal{O}_X \)-module is denoted by \( \text{Der}_S(\mathcal{O}_X, \mathcal{F}) \) and is called the sheaf of \( S \)-derivations of \( \mathcal{O}_X \) in \( \mathcal{F} \), and what we have seen is further expressed in the following corollary:

**Corollary (16.5.5).** — For every \( \mathcal{O}_X \)-module \( \mathcal{F} \), the homomorphism of \( \mathcal{O}_X \)-modules induced by \( u \mapsto u \circ d_{X/S} \)

\[(16.5.5.1) \quad \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{F}) \to \text{Der}_S(\mathcal{O}_X, \mathcal{F})
\]

is bijective.

**Corollary (16.5.6).** —

(i) If the morphism \( f : X \to S \) is of finite presentation and if \( \mathcal{F} \) is a quasi-coherent \( \mathcal{O}_X \)-module, then \( \text{Der}_S(\mathcal{O}_X, \mathcal{F}) \) is a quasi-coherent \( \mathcal{O}_X \)-module.

(ii) If in addition \( S \) is locally Noetherian and if \( \mathcal{F} \) is coherent, then \( \text{Der}_S(\mathcal{O}_X, \mathcal{F}) \) is a coherent \( \mathcal{O}_X \)-module.

**Proof.** The assertion (i) follows from the isomorphism (16.5.5.1), from (16.4.22), and (I, 3.12); the assertion (ii) follows from (0, 5.3.5). \( \square \)

(16.5.7). We set

\[(16.5.7.1) \quad \mathcal{O}_{X/S} = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X) = \text{Der}_S(\mathcal{O}_X, \mathcal{O}_X),
\]

and say that it is the sheaf of \( S \)-derivations of \( \mathcal{O}_X \) or even the tangent sheaf of \( X \) relative to \( S \): it is therefore the dual of the \( \mathcal{O}_X \)-module \( \Omega^1_{X/S} \). If \( f \) is locally of finite presentation, \( \mathcal{O}_{X/S} \) is a quasi-coherent \( \mathcal{O}_X \)-module; if in addition \( S \) is locally Noetherian, then \( \mathcal{O}_{X/S} \) is coherent \((16.5.6)\).

(16.5.8). Suppose in particular that \( \Omega^1_{X/S} \) is a locally free \( \mathcal{O}_X \)-module (of finite rank) (which will be the case then \( f \) is smooth \((17.2.3)\)); then \( \mathcal{O}_{X/S} \) is locally free \( \mathcal{O}_X \)-module of the same rank as \( \Omega^1_{X/S} \) at each point. More specifically, suppose that \( \Omega^1_{X/S} \) is of rank \( n \) at a point \( x \); then there are \( n \) sections \( s_i \) \((1 \leq i \leq n)\) of \( \mathcal{O}_X \) over an affine neighborhood \( U \) of \( x \) such that the canonical images of the \( ds_i \) in \( \Omega^1_{X/S} \otimes_{\mathcal{O}_X} k(x) \) form a basis if this \( k(x) \)-vector space; by virtue of Nakayama’s lemma, the germs \((ds_i)_x\) of the \( ds_i \) at the point \( x \) form a basis of the \( \mathcal{O}_x \)-module \((\Omega^1_{X/S})_x\), and therefore, by restricting \( U \), we can suppose that the \( ds_i \) form a basis of the \( \Gamma(U, \mathcal{O}_X) \)-module \( \Gamma(U, \Omega^1_{X/S}) \). So the \( \Gamma(U, \mathcal{O}_X) \)-module \( \Gamma(U, \mathcal{O}_{X/S}) \) is dual to the above; we denote by \((D_i)_{1 \leq i \leq n}\) or \((\frac{\partial}{\partial s_i})_{1 \leq i \leq n}\) the dual basis of \((ds_i)_{1 \leq i \leq n}\), so that, by (16.5.3), we have

\[(16.5.8.1) \quad D_is_j = \langle D_i, ds_j \rangle = \left\langle \frac{\partial}{\partial s_i}, ds_j \right\rangle = \delta_{ij} \quad \text{(Kronecker’s symbol)}.
\]

Every \( \Gamma(S, \mathcal{O}_S) \)-derivation of the \( \Gamma(S, \mathcal{O}_S) \)-algebra \( \Gamma(U, \mathcal{O}_X) \) is therefore written in an unique way as

\[(16.5.8.2) D = \sum_{i=1}^{n} a_i D_i = \sum_{i=1}^{n} a_i \frac{\partial}{\partial s_i},
\]

where the \( a_i \) \((1 \leq i \leq n)\) are sections of \( \mathcal{O}_X \) over \( U \). For every section \( g \in \Gamma(U, \mathcal{O}_X) \), if we put \( dg = \sum_{i=1}^{n} c_i ds_i \), then we have \( c_i = \langle D_i, dg \rangle = D_ig \) by virtue of (16.5.8.1), in other words,

\[(16.5.8.2) \quad dg = \sum_{i=1}^{n} (D_ig) ds_i = \sum_{i=1}^{n} \frac{\partial g}{\partial s_i} ds_i.
\]

(16.5.9). Let \( D_1, D_2 \) be two \( S \)-derivations of \( \mathcal{O}_X \). For every open \( U \) of \( X \), if \( D^U_1, D^U_2 \) are the corresponding derivations of the ring \( \Gamma(U, \mathcal{O}_X) \), the bracket

\[ [D^U_1, D^U_2] = D^U_1 \circ D^U_2 - D^U_2 \circ D^U_1
\]

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is also a derivation in this ring, and therefore the $\psi^* (\mathcal{O}_S)$-endomorphism of $\mathcal{O}_X$

(16.5.9.1) \[ [D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1 \]
is also an $S$-derivation; as we immediately check that this bracket satisfies the Jacobi identity, we have thus defined on $\text{Der}_S(\mathcal{O}_X, \mathcal{O}_X)$ a $\Gamma(S, \mathcal{O}_S)$-Lie algebra structure. Since the definition of this structure commutes with the restriction to an open of $X$, we thus see that $\mathfrak{g}_{X/S}$ is canonically equipped with a $\psi^* (\mathcal{O}_S)$-Lie algebra structure. Note that the mapping $(D_1, D_2) \mapsto [D_1, D_2]$ is not $\Gamma(X, \mathcal{O}_X)$-bilinear.

(16.5.10). For every base change $g : S' \to S$, if we set $X' = X \times_S S'$, then we see (16.4.5) that we have a canonical isomorphism

(16.5.10.1) \[ \Omega^1_{X/S} \otimes_S S' \cong \Omega^1_{X'/S'} \]
from which we deduce, by (16.5.10.1), a canonical homomorphism (Bourbaki, Alg., chap. II, 3rd ed., IV-4 | 30 §5, n°3)

(16.5.10.2) \[ \mathfrak{g}_{X/S} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \to \mathfrak{g}_{X'/S'} \]
which is neither injective nor surjective in general. However:

**Proposition (16.5.11).** —

(i) If $g : S' \to S$ is a flat morphism and if $f$ is locally of finite type (resp. locally of finite presentation), then the homomorphism (16.5.10.2) is injective (resp. bijective).

(ii) If $\Omega^1_{X/S}$ is a locally free $\mathcal{O}_X$-module of finite type, then the homomorphism (16.5.10.2) is bijective.

**Proof.** The assertion (ii) follows from Bourbaki, Alg., chap. II, 3rd ed., §5, n°3, prop. 7. The assertion (i) follows similarly from Bourbaki, Alg. Comm., chap. I, §2, n°10, prop. 11 and from the fact that if $f$ is locally of finite type (resp. locally of finite presentation), then $\Omega^1_{X/S}$ is an $\mathcal{O}_X$-module of finite type (resp. of finite presentation) (16.3.9) (16.4.22)). \( \square \)

(16.5.12). Since $\Omega^1_{X/S}$ is a quasi-coherent $\mathcal{O}_X$-module, we can consider the vector bundle over $X$ defined by $\Omega^1_{X/S}$ (II, 1.7.8)

(16.5.12.1) \[ T_{X/S} = \mathcal{V}(\Omega^1_{X/S}) \]
which is called the **tangent bundle of $X$ relative to $S$.** We have therefore a canonical bijection (II, 1.7.9)

\[ \Gamma(T_{X/S}/S) \cong \text{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X) = \Gamma(X, \mathfrak{g}_{X/S}) \]

by definition of $\mathfrak{g}_{X/S}$, and we can replace $X$ by an open set $U$ of $X$ in this isomorphism; so we can say that the tangent sheaf of $X$ relative to $S$ is isomorphic to the sheaf of germs of $S$-sections of the tangent bundle of $X$ relative to $S$. If $f : X \to Y$ is an $S$-morphism, we saw (16.4.19) that we have a canonical homomorphism $f^* : f^*(\Omega^1_{Y/S}) \to \Omega^1_{X/S}$, which, having in mind that

\[ \mathcal{V}(f^*(\Omega^1_{Y/S})) = \mathcal{V}(\Omega^1_{Y/S}) \times_Y X \quad (\text{II, 1.7.11}), \]
gives us an $X$-morphism $T_{X/S}(f) : T_{X/S} \to T_{Y/S} \times_Y X$. If $g : Y \to Z$ is a second $S$-morphism, we have $T_{X/S}(g \circ f) = (T_{Y/S}(g) \times 1_X) \circ T_{X/S}(f)$ (0, 20.5.4.1).

It follows from (16.5.10.1) and from (II, 1.7.11) that for every base change $g : S' \to S$ we have a canonical isomorphism

(16.5.12.2) \[ T_{X'/S'} \cong T_{X/S} \times_S S' = T_{X/S} \times X' \]

(16.5.13). For every point $x \in X$, we define the **tangent space of $X$ at the point $x$** (relative to $S$) to be the set of points in the fibre $T_{X/S} \times_X \text{Spec}(k(x))$ that are rational over $k(x)$, that is, the set

(16.5.13.1) \[ T_{X/S}(x) = \text{Hom}_{k(x)}(\Omega^1_{X/S} \otimes_{\mathcal{O}_S} k(x), k(x)), \]
which is the dual of the $k(x)$-vector space $\Omega^1_{\mathcal{O}_X/\mathcal{O}_S}/\mathfrak{m}_x \cdot \Omega^1_{\mathcal{O}_X/\mathcal{O}_S}$. When $\Omega^1_{X/S}$ is an $\mathcal{O}_X$-module of finite type, then $T_{X/S}(x)$ is a vector space of finite rank over $k(x)$, and for every base change $g : S \to S'$, and every point $x' \in X' = X \times_S S'$ over $x$, we have a canonical isomorphism

(16.5.13.2) \[ T_{X'/S'}(x') \cong T_{X/S} \otimes_{k(x)} k(x'). \]
If \( x \) is rational over \( k(s) \), where \( s = f(x) \) (so that \( k(s) \rightarrow k(x) \) is an isomorphism), it follows from (16.4.13) that we have a canonical isomorphism

\[
T_{X/S}(x) = T_{X/k(s)}(x) = \text{Hom}_{k(s)}(m_x^\nu/m_x^\mu, k(x)),
\]

where \( m_x^\nu \) is the maximal ideal of \( \mathcal{O}_{X,x} = \mathcal{O}_{X,k}/m_X \mathcal{O}_{X,k} \). In the case where \( S \) is the spectrum of a field \( k \), we recover the definition of the Zariski tangent space of a point \( x \in X \) rational over \( k \), as the dual of \( m_x/m_x^2 \).

Let \( Y \) be a second \( S \)-prescheme and let \( g : Y \rightarrow X \) be an \( S \)-morphism; then we have a canonical homomorphism of \( \mathcal{O}_Y \)-modules (16.4.19)

\[
g_Y^*/\mathcal{O}_{X/S} : g^*(\Omega^1_{X/S}) \rightarrow \Omega^1_{Y/S}.
\]

Now note that if \( y \in Y \) and \( x = g(y) \), then we have

\[
g^*(\Omega^1_{X/S}) \otimes_{\mathcal{O}_Y} k(y) = (\Omega^1_{X/S} \otimes_{\mathcal{O}_X} k(x)) \otimes_{k(x)} k(y)
\]

and consequently, if \( \Omega^1_{X/S} \) is an \( \mathcal{O}_X \)-module of finite type, then we can identify

\[
\text{Hom}_{k(y)}(g^*(\Omega^1_{X/S}) \otimes_{\mathcal{O}_Y} k(y), k(y))
\]

with \( T_{X/S}(x) \otimes_{k(x)} k(y) \). We therefore deduce from the homomorphism (16.5.13.4) a homomorphism of \( k(y) \)-vector spaces

\[
T_y(g) : T_{Y/S}(y) \rightarrow T_{X/S}(x) \otimes_{k(x)} k(y)
\]

called the linear map tangent to \( g \) at the point \( y \). When \( y \) is rational over \( k(s) \), we can identify \( k(s), k(y), \) and \( k(x) \), and \( T_y(g) \) is then a homomorphism of \( k(s) \)-vector spaces \( T_{Y/S}(y) \rightarrow T_{X/S}(x) \); also note that in this case, \( g^*(\Omega^1_{X/S}) \otimes_{\mathcal{O}_Y} k(y) \) is identified with \( \Omega^1_{X/S} \otimes_{\mathcal{O}_X} k(x) \), and the above homomorphism is therefore defined without any finiteness conditions on \( \Omega^1_{X/S} \) and it is none other than the homomorphism \( T_{Y/S}(g) \) (16.5.12) restricted to the fibre of \( T_{Y/S} \) at the point \( y \).

(16.5.14). The interpretation of derivations of an \( A \)-algebra \( B \) to a \( B \)-module \( L \), given in (0, 20.1.1), translates to the language of preschemes in the following way.

Consider two morphisms of preschemes \( f : X \rightarrow S, g : Y \rightarrow S \), and a closed subscheme \( Y_0 \) of \( Y \) defined by a zero-square ideal \( \mathcal{J} \) of \( \mathcal{O}_Y \) (so that \( Y \) and \( Y_0 \) have the same underlying subspace). Suppose we are given an \( S \)-morphism \( u_0 : Y_0 \rightarrow X \), so that we have a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{u_0} & Y_0 \\
\downarrow{f} & & \downarrow{j} \\
S & \xrightarrow{g} & Y
\end{array}
\]

and we suggest looking for an \( S \)-morphism \( u : Y \rightarrow X \) such that \( u_0 = u \circ j \) (in other words, if it is possible to complete the diagram above by the dotted arrow \( u \), keeping it commutative).

For that, consider an affine open \( U = \text{Spec}(C) \) of \( Y \); its inverse image \( j^{-1}(U) \) is the affine open \( U_0 = \text{Spec}(C/\mathcal{J}) \), where \( \mathcal{J} = \Gamma(U, \mathcal{J}) \), a zero-square ideal in \( C \); suppose that \( U \) is small enough so that \( u_0(U_0) \) is contained in an affine open \( V = \text{Spec}(B) \) of \( X \) and that \( g(U) = f(u_0(U_0)) \) is contained in an affine open \( W = \text{Spec}(A) \) of \( S \), so that \( B \) and \( C \) are \( A \)-algebras and \( u_0|U_0 \) corresponds to an \( A \)-homomorphism \( \psi \) from \( B \) to \( C/\mathcal{J} \); Let \( P(U_0) \) be the set of restrictions \( u|U \) of the sought homomorphisms, which corresponds canonically to \( A \)-homomorphisms of algebras \( \phi : B \rightarrow C \) such that the composite \( B \xrightarrow{\psi} C \rightarrow C/\mathcal{J} \) is equal to \( \psi \). So we know (0, 20.1.1) that the set of such homomorphisms is either empty or of the form \( \phi_1 \circ \text{Der}_A(B, \mathcal{J}) \); when \( P(U_0) \) is not empty, the additive group \( \text{Der}_A(B, \mathcal{J}) \) acts by addition on \( P(U_0) \), which is therefore an affine space for the additive group \( \text{Der}_A(B, \mathcal{J}) \) (or even a principal homogeneous space (or torsor) under \( \text{Der}_A(B, \mathcal{J}) \)).

Now notice that, since \( \mathcal{J} \) is equipped with a \( B \)-module structure via \( \psi \), we have an isomorphism \( \nu : \text{Hom}_B(\Omega^1_{B/A}, \mathcal{J}) \) onto \( \text{Der}_A(B, \mathcal{J}) \) (0, 20.4.8). Besides, as \( \mathcal{J} \) is square-zero, therefore a \( C/\mathcal{J} \)-module, every \( B \)-homomorphism \( \nu : \Omega^1_{B/A} \rightarrow \mathcal{J} \) can be considered as a \( C/\mathcal{J} \)-homomorphism \( \Omega^1_{B/A} \otimes_B (C/\mathcal{J}) \rightarrow \mathcal{J} \). As \( \mathcal{J} \) is square-zero, it can be considered as a quasi-coherent
In general, when we are given a sheaf of sets \( G \) over a topological space \( Z \), we make an affine scheme, \( \mathcal{O}_Z \) is a sheaf of sets \( \mathcal{O}_Z \)-module; let’s introduce the \( \mathcal{O}_Z \)-module \( \mathcal{O}_{Y_0} \).

(16.5.14.2) \[ \mathcal{O}_{Y_0} = \mathcal{O}_m(\mathcal{O}_{X/S}^1, \mathcal{O}_X) \] it follows from the fact that \( \Omega^1_{B/A} = \Gamma(V, \Omega_X^1) \) (16.3.7) that we can write \( \text{Der}_A(B, \Sigma) = \Gamma(U_0, \mathcal{O}_Z) \).

As \( P(U_0) \) is defined as a set of \( S \)-morphisms \( U \to X \), it is clear that \( U_0 \to P(U_0) \) is a sheaf of sets \( \mathcal{P} \) on \( Y_0 \). We can use this fact to prove that the map \( h : \Gamma(U_0, \mathcal{O}_Z) \times P(U_0) \to P(U_0) \) defining the torsor structure on \( P(U_0) \) is independent of choice of \( V \) and \( W \) and also that, if \( U' \subset U \) is a second affine open of \( Y, U_0 \) its inverse image in \( Y_0 \), then the diagram

\[
\begin{array}{ccc}
\Gamma(U_0, \mathcal{O}_Z) \times P(U_0) & \xrightarrow{h} & P(U_0) \\
\downarrow & & \downarrow \\
\Gamma(U_0', \mathcal{O}_Z) \times P(U_0') & \xrightarrow{h'} & P(U_0')
\end{array}
\]

is commutative (the vertical arrows being the restrictions). In light of the above remark, we reduce to proving the commutativity of the above diagram when \( h \) is defined as such from affine opens \( V, W \) and \( h' \) from affine opens \( V' \subset V \) and \( W' \subset W \). But because of the preceding description of \( h \), this follows from the commutativity of the diagram (0, 205.4.2).

The mapping \( \Gamma(U_0, \mathcal{O}_Z) \times P(U_0) \to P(U_0) \) therefore define a homomorphism of sheaf of sets \( m : \mathcal{O}_Z \times \mathcal{P} \times \mathcal{P} \) such that, for all open sets \( U_0 \) for which \( \Gamma(U_0, \mathcal{P}) \neq \emptyset \), \( m_{U_0} : \Gamma(U_0, \mathcal{O}_Z) \times \Gamma(U_0, \mathcal{P}) \to \Gamma(U_0, \mathcal{P}) \) is an external law defining in \( \Gamma(U_0, \mathcal{P}) \) a torsor structure for the group \( \Gamma(U_0, \mathcal{O}_Z) \).

(16.5.15). In general, when we are given a sheaf of sets \( \mathcal{P} \) over a topological space \( Z \), a sheaf of groups \( \mathcal{O}_Z \) (not necessarily commutative), and a homomorphism of sheaves of sets \( m : \mathcal{O}_Z \times \mathcal{P} \to \mathcal{P} \) such that, for every open \( U \subset Z \) such that \( \Gamma(U, \mathcal{P}) \neq \emptyset \), \( m_U : \Gamma(U, \mathcal{O}_Z) \times \Gamma(U, \mathcal{P}) \to \Gamma(U, \mathcal{P}) \) makes \( \Gamma(U, \mathcal{P}) \) a torsor under the group \( \Gamma(U, \mathcal{O}_Z) \), then we say that \( \mathcal{P} \) is a pseudo-torsor (or formally principal homogeneous sheaf) under the sheaf of groups \( \mathcal{O}_Z \). We say that \( \mathcal{P} \) is a torsor (or principal homogeneous sheaf) under \( \mathcal{O}_Z \) if in addition \( \Gamma(U, \mathcal{P}) \neq \emptyset \) for every open \( U \neq \emptyset \) in a suitable basis for the topology of \( Z \).

For the general theory of torsors, we refer to [eAG64]; we will limit ourselves to recalling the canonical correspondence between isomorphism classes of torsors (for a given \( \mathcal{O}_Z \)) and elements from the cohomology set \( H^1(Z, \mathcal{O}_Z) \). Consider a torsor \( \mathcal{P} \) under \( \mathcal{O}_Z \) and an open cover \( (U_\lambda) \) of \( Z \) such that \( \Gamma(U_\lambda, \mathcal{P}) \neq \emptyset \) for every \( \lambda \); denote by \( p_\lambda \) an element of \( \Gamma(U_\lambda, \mathcal{O}_Z) \). For every pair of indices \( \lambda, \mu \) such that \( U_\lambda \cap U_\mu \neq \emptyset \), there exists a unique element \( \gamma_{\lambda\mu} \) of \( \Gamma(U_\lambda \cap U_\mu, \mathcal{O}_Z) \), \( \gamma_{\lambda\mu} : \mathcal{P}_{\lambda} \cap \mathcal{P}_{\mu} \to \mathcal{P}_{\lambda} \cap \mathcal{P}_{\mu} \), such that \( \gamma_{\lambda\mu} \cdot p_\lambda|_{U_\lambda \cap U_\mu} = p_\mu|_{U_\lambda \cap U_\mu} \); in addition, if \( \lambda, \mu, \nu \) are three indices such that \( U_\lambda \cap U_\mu \cap U_\nu \neq \emptyset \), then the restrictions \( \gamma_{\lambda\mu}, \gamma_{\mu\nu}, \gamma_{\lambda\nu} \) of \( \gamma_{\lambda\mu}, \gamma_{\mu\nu}, \gamma_{\lambda\nu} \) to \( U_\lambda \cap U_\mu \cap U_\nu \) satisfy the condition \( \gamma_{\lambda\nu} = \gamma_{\lambda\mu} \gamma_{\mu\nu} \); in other words, \( (\lambda, \mu) \mapsto \gamma_{\lambda\mu} \) is a 1-cocycle of the cover \( (U_\lambda) \) with values in \( \mathcal{O}_Z \). If, for every \( \lambda, p_\lambda \) is a second element of \( \Gamma(U_\lambda, \mathcal{O}_Z) \), then there exists a unique element \( \beta_\lambda \in \Gamma(U_\lambda, \mathcal{O}_Z) \) such that \( p_\lambda' = \beta_\lambda \cdot p_\lambda \), and the 1-cocycle \( (\gamma_{\lambda\mu}') \) corresponding to the family \( (p_\lambda') \) is given by \( \gamma_{\lambda\mu}' = \beta_\lambda \gamma_{\lambda\mu} \beta_\mu^{-1} \); that is, it is cohomologous to \( \gamma_{\lambda\mu} \). Conversely, the data of a 1-cocycle \( (\gamma_{\lambda\mu}) \) defines, for every pair \( (\lambda, \mu) \), an automorphism \( \theta_{\lambda\mu} \) of the sheaf of sets \( \mathcal{O}_Z|_{U_\lambda \cap U_\mu} \), namely the right translation by \( \gamma_{\lambda\mu} \), and the fact that it is a cocycle shows that we can glue the sheaves of sets \( \mathcal{O}_Z|_{U_\lambda \cap U_\mu} \) via the automorphisms \( \theta_{\lambda\mu} \); we thus obtain a torsor under \( \mathcal{O}_Z \), denoted \( \mathcal{P} \), and if we take for \( p_\lambda \) the unit section over \( U_\lambda \), then the corresponding 1-cocycle is none other than the given 1-cocycle \( (\gamma_{\lambda\mu}) \); in addition, if we replace \( \gamma_{\lambda\mu} \) by a 1-cocycle \( \gamma_{\lambda\mu}' = \beta_\lambda \gamma_{\lambda\mu} \beta_\mu^{-1} \) cohomologous to it, then we check immediately that the torsor obtained is isomorphic to \( \mathcal{P} \).

In particular, if \( (\gamma_{\lambda\mu}) \) is a 1-coboundary, in other words of the form \( \gamma_{\lambda\mu} = \beta_\lambda \beta_\mu^{-1} \), then the torsor \( \mathcal{P} \) obtained is isomorphic to \( \mathcal{P} \) (considered as a torsor under itself by left translations); we say in this case that \( \mathcal{P} \) is trivial, and the converse is evident.

In particular, it follows from (III, 1.3.1) that we have:

**Proposition (16.5.16).** — Let \( Z \) be an affine scheme, \( \mathcal{O}_Z \) a quasi-coherent \( \mathcal{O}_Z \)-module; then every torsor over \( \mathcal{O}_Z \) is trivial.

\[\text{[Trans.] This is nowadays more commonly called a \( \mathcal{O}_Z \)-torsor rather than a torsor under \( \mathcal{O}_Z \).}\]
Returning to the problem considered in (16.5.13), we thus obtain:

**Proposition (16.5.17).** — Let \( X, Y \) be two \( S \)-preschemes, \( Y_0 \) a closed subprescheme of \( Y \) defined by a quasi-coherent ideal \( \mathcal{I} \) of \( \mathcal{O}_Y \) such that \( \mathcal{I}^2 = 0 \), \( j : Y_0 \to Y \) the canonical injection. Let \( u_0 : Y_0 \to X \) be an \( S \)-morphism, and \( \mathcal{P} \) the sheaf of sets on \( Y \) such that, for every open \( U \) of \( Y \), \( \Gamma(U, \mathcal{P}) \) is the set of \( S \)-morphisms \( u : U \to X \) such that \( u_0|U = u \circ (j|U_0) \), where \( U_0 = j^{-1}(U) \). Then there exists on \( \mathcal{P} \) the structure of a pseudo-torsor over the \( \mathcal{O}_{Y_0} \)-module \( \mathcal{P} = \mathcal{H}om_{\mathcal{O}_{Y_0}}(u_0^*(\Omega^1_X/S), \mathcal{I}) \).

In particular:

**Corollary (16.5.18).** — With the notation of (16.5.16), suppose that \( Y \) is affine and \( \Omega^1_{X/S} \) is of finite presentation; if there is an open cover \( (U_\alpha) \) of \( Y \), and, for every index \( \alpha \), an \( S \)-morphism \( v_\alpha : U_\alpha \to X \) such that, if \( U^0_\alpha = j^{-1}(U_\alpha) \), we have \( v_\alpha \circ (j|U^0_\alpha) = u_0|U^0_\alpha \), then there is an \( S \)-morphism \( u : Y \to X \) such that \( u \circ j = u_0 \).

**Proof.** Indeed, \( \mathcal{P} \) is a quasi-coherent \( \mathcal{O}_{Y_0} \)-module (I, 1.3.12); by (16.5.16) and the fact that \( Y_0 \) is then affine, the sheaf \( \mathcal{P} \), which is by hypothesis a torsor over \( \mathcal{P} \), and not only a pseudo-torsor, is trivial; but if \( w \) is an isomorphism from \( \mathcal{P} \) to \( \mathcal{P} \) (as it is a torsor over \( \mathcal{P} \)), the image under \( w \) of the zero section of \( \mathcal{P} \) is the \( S \)-morphism we want. \( \square \)

### 16.6. Sheaf of \( p \)-differentials and exterior differentials.

(16.6.1). Let \( f : X \to S \) be a morphism of preschemes. We define the sheaf of \( p \)-differentials of \( X \) relative to \( S \) \((p \text{ integer})\) to be the \( p \)'th exterior power \((0, 4.1.5)\) of the \( \mathcal{O}_X \)-module \( \Omega^1_{X/S} \), denoted by

\[
\Omega^p_{X/S} = \bigwedge^p (\Omega^1_{X/S}).
\]

So we have \( \Omega^0_{X/S} = \mathcal{O}_X \), and \( \Omega^p_{X/S} = 0 \) for \( p < 0 \); the \( \Omega^p_{X/S} \) are the homogeneous components of the exterior algebra of \( \Omega^1_{X/S} \)

\[
\Omega^*_X = \bigwedge (\Omega^1_{X/S}) = \bigoplus_{p \in \mathbb{Z}} \bigwedge^p (\Omega^1_{X/S}),
\]

which is therefore a graded quasi-coherent anti-commutative \( \mathcal{O}_X \)-algebra whose elements of degree 1 are square-zero. For every affine \( U \) of \( X \), we have \( \Gamma(U, \Omega^*_X) = \bigwedge (\Gamma(U, \Omega^1_{X/S})) \), where \( \Gamma(U, \Omega^1_{X/S}) \) is considered as a \( \Gamma(U, \mathcal{O}_X) \)-module.

When \( S = \text{Spec}(A) \) and \( X = \text{Spec}(B) \) are affines, \( B \) being then an \( A \)-algebra, we have \((0, 4.1.5)\)

\[
\Omega^p_{X/S} = (\Omega^p_{B/A})^\wedge, \text{ by putting } \Omega^p_{B/A} = \bigwedge^p \Omega^1_{B/A}.
\]

**Theorem (16.6.2).** — There is one and only one endomorphism \( d \) of the sheaf of additive groups \( \Omega^*_X \) with the following properties:

(i) \( d \circ d = 0 \).

(ii) For every open set \( U \) of \( X \) and every section \( f \in \Gamma(U, \mathcal{O}_X) \) we have \( df = df_X \).

(iii) For every open set \( U \) of \( X \), every pair of integers \( p, q \) and every pair of sections \( \omega'_p \in \Gamma(U, \Omega^p_{X/S}) \), \( \omega''_q \in \Gamma(U, \Omega^q_{X/S}) \), we have

\[
d(\omega'_p \wedge \omega''_q) = (d\omega'_p) \wedge \omega''_q + (-1)^p \omega'_p \wedge d\omega''_q.
\]

Also, \( d \) is an endomorphism of graded \( \psi^*(\mathcal{O}_X) \)-modules of degree +1.

**Proof.** Suppose that we have proved the existence of an endomorphism \( d \). For every open affine \( U \) of \( X \), every section of \( \Omega^p_{X/S} \) over \( U \) is (because of (ii)) a linear combination if a finite number of elements of the form \( g(df_1 \wedge df_2 \wedge \cdots \wedge df_p) \), where \( g \) and the \( f_i \) are sections of \( \mathcal{O}_X \) over \( U \). The conditions (i) and (iii) then show, by induction on \( p \), that we necessarily have

\[
d(g(df_1 \wedge df_2 \wedge \cdots \wedge df_p)) = dg \wedge df_1 \wedge df_2 \wedge \cdots \wedge df_p.
\]

This therefore proves the uniqueness of \( d \) and the last claim of the theorem. By virtue of this uniqueness property, to show the existence of \( d \), we can restrict ourselves to the case where \( S = \text{Spec}(A) \) and \( X = \text{Spec}(B) \) are affines. Now (Bourbaki, Alg., chap. III, 3rd ed., §10) to define an
A-derivation $D$ of degree +1 of an exterior algebra $\Lambda(M)$ (where $M$ is a $B$-module and an $A$-algebra), such derivation taking its values in a graded anti-commutative $A$-algebra $C = \bigoplus_{n=0}^{\infty} C_n$, whose elements of degree 1 are square-zero, it suffices to give arbitrarily an $A$-derivation $D_0$ of $B$ in $C_1$ and an $A$-homomorphism $D_1$ of $M$ in $C_2$; then it exists one and only one $A$-anti-derivation $D$ of $\Lambda(M)$ in $C$ coinciding with $D_0$ in $B$ and $D_1$ in $M$.

In the present case, $D_0$ is necessarily equal to $d_{B/A}$ by (ii); we reduce to seeing, having (16.6.2.2) in mind, that there is an $A$-homomorphism $u$ of $\Omega^1_{B/A}$ in $\Omega^2_{B/A}$ such that

$$u(g,d(f)) = dg \wedge df$$

for whichever $f, g$ in $A$; it suffices to show that there is an $A$-homomorphism $v : B \otimes_A \Omega^1_{B/A} \rightarrow \Omega^2_{B/A}$ such that

$$v(g,\omega) = dg \wedge \omega$$

for $g \in B$ and $\omega \in \Omega^1_{B/A}$. Finally, since $\Omega^1_{B/A} = \mathcal{I}/\mathcal{I}^2$ (where $\mathcal{I} = \mathcal{I}_{B/A}$ is the kernel of the canonical homomorphism $B \times_A B \rightarrow B$) and that $\Omega^1_{B/A}$ is generated by elements of the form $g.df$, it is enough to define an $A$-homomorphism $w : B \otimes_A (B \otimes_A B) \rightarrow \Omega^2_{B/A}$ such that

$$w(g' \otimes g \otimes f) = dg' \wedge (g.df)$$

and such that $w$ is zero on the image of $B \otimes A \mathcal{I}^2$. Or, since the second member of (16.6.2.5) is $A$-trilinear in $g', g$ and $f$, the existence of $w$ verifying (16.6.2.5) is immediate. Since, on the other hand, $\mathcal{I}$ is generated by elements of the form $1 \otimes x - x \otimes 1$ ($x \in B$), we reduce to checking that when $z = (1 \otimes x - x \otimes 1)(1 \otimes y - y \otimes 1)$ we have $w(g' \otimes z) = 0$. Or, since $z = 1 \otimes (xy) + (xy) \otimes 1 - x \otimes y - y \otimes x$, the formula (16.6.2.4) shows that it is enough to see that we have $d(xy) - x.dy - y.dx = 0$.

It remains to be shown that $d$ verifies the condition (i). Now, the square of an anti-derivation is a derivation (Bourbaki, loc. cit.), and since $\Omega^1_{B/A}$ is generated by $\Omega^1_{B/A}$ as a $B$-algebra, it is enough to verify that $d(dz) = 0$ for $z \in B$ and $x \in \Omega^1_{B/A}$; in the first case, this follows from the formula (16.6.2.3) when $g = 1$; for the second, we can restrict ourselves to the case where $z = g.df$ with $f, g$ in $B$, and then we have, because of (16.6.2.1) and (16.6.2.3),

$$d(d(g.df)) = d(dg \wedge df) = (d(dg)) \wedge (df) - (dg) \wedge (d(df)) = 0.$$

\[\square\]

**Definition (16.6.3).** — The anti-derivation $d$ defined in (16.6.2) (also denoted by $d_{X/S}$) is called the exterior differential on $X$ (relative to $S$).

**Proposition (16.6.4).** — For every base change $g : S' \rightarrow S$, if we put $X' = X \times_S S'$, the canonical morphism

$$\Omega^*_{X/S} \otimes_S S' \rightarrow \Omega^*_{X'/S'}$$

deduced from the isomorphism (16.5.10.1) is bijective. Also, if $s$ is a section of $\Omega^*_{X/S}$ over an open set $U$ of $X$, $s \otimes 1$ its inverse image, section of $\Omega^*_{X'/S'}$ over the inverse image $U'$ of $U$ in $X'$, we have $d_{X'/S'}(s \otimes 1) = d_{X/S}(s) \otimes 1$.

**Proof.** The first claim is immediate, the formation of the exterior algebra of a module commutes with extending the scalar ring. To prove the second, we can, because of (16.6.2.2), restrict ourselves to the case where $s \in \Gamma(U, \mathcal{O}_X)$, and in this case the claim has already been proven (16.4.3.7).

(16.6.5.) Suppose that $\Omega^1_{X/S}$ is an locally free $\mathcal{O}_X$-module of rank $n$ in a point $x$, so that we have $n$ sections $s_i \in \Gamma(U, \mathcal{O}_X)$ such that the $s_i$ form a basis for the $\Gamma(U, \mathcal{O}_X)$-module $\Gamma(U, \Omega^1_{X/S})$ (16.5.8). Then, for every integer $p \geq 1$, the $p$-differentials $ds_{i_1} \wedge ds_{i_2} \wedge \cdots \wedge ds_{i_p}$ (for $i_1 \leq i_2 \leq \cdots \leq i_p$ elements of $[1, n]$) form a basis of $\left(\begin{array}{c}n \\ p \end{array}\right)$ elements of $\Gamma(U, \Omega^p_{X/S})$ over $\Gamma(U, \mathcal{O}_X)$. Also the formula (16.6.2.2) shows that for every section $g \in \Gamma(U, \mathcal{O}_X)$, we have

$$d(g, ds_{i_1} \wedge ds_{i_2} \wedge \cdots \wedge ds_{i_p}) = \sum_k (-1)^k \frac{\partial g}{\partial s_k} ds_{i_1} \wedge \cdots \wedge [\alpha ds_{i_k} \wedge ds_{i_{k+1}} \wedge \cdots \wedge ds_{i_p}$$
where, in the second member, \( m \) varies in the set of the \( n - p \) indexes different from the \( i_0, i_1 \) being the biggest index \( < k \).

We note that the relation \( d (dg) = 0 \) for every section \( g \in \Gamma(U, \mathcal{O}_X) \) expresses itself in the form
\[
D_i (D_j g) = D_j (D_i g) \quad \text{for} \ i \neq j;
\]
in other words, the derivations \( D_i \) defined in (16.5.7) commute with each other.

16.7. The \( \mathcal{P}^n_{X/S} (\mathcal{F}) \).

(16.7.1). Let \( f : X \to S \) be a morphism of preschemes, \( \mathcal{F} \) an \( \mathcal{O}_X \)-module. We denote by \( X^{(n)} \) the \( n \)'th infinitesimal neighborhood of \( X \) via the diagonal morphism \( \Delta_f : X \to X \times_S X \), by \( h_n : X^{(n)} \to X \) the canonical morphism (16.1.2), and consider the two composite morphisms
\[
p_1^{(n)} : X^{(n)} \xrightarrow{h_n} X \times_S X \xrightarrow{p_1} X, \quad p_2^{(n)} : X^{(n)} \xrightarrow{h_n} X \times_S X \xrightarrow{p_2} X
\]
so that, by definition, \( p_1^{(n)} \) corresponds to the homomorphism of sheaves of rings \( \mathcal{O}_X \to \mathcal{P}^n_{X/S} \) which we have chosen to define the \( \mathcal{O}_X \)-algebra structure on \( \mathcal{P}^n_{X/S} \) (16.3.5), and \( p_2^{(n)} \) to the homomorphism of sheaves of rings \( d^n_{X/S} : \mathcal{O}_X \to \mathcal{P}^n_{X/S} \) (16.3.6). Since \( X^{(n)} \) and \( X \) have the same underlying subspace, we can write
\[
\mathcal{P}^n_{X/S} = (p_1^{(n)})_*((p_2^{(n)})^*(\mathcal{O}_X)).
\]
More generally, we define
\[
\mathcal{P}^n_{X/S} (\mathcal{F}) = (p_1^{(n)})_*((p_2^{(n)})^*(\mathcal{O}_X)).
\]
so that \( \mathcal{P}^n_{X/S} = \mathcal{P}^n_{X/S} (\mathcal{O}_X) \); by definition, \( \mathcal{P}^n_{X/S} (\mathcal{F}) \) is an \( \mathcal{O}_X \)-module.

(16.7.2). If we come back to the definition of the inverse image of modules on ringed spaces (0.4.3.1) and having in mind that \( X^{(n)} \) and \( X \) have the same underlying space, we see that we can write the definition (16.7.1.2) in the form
\[
\mathcal{P}^n_{X/S} (\mathcal{F}) = \mathcal{P}^n_{X/S} (\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F},
\]
but where you have to be careful that, in the interpretation of the symbol \( \otimes \), \( \mathcal{P}^n_{X/S} \) is endowed with the structure of \( \mathcal{O}_X \)-module defined by the homomorphism of sheaves of rings \( d^n_{X/S} : \mathcal{O}_X \to \mathcal{P}^n_{X/S} \). It follows immediately from such formula (or directly from (16.7.1.2)) that \( \mathcal{P}^n_{X/S} (\mathcal{F}) \) is canonically equipped with a \( \mathcal{P}^n_{X/S} \)-module structure.

**Proposition (16.7.3).** —

(i) The functor \( \mathcal{F} \mapsto \mathcal{P}^n_{X/S} (\mathcal{F}) \) from the category of \( \mathcal{O}_X \)-modules to the category of \( \mathcal{P}^n_{X/S} \)-modules is right exact, and commutes with arbitrary inductive limits; it is exact when \( \mathcal{P}^n_{X/S} \) is flat.

(ii) If \( \mathcal{F} \) is a quasi-coherent \( \mathcal{O}_X \)-module (resp. of finite type, resp. of finite presentation), then \( \mathcal{P}^n_{X/S} (\mathcal{F}) \) is quasi-coherent (resp. of finite type, resp. of finite presentation).

**Proof.** The claims from (i) follow from (16.7.2.1) and the consideration of the symmetry of \( \mathcal{P}^n_{X/S} \) (16.3.4). The claims from (ii) follow from the right exactness of the functor \( \mathcal{F} \mapsto \mathcal{P}^n_{X/S} (\mathcal{F}) \).

(16.7.4). The two structures of \( \mathcal{O}_X \)-module on \( \mathcal{P}^n_{X/S} \) define in \( \mathcal{P}^n_{X/S} (\mathcal{F}) \) two structures of \( \mathcal{O}_X \)-modules, which happen to be permutable, and therefore a \( \mathcal{O}_X \)-bimodule structure. It is convenient to denote the structure coming from the structure homomorphism \( \mathcal{O}_X \to \mathcal{P}^n_{X/S} \) (chosen in (16.3.5)) on the left and the one coming from the homomorphism \( d^n_{X/S} : \mathcal{O}_X \to \mathcal{P}^n_{X/S} \) on the right. On other words, for every open \( U \) of \( X \), and every triplet \( a \in \Gamma(U, \mathcal{O}_X) \), \( b \in \Gamma(U, \mathcal{P}^n_{X/S}) \), \( t \in \Gamma(U, \mathcal{F}) \), we have by definition
\[
a(b \otimes t) = (ab) \otimes t, \quad (b \otimes t)a = (b.d^m a) \otimes t = b \otimes (at) = (d^m a). (b \otimes t).
\]
The \( \mathcal{O}_X \)-module structure coming from the definition (16.7.1.2) is therefore, under these conventions, the left \( \mathcal{O}_X \)-module structure. If \( \mathcal{F} \) is a quasi-coherent \( \mathcal{O}_X \)-module, then the same is true for \( \mathcal{P}^n_{X/S} (\mathcal{F}) \) for any one of its \( \mathcal{O}_X \)-module structures. If also \( \mathcal{F} \) is of finite type (resp. of finite presentation) and...
We have therefore a projective system of $O$ which are homomorphisms of

Also, this shows that the homomorphisms (16.7.5.1) form a projective system of homomorphisms, $\text{Alg}$.

The canonical homomorphisms of sheaves of rings (16.7.7).

(16.7.7.1) $d^n_{X/S,\mathcal{F}} : \mathcal{F} \to \mathcal{P}^n_{X/S}(\mathcal{F})$ (also denoted $d^n_{X/S}$) such that, in the notations of (16.7.4), we have

and consequently, because of (16.7.4.1)

(16.7.5.3) $d^n_{X/S,\mathcal{F}}(at) = (1 \otimes t)a = (d^n_{X/S,\mathcal{F}}(t)).a$

(IV.16.7.5.4) $d^n_{X/S,\mathcal{F}}(at) = (d^n_{X/S,\mathcal{F}}(a)).(1 \otimes t) = (d^n_{X/S,\mathcal{F}}(a))(d^n_{X/S,\mathcal{F}}(t))$. Therefore it is $O_X$-linear for the structure of right $O_X$-module on $\mathcal{P}^n_{X/S}(\mathcal{F})$, and semilinear (relative to the homomorphism $\sigma$ (16.3.4)) for the left $O_X$-module structure.

**Proposition (16.7.6).** — The right $O_X$-module $\mathcal{P}^n_{X/S}(\mathcal{F})$ is generated by the image of $\mathcal{F}$ by the homomorphism $d^n_{X/S,\mathcal{F}}$.

**Proof.** This is an immediate consequence of (16.7.5.3) and of the particular case $\mathcal{F} = O_X$ (16.3.8). □

(16.7.7). The canonical homomorphisms of sheaves of rings

$$\varphi_{nm} : \mathcal{P}^n_{X/S} \to \mathcal{P}^m_{X/S}$$

for $n \leq m$ (16.1.2) define, because of (16.7.2.1), canonical homomorphisms

$$\mathcal{P}^n_{X/S}(\mathcal{F}) \to \mathcal{P}^m_{X/S}(\mathcal{F}) \quad (n \leq m)$$

which are homomorphisms of $O_X$-bimodules in light of (16.1.6) and (7.4.1); also we have commutative diagrams

We have therefore a projective system of $O_X$-bimodules ($\mathcal{P}^n_{X/S}(\mathcal{F})$), and we define

(16.7.7.1) $\mathcal{P}^\infty_{X/S}(\mathcal{F}) = \lim_{\longrightarrow} \mathcal{P}^n_{X/S}(\mathcal{F})$.

Also, this shows that the homomorphisms (16.7.5.1) form a projective system of homomorphisms, and therefore define a canonical homomorphism

(16.7.7.2) $d^\infty_{X/S,\mathcal{F}} : \mathcal{F} \to \mathcal{P}^\infty_{X/S}(\mathcal{F})$.

(16.7.8). Let $\mathcal{F}, \mathcal{G}$ be two $O_X$-modules; it follows immediately from the definition (16.7.2.1) that we have a canonical isomorphism of $\mathcal{P}^n_{X/S}$-modules

(16.7.8) $\mathcal{P}^n_{X/S}(\mathcal{F} \otimes_{O_X} \mathcal{G}) \simeq \mathcal{P}^n_{X/S}(\mathcal{F}) \otimes_{\mathcal{O}_{X/S}} \mathcal{P}^n_{X/S}(\mathcal{G})$

(Bourbaki, Alg., chap. II, 3rd ed., §5, n.1, prop. 3).

We conclude in particular (or we see directly from the definition (16.7.2.1)) that if $\mathcal{F}$ has an $O_X$-algebra structure (not necessarily associative), $\mathcal{P}^n_{X/S}(\mathcal{F})$ has a canonical $O_X$-algebra structure; the latter is associative (resp. commutative, res. unital, resp. a Lie algebra) if $\mathcal{F}$ is so. Also the canonical homomorphisms $\mathcal{P}^n_{X/S}(\mathcal{F}) \to \mathcal{P}^n_{X/S}(\mathcal{F})$ for $n \leq m$ (16.7.7) are then algebra di-homomorphisms; similarly, (16.7.5.1) is then an $O_X$-algebra homomorphisms when $\mathcal{P}^n_{X/S}(\mathcal{F})$ is equipped with the $O_X$-algebra structure from its structure of right $O_X$-module.
With the same notations, we equally have a canonical homomorphisms of $\mathcal{P}^n_{X/S}$-modules
\begin{equation}
\mathcal{P}^n_{X/S}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})) \longrightarrow \mathcal{H}om_{\mathcal{O}_{X/S}}(\mathcal{P}^n_{X/S}(\mathcal{F}),\mathcal{P}^n_{X/S}(\mathcal{G}))
\end{equation}
(Bourbaki, Alg., chap. II, 3rd ed., §5, n.3), which is bijective when $\mathcal{P}^n_{X/S}$ is locally free of finite type (loc. cit., prop. 7).

(16.7.9). Suppose we are in the situation described in (16.4.1); then from the canonical homomorphism $P^n(u)$ (16.4.3.3) we deduce immediately a canonical homomorphism of $\mathcal{O}_X$-bimodules
\begin{equation}
u^*(\mathcal{P}^n_{X/S}(\mathcal{F})) \longrightarrow \mathcal{P}^n_{X/S}(\mathcal{F})\end{equation}
We leave it to the reader to extend the properties seen in (16.4) in the case $\mathcal{F} = \mathcal{O}_X$.

**Remark (16.7.10).** — The definition of $\mathcal{P}^n_{X/S}(\mathcal{F})$ in the form (16.7.1.2) still makes sense when $\mathcal{F}$ is a sheaf of sets (the inverse image of a sheaf of sets by $p_{2}^{(n)}$ being defined in (0, 3.7.1)); a variant of this definition allows us to define the “jet schemes” (relatively to $S$) for any prescheme $X$.

16.8. Differential operators. 4

**Definition (16.8.1).** — Let $f = (\psi, \theta) : X \rightarrow S$ be a morphism of preschemes, $\mathcal{F}, \mathcal{G}$ two $\mathcal{O}_X$-modules, $n$ an integer $\geq 0$. We say that a morphism $D : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves of additive groups is a differential operator of order $\leq n$ (relative to $S$) if there is a homomorphism of $\mathcal{O}_X$-modules $u : \mathcal{P}^n_{X/S}(\mathcal{F}) \rightarrow \mathcal{G}$ (where $\mathcal{P}^n_{X/S}(\mathcal{F})$ is equipped with the structure of left $\mathcal{O}_X$-modules (16.7.4)) such that we have
\[ D = u \circ d^n_{X/S,\mathcal{G}}. \]

It is clear, because of the existence of canonical morphisms
\[ \mathcal{P}^m_{X/S}(\mathcal{F}) \longrightarrow \mathcal{P}^n_{X/S}(\mathcal{F}) \]
for $n \leq m$ (16.7.7), that a differential operator of order $\leq n$ is a differential operator of order $\leq m$ for $n < m$. If $D : \mathcal{F} \rightarrow \mathcal{G}$ is a differential operator of order $\leq n$, then, for every open set $U$ of $X$, $D|U : \mathcal{F}|U \rightarrow \mathcal{G}|U$ is a differential operator of order $\leq n$.

We say that a homomorphism $D : \mathcal{F} \rightarrow \mathcal{G}$ is a differential operator (relative to $S$) if, for every $x \in X$, there is an open neighborhood $U$ of $x$ and an integer $n \geq 0$ such that $D|U : \mathcal{F}|U \rightarrow \mathcal{G}|U$ is a differential operator of order $\leq n$. The order of a differential operator is the upper bound of all integers $n$ so that $D$ is a differential operator of order $\leq n$ (and therefore $+\infty$ if there is no such integer); such order is always finite if $X$ is quasi-compact. The differential operator of order $0$ are exactly the homomorphisms of $\mathcal{O}_X$-modules $\mathcal{F} \rightarrow \mathcal{G}$; the operators of order $< 0$ are zero by convention. For $n \geq 0$ a differential operator of order $n$ is not in general a homomorphism of $\mathcal{O}_X$-modules but always a homomorphism of $\psi^*(\mathcal{O}_S)$-modules.

When $\mathcal{F} = \mathcal{O}_X$, a differential operator of order $\leq 1$ of $\mathcal{O}_X$ to $\mathcal{G}$ can be put in the form of $v + D$, where $v : \mathcal{O}_X \rightarrow \mathcal{G}$ is an $\mathcal{O}_X$-homomorphism, and $D$ is an $S$-derivation (16.5.1) of $\mathcal{O}_X$ to $\mathcal{G}$: this results from the structure of $P^n_{B/A}$ (0, 20.4.8).

(16.8.2). To describe in a more precise manner a differential operator of order $\leq n$, $D : \mathcal{F} \rightarrow \mathcal{G}$, it suffices, for every open set $U$ of $X$ whose image in $S$ is contained in an affine open set $V$, to characterize the homomorphism $D = D_U : \Gamma(U, \mathcal{F}) \rightarrow \Gamma(U, \mathcal{G})$. If we put $\Gamma(V, \mathcal{O}_S) = A$, $\Gamma(U, \mathcal{O}_X) = B$, so that $B$ is an $A$-algebra, we have $\Gamma(U, \mathcal{P}^n_{X/S}(\mathcal{F})) = (B \otimes_A B)^{1/n+1}$, where we abbreviate $\mathcal{T} = \mathcal{T}_{B/A}$. Also put $M = \Gamma(U, \mathcal{F})$, $N = \Gamma(U, \mathcal{G})$; then the definition of $D$ means that for every pair $(U, V)$ satisfying the above, the $A$-homomorphism $D : M \rightarrow N$ factors through
\[ M \longrightarrow ((B \otimes_A B)^{1/n+1}) \otimes_B M \overset{v}{\rightarrow} N \]
where the first arrow is the canonical morphism $t \mapsto 1 \otimes t$, and $v$ is a $B$-homomorphism, the structure of $B$-module coming from the first factor (whereas we recall that in the formation of the tensor product over $B$, the structure of $B$-module on $(B \otimes_A B)^{1/n+1}$ comes from the second factor $B$). Note also that the $B$-module $((B \otimes_A B)^{1/n+1}) \otimes_B M$ is isomorphic to $(B \otimes_A M)^{1/n+1}(B \otimes_A M)$, where $(B \otimes_A M)$ is considered as a $(B \otimes_A B)$ module and its structure of $B$-module comes from $t \mapsto 1 \otimes t$ of $B$ in $B \otimes_B B$. Let then $D'$ be the $B$-homomorphism of $B \otimes_A M$ to $N$ such that $D'(b \otimes t) = bDt(t)$; then condition of factorization of $D$ is to say that $D'$ must be zero on the $B$-module $\mathcal{T}^{n+1}(B \otimes_A M)$.

\footnote{For a more general formalism, see the exposé VII of [eAG64] (due to P. Gabriel).}
(16.8.3). It is clear that the set of differential operators of order \(\leq n\) from \(\mathcal{F}\) to \(\mathcal{G}\) forms an additive group, denoted by \(\text{Diff}^n_{X/S}(\mathcal{F}, \mathcal{G})\); when \(\mathcal{F} = \mathcal{G} = \mathcal{O}_X\), we also write \(\text{Diff}^n_{X/S}\) instead of \(\text{Diff}^n_{X/S}(\mathcal{O}_X, \mathcal{O}_X)\).

We have seen (16.8.1), that given two open sets \(U \supset V\) of \(X\), we have a restriction homomorphism

\[
\text{Diff}^n_{X/S}(\mathcal{F}|U, \mathcal{G}|U) \rightarrow \text{Diff}^n_{X/S}(\mathcal{F}|V, \mathcal{G}|V)
\]

from which we deduce that \(U \mapsto \text{Diff}^n_{U/S}(\mathcal{F}|U, \mathcal{G}|U)\) is a presheaf of additive groups; in fact, it is actually a sheaf, since for an open set \(U\) varying in \(X\), the homomorphisms \(u \mapsto u \circ d^n_{U/S, \mathcal{F}}|U\) are isomorphisms of sheaves of additive groups (16.8.3.1)

\[
\text{Hom}_{\mathcal{O}_U}(\mathcal{P}^n_{U/S}(\mathcal{F}|U, \mathcal{G}|U)) \simeq \text{Diff}^n_{U/S}(\mathcal{F}|U, \mathcal{G}|U),
\]

because of the fact that the image of \(\mathcal{F}\) by \(d^n_{U/S, \mathcal{F}}\) generates \(\mathcal{P}^n_{U/S}(\mathcal{F})\) (16.7.6). We denote this sheaf by \(\text{Diff}^n_{X/S}(\mathcal{F}, \mathcal{G})\), and we have:

**Proposition (16.8.4).** — The isomorphisms (16.8.3.1) define an isomorphism of sheaves of additive groups

(16.8.4.1)

\[
\text{Hom}_{\mathcal{O}_X}(\mathcal{P}^n_{X/S}(\mathcal{F}), \mathcal{G}) \simeq \text{Diff}^n_{X/S}(\mathcal{F}, \mathcal{G}).
\]

When \(\mathcal{F} = \mathcal{G} = \mathcal{O}_X\), we also write \(\text{Diff}^n_{X/S}\) instead of \(\text{Diff}^n_{X/S}(\mathcal{O}_X, \mathcal{O}_X)\); it results from (16.8.4) that \(\text{Diff}^n_{X/S}\) is the dual of \(\mathcal{P}^n_{X/S}\); so we also write \(\langle t, D \rangle\) instead of \(u(t)\) if \(t\) is a section of \(\mathcal{P}^n_{X/S}\) over an open set and if \(u\) is a homomorphism from \(\mathcal{P}^n_{X/S}\) to \(\mathcal{O}_X\) corresponding to \(D\).

(16.8.5). When \(\mathcal{P}^n_{X/S}(\mathcal{F})\) has a \(\mathcal{O}_X\)-bimodule structure (16.7.4), we deduce canonically a \(\mathcal{O}_X\)-bimodule structure on \(\text{Hom}_{\mathcal{O}_X}(\mathcal{P}^n_{X/S}(\mathcal{F}), \mathcal{G})\), and therefore on \(\text{Diff}^n_{X/S}(\mathcal{F}, \mathcal{G})\) because of (16.8.4.1). More precisely, to the left \(\mathcal{O}_X\)-module structure on \(\mathcal{P}^n_{X/S}(\mathcal{F})\) corresponds, because of (16.8.1), the left \(\mathcal{O}_X\)-module structure on \(\text{Diff}^n_{X/S}(\mathcal{F}, \mathcal{G})\) explained as follows: for every open set \(U\) of \(X\), every section \(a \in \Gamma(U, \mathcal{O}_X)\) and every differential operator \(D : \mathcal{F}|U \rightarrow \mathcal{G}|U\), \(aD\) is the differential operator which, for every section \(t \in \Gamma(U, \mathcal{F})\), makes correspond the section

(16.8.5.1)

\[
(aD)(t) = a(D(t))
\]

of \(\Gamma(U, \mathcal{G})\). Similarly, the right \(\mathcal{O}_X\)-module structure on \(\text{Diff}^n_{X/S}(\mathcal{F}, \mathcal{G})\) is made explicit as follows: under the same notations as above, \(Da\) is the operator which, to every \(t \in \Gamma(U, \mathcal{F})\), makes correspond the section

(16.8.5.2)

\[
(Da)(t) = D(at).
\]

**Proposition (16.8.6).** — If \(f : X \rightarrow S\) is a morphism locally of finite presentation, \(\mathcal{F}\) a quasi-coherent \(\mathcal{O}_X\)-module of finite presentation and \(\mathcal{G}\) a quasi-coherent \(\mathcal{O}_X\)-module, then \(\text{Diff}^n_{X/S}(\mathcal{F}, \mathcal{G})\) is a quasi-coherent \(\mathcal{O}_X\)-module for any of the structures defined in (16.8.5).

**Proof.** The proposition follows from the fact that, under these hypothesis, \(\mathcal{P}^n_{X/S}\) is a quasi-coherent \(\mathcal{O}_X\)-module of finite presentation (16.7.4) and of (I, 3.12) \(\square\)

(16.8.6). The set of differential operators (of unspecified order (16.8.1)) is denoted by \(\text{Diff}_{X/S}(\mathcal{F}, \mathcal{G})\); we also see as in (16.8.3) that \(U \mapsto \text{Diff}_{U/S}(\mathcal{F}|U, \mathcal{G}|U)\) is a sheaf of additive groups, which we will denote by \(\text{Diff}^n_{U/S}(\mathcal{F}, \mathcal{G})\). It is immediate that \(\text{Diff}^n_{U/S}(\mathcal{F}, \mathcal{G})\) is the reunion of the increasing filtered family of its subsheaves \(\text{Diff}^n_{U/S}(\mathcal{F}, \mathcal{G})\); if \(X\) is quasi-compact, \(\text{Diff}^n_{X/S}(\mathcal{F}, \mathcal{G})\) is similarly the union of its subgroups \(\text{Diff}^n_{X/S}(\mathcal{F}, \mathcal{G})\) (16.8.1). The \(\mathcal{O}_X\)-bimodule structure on the \(\text{Diff}^n_{X/S}(\mathcal{F}, \mathcal{G})\) induce therefore a \(\mathcal{O}_X\)-bimodule structure on \(\text{Diff}^n_{X/S}(\mathcal{F}, \mathcal{G})\), further explained in (16.8.5.1) and (16.8.5.2).

Note that, for \(n \leq m\), we have a commutative diagram

(16.8.7.1)

\[
\begin{align*}
\text{Hom}_{\mathcal{O}_X}(\mathcal{P}^m_{X/S}(\mathcal{F}), \mathcal{G}) & \xrightarrow{\sim} \text{Diff}^m_{X/S}(\mathcal{F}, \mathcal{G}) \\
\text{Hom}_{\mathcal{O}_X}(\mathcal{P}^n_{X/S}(\mathcal{F}), \mathcal{G}) & \xrightarrow{\sim} \text{Diff}^n_{X/S}(\mathcal{F}, \mathcal{G})
\end{align*}
\]

where the horizontal arrows are the isomorphisms (16.8.4.1) and the horizontal arrow on the left comes from the canonical morphism \(\mathcal{P}^m_{X/S}(\mathcal{F}) \rightarrow \mathcal{P}^n_{X/S}(\mathcal{F})\) (16.7.7). For every open set \(U\) of \(X\), we
then endow $\Gamma(U, \mathcal{P}_X^n(\mathcal{F})) = \lim \Gamma(U, \mathcal{P}_{X/S}^n(\mathcal{F}))$ of the projective limit topology of the discrete topologies on $\Gamma(U, \mathcal{P}_{X/S}^n(\mathcal{F}))$, which defines on $\Gamma(U, \mathcal{P}_X^\infty(\mathcal{F}))$ a topological $\Gamma(U, \mathcal{O}_X)$-bimodule structure, so that $\mathcal{P}_X^\infty(\mathcal{F})$ shows itself as a sheaf valued in the category of topological commutative groups $(0, 3.2.6)$. So [God58, II.1.11], the limit of the inductive system of sheaves of commutative groups $(\text{Hom}_{\mathcal{O}_X}(\mathcal{P}_X^n(\mathcal{F}), \mathcal{G}))$ is precisely the sheaf of continuous germs of homomorphisms from $\mathcal{P}_X^\infty(\mathcal{F})$ to $\mathcal{G}$ (the latter equipped with the discrete topology); the continuous homomorphisms $\Gamma(U, \mathcal{P}_X^\infty(\mathcal{F}))$ into the discrete group $\mathcal{G}$ indeed correspond bijectively to the inductive systems of group homomorphisms $\Gamma(U, \mathcal{P}_X^n(\mathcal{F})) \to \Gamma(U, \mathcal{G})$. We can furthermore express (16.8.4) by saying there is a canonical isomorphism

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{P}_X^\infty(\mathcal{F}), \mathcal{G}) \simeq \text{Diff}_{X/S}(\mathcal{F}, \mathcal{G})$$

where the first member denotes the sheaf of germs of continuous homomorphisms from $\mathcal{P}_X^\infty(\mathcal{F})$ to $\mathcal{G}$.

**Proposition (16.8.8).** — Let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_X$-modules, $D : \mathcal{F} \to \mathcal{G}$ a homomorphism of $\psi^*(\mathcal{O}_S)$-modules, $n$ an integer $\geq 0$. The following conditions are equivalent:

(a) $D$ is a differential operator of order $\leq n$.

(b) For all sections $s$ of $\mathcal{O}_X$ over an open set $U$, the homomorphism $D_s : \mathcal{F}|_U \to \mathcal{G}|_U$ such that, for every section $t$ of $\mathcal{F}$ over an open set $V \subset U$, we have

$$D_s(t) = D(at) - aD(t)$$

is a differential operator of order $\leq n - 1$.

(c) For every open set $U$ of $X$, every family $(a_i)_{1 \leq i \leq n+1}$ of $n + 1$ sections of $\mathcal{O}_X$ over $U$ and every section $t$ of $\mathcal{F}$ over $U$, we have the identity

$$\sum_{H \subset I_{n+1}} (-1)^{\text{Card}(H)} \left( \prod_{i \in H} a_i \right) D(\left( \prod_{i \notin H} a_i \right)t) = 0$$

(where $I_{n+1}$ is the interval $1 \leq i \leq n + 1$ of $\mathbb{N}$).

**Proof.** Let us prove first the equivalence of (a) and (b). By definition, to prove that $D$ is a differential operator of order $\leq n$, it suffices to prove so for the restriction $D|U : \mathcal{F}|_U \to \mathcal{G}|_U$ to any affine open set $U$ of $X$, and on the other hand the property (c) is valid for every open set $U$ if it is so on every affine open set. We can therefore restrict ourselves to the case where $S = \text{Spec}(A)$ and $X = \text{Spec}(B)$ are affines. Because of (16.8.2) (where we use the same notations), the condition (a) means that the $A$-homomorphism $D' : B \otimes_A M \to N$ such that $D'(b \otimes t) = bD(t)$ is zero on $\mathfrak{m}_n+1(B \otimes_A M)$, which is, because of (0, 20.44), equivalent to saying that $D'$ annihilates every element of the form

$$\left( \prod_{i=1}^{n+1} (a_i \otimes 1 - 1 \otimes a_i) \right)(1 \otimes t)$$

where $a_i \in B$ and $t \in M$. Now, this element can be written as $\sum_{H \subset I_{n+1}} \left( \prod_{i \in H} a_i \right)(1 \otimes (\prod_{i \notin H} a_i)t)$, and the image under $D'$ of this element is the first member of (16.8.8.2), which proves the equivalence of (a) and (c).

Let us prove now the equivalence of (b) and (c). Let us reason by induction on $n$, the statement being trivial for $n = 0$. Writing $a_{n+1}$ instead of $a$ in the condition (b), we see, by the induction hypothesis, that condition (b) means that for every family $(a_i)_{1 \leq i \leq n}$ of $n$ sections of $\mathcal{O}_X$ over $U$ and every section $t$ of $\mathcal{F}$ over $U$, that

$$\sum_{H' \subset I_{n+1}} (-1)^{\text{Card}(H')} \left( \prod_{i \in H'} a_i \right)D_{a_{n+1}}(\left( \prod_{i \notin H'} a_i \right)t) = 0.$$

But if we replace on this relation $D_{a_{n+1}}$ by the definition (16.8.8.1), we check immediately that we have, up to sign, the first member of (16.8.8.2); from which we conclude. $\square$

**Proposition (16.8.9).** — If $D : \mathcal{F} \to \mathcal{G}$ is a differential operator of order $\leq n$, and $D' : \mathcal{G} \to \mathcal{H}$ a differential operator of order $\leq n'$, then $D' \circ D : \mathcal{F} \to \mathcal{H}$ is a differential operator of order $\leq n + n'$.
Lemma (16.8.9.1). — There is one and only one \( \mathcal{O}_X \)-homomorphism

\[ \delta : \mathcal{O}^{n+n'}_{X/S} \rightarrow \mathcal{O}^{n+n'}_{X/S}(\mathcal{F}) = \mathcal{O}^{n+n'}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}^n_{X/S} \]

making the following diagram commute:

\[
\begin{array}{ccc}
\mathcal{O}_X & \xrightarrow{d^{n+n'}_{X/S}} & \mathcal{O}^{n+n'}_{X/S} \\
d_X & \downarrow & \delta \\
\mathcal{O}^n_{X/S} & \xrightarrow{d^n_{X/S}} & \mathcal{O}^{n+n'}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}^n_{X/S} (\mathcal{F})
\end{array}
\]

We will then have, indeed, a commutative diagram deduced from (16.8.9.3) by tensorization with \( \mathcal{F} \):

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\mathcal{F} \otimes_{\mathcal{O}^n_{X/S}} d^{n+n'}_{X/S}} & \mathcal{O}^{n+n'}_{X/S} (\mathcal{F}) \\
d^{n}_{X/S,\mathcal{F}} & \downarrow & \delta \otimes 1 \\
\mathcal{O}^n_{X/S} (\mathcal{F}) & \xrightarrow{d^n_{X/S}} & \mathcal{O}^{n+n'}_{X/S} (\mathcal{F})(\mathcal{F})
\end{array}
\]

and on the other hand, we verify immediately the from definition (16.7.5) that the diagram

\[
\begin{array}{ccc}
\mathcal{O}^n_{X/S} (\mathcal{F}) & \xrightarrow{u} & \mathcal{O}^{n+n'}_{X/S} (\mathcal{F}) \\
\mathcal{O}^{n+n'}_{X/S} (\mathcal{F})(\mathcal{F}) & \xrightarrow{\delta \otimes 1} & \mathcal{O}^{n+n'}_{X/S} (\mathcal{F})(\mathcal{F}) \\
\mathcal{O}^{n+n'}_{X/S} (\mathcal{F})(\mathcal{F}) & \xrightarrow{1 \otimes u} & \mathcal{O}^{n+n'}_{X/S} (\mathcal{F})(\mathcal{F})
\end{array}
\]

is commutative. We finish the proof by taking \( w \) to be the composite \( \mathcal{O}_X \)-homomorphism

\[ \mathcal{O}^{n+n'}_{X/S} (\mathcal{F}) \xrightarrow{\delta \otimes 1} \mathcal{O}^{n+n'}_{X/S} (\mathcal{F})(\mathcal{F}) \xrightarrow{1 \otimes u} \mathcal{O}^{n+n'}_{X/S} (\mathcal{F})(\mathcal{F}). \]

□

Proof (16.8.9.3). It remains to prove the lemma (16.8.9.3). Considering (16.7.6), which proves the uniqueness of \( \delta \), we are brought back to the case where \( S = \text{Spec}(A) \) and \( X = \text{Spec}(B) \) are affines; letting \( \mathcal{I} = \mathcal{I}_B/A \), it suffices to define a canonical homomorphism of \( B \)-modules

\[ q : (B \otimes_A B) / \mathcal{I}^{n+n'+1} \rightarrow ((B \otimes_A B) / \mathcal{I}^{n'+1}) \otimes_B ((B \otimes_A B) / \mathcal{I}^{n+1}) \]

the \( B \)-module structure of the two members coming from the first \( B \) factor; recall that on tensor product of the second member, \( (B \otimes_A B) / \mathcal{I}^{n'+1} \) must be considered as a right \( B \)-module by its second \( B \) factor, and \( (B \otimes_A B) / \mathcal{I}^{n+1} \) as a left \( B \)-module by its first \( B \) factor (16.7.2). It is the same to define a homomorphism of \( B \)-modules

\[ q_0 : B \otimes_A B \rightarrow ((B \otimes_A B) / \mathcal{I}^{n'+1}) \otimes_B ((B \otimes_A B) / \mathcal{I}^{n+1}) \]
and prove it is zero on $2^{n+n'+1}$. Now, we immediately define a homomorphism by the condition that

$$q_0(b \otimes b') = \pi_{n'}(b \otimes 1) \otimes \pi_n(1 \otimes b')$$

for $b, b'$ in $B$. Under the notations of (16.3.7). Also, it is immediate that $q_0$ is a homomorphism of rings. Now, we can write

$$q_0(b \otimes 1 - 1 \otimes b) = \pi_{n'}(b \otimes 1 - 1 \otimes b) \otimes \pi_n(1 \otimes 1) + \pi_{n'}(1 \otimes b) \otimes \pi_n(1 \otimes 1) - \pi_{n'}(1 \otimes 1) \otimes \pi_n(1 \otimes b)$$

and we have

$$\pi_{n'}(1 \otimes 1) \otimes \pi_n(1 \otimes 1) = \pi_{n'}(1 \otimes 1) \otimes b \pi_n(1 \otimes 1) = \pi_{n'}(1 \otimes 1) \otimes \pi_n(b \otimes 1)$$

from which, finally

$$(16.8.9.4) \quad q_0(b \otimes 1 - 1 \otimes b) = \pi_{n'}(b \otimes 1 - 1 \otimes b) \otimes \pi_n(1 \otimes 1) + \pi_{n'}(1 \otimes 1) \otimes \pi_n(b \otimes 1 - 1 \otimes b).$$

A product of $n+n'$ and terms of the form (16.8.9.4) is therefore necessarily zero, because the same is true for the product of $n+1$ terms of the form $\pi_n(b \otimes 1 - 1 \otimes b)$ and of $n'+1$ terms of the form $\pi_{n'}(b \otimes 1 - 1 \otimes b)$. The conclusion therefore results from (0, 20.4.4).

**Corollary (16.8.10).** — The sheaf $\mathcal{Diff}^{n}_{X/S}(\mathcal{O}_X, \mathcal{O}_X)$ (also denoted $\mathcal{Diff}^n_{X/S}$) is canonically endowed with the structure of sheaf of rings, and the $\mathcal{Diff}^n_{X/S}$ form an increasing filtration compatible with such structure.

In particular, $\mathcal{Diff}^n_{X/S}$ is a sheaf of subrings of $\mathcal{Diff}^{n}_{X/S}$, which is canonically identified with $\mathcal{O}_X$ (16.8.1). The formulas (16.8.5.1) and (16.8.5.2) show that the structure of $\mathcal{O}_X$-bimodule of $\mathcal{Diff}^{n}_{X/S}$ comes from the multiplication on the left and on the right by sections of $\mathcal{O}_X$ considered as a sheaf of subrings of $\mathcal{Diff}^{n}_{X/S}$.

**Remarks (16.8.11).** —

(i) Suppose that $\mathcal{F} = \oplus_{\lambda \in L} \mathcal{F}_\lambda$; then it is clear (16.7.2.1) that $\mathcal{P}^n_{X/S}(\mathcal{F}) = \oplus_{\lambda \in L} \mathcal{P}^n_{X/S}(\mathcal{F}_\lambda)$; since the functor $U \mapsto \Gamma(U, \mathcal{F})$ commutes with the formation of arbitrary direct sums, $d^n_{X/S, \mathcal{F}}$ is the homomorphism whose restriction to each $\mathcal{F}_\lambda$ is $d^n_{X/S, \mathcal{F}} : \mathcal{F}_\lambda \rightarrow \mathcal{P}^n_{X/S}(\mathcal{F}_\lambda)$; then we conclude immediately that we have

$$\mathcal{Diff}^n_{X/S}(\mathcal{F}, \mathcal{G}) = \prod_{\lambda \in L} \mathcal{Diff}^n_{X/S}(\mathcal{F}_\lambda, \mathcal{G}),$$

and therefore also (0, 3.2.6)

$$\mathcal{Diff}^n_{X/S}(\mathcal{F}, \mathcal{G}) = \prod_{\lambda \in L} \mathcal{Diff}^n_{X/S}(\mathcal{F}, \mathcal{G}).$$

Moreover, if $\mathcal{G} = \prod_{\mu \in M} \mathcal{G}_\mu$ (0, 3.2.6), we have

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{P}^n_{X/S}(\mathcal{F}), \mathcal{G}) = \prod_{\mu \in M} \text{Hom}_{\mathcal{O}_X}(\mathcal{P}^n_{X/S}(\mathcal{F}), \mathcal{G}_\mu),$$

every homomorphism $u$ from $\mathcal{P}^n_{X/S}(\mathcal{F})$ to $\mathcal{G}$ corresponds bijectively to the family of its composites $u_\mu : \mathcal{P}^n_{X/S}(\mathcal{F}) \rightarrow \mathcal{G} \rightarrow \mathcal{G}_\mu$. We have therefore

$$\text{Diff}^n_{X/S}(\mathcal{F}, \mathcal{G}) = \prod_{\mu \in M} \text{Diff}^n_{X/S}(\mathcal{F}, \mathcal{G}_\mu),$$

and consequently also

$$\mathcal{Diff}^n_{X/S}(\mathcal{F}, \mathcal{G}) = \prod_{\mu \in M} \mathcal{Diff}^n_{X/S}(\mathcal{F}, \mathcal{G}_\mu).$$

(ii) So far, we have hardly encountered differential operators $\mathcal{F} \rightarrow \mathcal{G}$ where $\mathcal{F}$ and $\mathcal{G}$ are not locally free of finite rank, in which case the structure is reduced locally, because of (i), to the case of the sheaf $\mathcal{Diff}^{n}_{X/S}$; the latter will be studied later (16.11) in a particular case.
16.9. Regular and quasi-regular immersions

Definition (16.9.1). — Let \( X \) be a ringed space. We say that an ideal \( \mathcal{I} \) of \( \mathcal{O}_X \) is regular (resp. quasi-regular) if, for every point \( x \in \text{Supp}(\mathcal{O}_X/\mathcal{I}) \), there is an open neighborhood of \( x \) in \( X \) and a regular sequence \((0, 15.2.2)\) (resp. quasi-regular \((0, 15.2.2)\)) of elements of \( \Gamma(U, \mathcal{O}_X) \) which generates \( \mathcal{I}|U \).

We say that a regular (resp. quasi-regular) sequence of sections of \( \mathcal{O}_X \) over \( U \) which generates \( \mathcal{I}|U \) is called a regular system (resp. quasi-regular system) of generators of \( \mathcal{I}|U \).

Definition (16.9.2). — Let \( j : Y \to X \) be an immersion of preschemes and let \( U \) be an open set such that \( j(Y) \subset U \) and that \( j \) is a closed immersion of \( Y \) in \( U \). We say that \( j \) is regular (resp. quasi-regular) if the closed subprescheme \( j(Y) \) of \( U \) associated to \( j \) is defined by a regular (resp. quasi-regular) ideal of \( \mathcal{O}_U \) (condition independent of the choice of \( U \)).

We say that a subprescheme \( Y \) of a prescheme \( X \) is regularly immersed (resp. quasi-regularly immersed) if the canonical injection \( j : Y \to X \) is a regular immersion (resp. quasi-regular immersion). If \( Y \) is a closed subprescheme and \( \mathcal{I} \) is the ideal of \( \mathcal{O}_X \) that defines \( Y \), it is the same as asking \( \mathcal{I} \) to be regular (resp. quasi-regular).

For example, if \( A \) is an integral ring, \( f \) and element \( \neq 0 \) of \( A \), the closed subprescheme \( V(f) \) of \( \text{Spec}(A) \) (isomorphic to \( \text{Spec}(A/(f)) \)) is regularly immersed in \( \text{Spec}(A) \).

Every regular ideal is quasi-regular \((0, 15.2.2)\); every regular immersion is quasi-regular (cf. \((16.9.11)\) for a converse).

Proposition (16.9.3). — Let \( X \) be a ringed space, \( \mathcal{I} \) an ideal of \( \mathcal{O}_X \), \((f_i)_{1 \leq i \leq m}\) a finite sequence of sections of \( \mathcal{O}_X \) over \( X \) generating \( \mathcal{I} \). For the \((f_i) \) to be a quasi-regular sequence \((0, 15.2.2)\), it is necessary and sufficient that the following conditions are verified:

\begin{enumerate}[(i)]
  \item The canonical images of \( f_i \) in \( \mathcal{I}/\mathcal{I}^2 \) form a basis of this \( \mathcal{O}_X/\mathcal{I} \)-module.
  \item The canonical surjective homomorphism \((16.1.2.2)\)
    \[
    S^*_{\mathcal{O}_X/\mathcal{I}}(\mathcal{I}/\mathcal{I}^2) \to \mathcal{R}\!_{\mathcal{I}}^*(\mathcal{O}_X)
    \]
    is bijective.
\end{enumerate}

Also, if this is true, then every sequence \((f'_i)_{1 \leq i \leq n}\) of \( n \) sections of \( \mathcal{I} \) over \( X \) which generates \( \mathcal{I} \) is quasi-regular.

Proof. The two conditions stated above only translate the ones in \((0, 15.2.2)\), given the definition of the canonical homomorphisms \((16.2.1.1)\). The last claim follows from the fact that, if a module \( M \) over a commutative ring \( A \) admits a basis of \( n \) elements, every system of \( n \) generators of \( M \) is a basis (Bourbaki, Alg. comm., chap. II, §3, cor. 5 of th. 1).

Corollary (16.9.4). — Let \( X \) be a locally ringed space, \( \mathcal{I} \) an ideal of \( \mathcal{O}_X \). For \( \mathcal{I} \) to be quasi-regular, it is necessary and sufficient that it verifies the following conditions:

\begin{enumerate}[(i)]
  \item \( \mathcal{I} \) is of finite type.
  \item \( \mathcal{I}/\mathcal{I}^2 \) is a locally free \( \mathcal{O}_X/\mathcal{I} \)-module.
  \item The canonical homomorphism
    \[(16.9.4.1)\]
    \[
    S^*_{\mathcal{O}_X/\mathcal{I}}(\mathcal{I}/\mathcal{I}^2) \to \mathcal{R}\!_{\mathcal{I}}^*(\mathcal{O}_X)
    \]
    is bijective.
\end{enumerate}

Proof. The necessity of the conditions follows immediately from \((16.9.3)\). To see that they are sufficient, it is enough to show, because of \((16.9.3)\), that if, in a point \( x \in \text{Supp}(\mathcal{O}_X/\mathcal{I}) \), there is a neighborhood \( U \) of \( x \) in \( X \) and \( n \)-sections \( f_i \) \((1 \leq i \leq n)\) of \( \mathcal{I} \) over \( U \) such that the image in \( \mathcal{I}/\mathcal{I}^2 \) form a basis of \( (\mathcal{I}/\mathcal{I}^2)|U \) over \( (\mathcal{O}_X/\mathcal{I})|U \), so there will be a neighborhood \( V \subset U \) of \( x \) such that the \( f_i|V \) generate \( \mathcal{I}|V \). Now, by hypothesis, we have \( \mathcal{I}_x \) \( \neq \mathcal{O}_x \), so that \( \mathcal{I}_x \) is contained in the maximal ideal of \( \mathcal{O}_x \); since \( \mathcal{I}_x \) is a \( \mathcal{O}_x \) module of finite type and the classes of \( (f_i)_x \) in \( \mathcal{I}_x/\mathcal{I}_x^2 \) generate this \( (\mathcal{O}_x/\mathcal{I}_x)|\text{Supp}(\mathcal{I}_x) \)-module, Nakayama’s lemma shows that the \( (f_i)_x \) generate \( \mathcal{I}_x \). Since \( \mathcal{I} \) is of finite type, we conclude by \((0, 5.2.2)\).
Corollary (16.9.5). — Let $X$ be a locally ringed space, $\mathcal{I}$ a quasi-regular ideal of $\mathcal{O}_X$, $(f_i)_{1 \leq i \leq n}$ a sequence of sections of $\mathcal{I}$ over $X$, $x$ a point of $\text{Supp}(\mathcal{O}_X/\mathcal{I})$. The following conditions are equivalent:

(a) There is a neighborhood of $x$ in $X$ such that $f_i|U$ form a quasi-regular sequence of elements $\Gamma(U, \mathcal{O}_X)$ generating $\mathcal{I}|U$.

(b) The $(f_i)_y$ form a system of generators of $\mathcal{I}_x$ whose size is as small as possible.

(b') The $(f_i)_x$ form a minimal set of generators of $\mathcal{I}_x$.

(c) If $f_i$ is the canonical image of $f_i$ in $\Gamma(X, \mathcal{I}/\mathcal{I}^2)$, the $(f_i)_x$ form a basis for the $(\mathcal{O}_x/\mathcal{I}_x)$-module $\mathcal{I}_x/\mathcal{I}_x^2$.

**Proof.** By hypothesis, $\mathcal{O}_x$ is a local ring, $\mathcal{I}_x$ an ideal of finite type of $\mathcal{O}_x$ contained in the maximal ideal of $\mathcal{O}_x$; the equivalence of (b), (b') and (c) results from Nakayama's lemma (Bourbaki, Alg. comm., chap. II, §3, n°2, prop. 5). It is clear that (a) implies (c) because of (16.9.3); on the other hand, from (0, 5.2.2) it follows that if condition (c) is verified (and therefore so is (b)), there is a neighborhood $U$ of $x$ in $X$ such that $(\mathcal{I}/\mathcal{I}^2)|U$ has constant rank equal to $n$, and that the $f_i|U$ generate $\mathcal{I}|U$; it suffices now to apply the last assertion of (16.9.3) to $U$. □

Remarks (16.9.6). —

(i) Under the general hypothesis of (16.9.5), for the sequence $(f_i)$ to generate $\mathcal{I}$, it is not enough that the $(f_i)_y$ form a basis of the $(\mathcal{O}_y/\mathcal{I}_y)$-module $(\mathcal{O}_y/\mathcal{I}_y^2)$ for all $y \in X$. We have an example by taking $X = \text{Spec}(A)$, where $A$ is a Dedekind ring, and $\mathcal{I} = \mathfrak{I}$, where $\mathfrak{I}$ is a non-principal ideal of $A$; then indeed $\mathcal{I}_y/\mathcal{I}_y^2 = 0$ in every point $y$ different from $x \in X$ corresponding to $\mathfrak{I}$, and $\mathcal{I}_x/\mathcal{I}_x^2$ has rank 1 over the field $\mathcal{O}_x/\mathcal{I}_x$; also, $\mathcal{I}$ is evidently a regular ideal.

(ii) In (16.9.5), one cannot replace “quasi-regular” by “regular”, even when $X$ is a prescheme (cf. (16.9.12)). Indeed, denote by $B$ the ring of germs of infinitely differentiable functions on the point 0 on $\mathbb{R}$; it has a maximal ideal $\mathfrak{m}$ generated by the germ of $t$ of the identity mapping to 0, and the intersection $n$ of the $\mathfrak{m}^k$ for $k > 0$ is not reduced to 0. Now let $A$ be the quotient ring $B[T]/nTB[T]$, and let $f_1, f_2$ be the canonical images in $A$ of the elements $t$ and $T$ of $B[T]$. The sequence $(f_1, f_2)$ is regular in $A$: indeed, $f_1$ is a zero divisor in $A$, because the relation $tP[T] \in nTB[T]$, for a polynomial $P \in B[T]$, implies that the products of $t$ by the coefficients of $P$ belong to the ideal $n$, and it results immediately that the coefficients are the same in $n$, so $P[T] \in nTB[T]$. Since $B/\mathfrak{m}$ is isomorphic to $\mathfrak{m}$, $A/\mathfrak{m}A$ is isomorphic to the ring of polynomials $R[T]$, therefore integral, and the image of $f_2$ in $A/\mathfrak{m}A$, being equal to $T$, is a zero divisor in $A$, so that our claim is true. However, $f_2$ is a zero divisor in $A$, since for every non-zero element $x \in a$, the image of $x$ in $A$ is not reduced to 0, but the image of $xt$ is zero. We conclude that the sequence $(f_2, f_1)$ is not regular in $A$; on the other hand, the ideal $\mathfrak{I} = f_1A + f_2A$ is distinct from $A$, so the conditions (b), (b') and (c) of (16.9.5) do not imply the condition (a) when we replace “quasi-regular” by “regular”.

(16.9.7). If $X = \text{Spec}(A)$ is an affine scheme, we’ll say that the ideal $\mathfrak{I}$ of $A$ is regular (resp. quasi-regular) if the ideal $\mathcal{I} = \mathfrak{I}$ of $\mathcal{O}_X$ is regular (resp. quasi-regular); we note that this notion is local and does not imply the existence of a system of generators of $\mathfrak{I}$ forming in $A$ a regular (resp. quasi-regular) sequence as the example (16.9.5) shows; however this is true if $A$ is local (16.9.5).

The proposition (16.9.4) can be translated in terms of quasi-regular immersions in the following manner:

Proposition (16.9.8). — Let $j : Y \to X$ be a morphism of preschemes; for $j$ to be a quasi-regular immersion, it is necessary and sufficient that $j$ satisfies the following conditions:

(i) $j$ is an immersion locally of finite presentation.

(ii) The conormal sheaf $\mathcal{N}^1_Y(j) = \mathcal{N}_{Y/X}$ (16.1.2) is a locally free $\mathcal{O}_Y$-module.

(iii) The canonical homomorphism

$$S_{\mathcal{O}_Y}^1(\mathcal{N}^1_Y(j)) \to \mathcal{N}_Y^1(j)$$

(16.1.2.2) is bijective.

**Proof.** The problem being local on $Y$, we can restrict ourselves to the case where $j$ is the canonical injection of a closed subscheme $Y$ of $X$, so the translation of (16.9.4) into (16.9.8) results...
We can therefore suppose $X \subseteq (16.1.3, (ii))$. 

(16.9.13.1) $\rightarrow$ the homomorphisms of $(16.9.13.1)$: prescheme of $Y$, such that the canonical injection $j$.

Remarks (16.9.12). — (i) We note that a regular immersion is not in general a flat morphism, regular.

Let $X$ be a locally Noetherian prescheme; then every quasi-regular ideal of $Y$, such that the canonical injection, $y$ a point of $Y$.

Proposition (16.9.10). — Let $X$ be a locally Noetherian prescheme, $Y$ a subprescheme of $X$, $j : Y \rightarrow X$ the canonical injection, $y$ a point of $Y$.

(i) For there to exist an open neighborhood $U$ of $y$ in $X$ such that the restriction $Y \cap U \rightarrow U$ of $j$ be a regular immersion, it is necessary and sufficient that the kernel $\mathcal{J}_y$ of the surjective homomorphism $\mathcal{O}_{X,Y} \rightarrow \mathcal{O}_{Y,y}$ is generated by a regular sequence of elements of $\mathcal{O}_{X,Y}$.

(ii) For the immersion $j$ to be regular, it is necessary and sufficient that it is quasi-regular.

Proof. (i) We can reduce to the case where $Y$ is a subprescheme defined by a coherent ideal $\mathcal{I}$ of $\mathcal{O}_X$. The condition is clearly necessary. Conversely, if $\mathcal{J}_y$ is generated by a regular sequence $(s_i)_y$, where the $s_i$ are sections of $\mathcal{I}$ over an open neighborhood $U$ of $y$ in $X$, we can suppose that the $s_i$ generate $\mathcal{I}|U$ (0, 5.5.2) and form a regular sequence (0, 15.2.4), from which the claim follows.

(ii) The fact that a quasi-regular immersion is regular follows from (i) and the identification of quasi-regular and regular sequences of $\mathcal{O}_{X,Y}$, formed from elements of the maximal ideal (0, 15.1.11)

Corollary (16.9.11). — Let $X$ be a locally Noetherian prescheme; then every quasi-regular ideal of $\mathcal{O}_X$ is regular.

Remarks (16.9.12). — (i) We note that a regular immersion is not in general a flat morphism, and therefore a fortiori neither are quasi-regular morphisms in the sense of (6.8.1).

(ii) Let $A$ be a local Noetherian ring; it follows immediately from (16.9.4) and from (0, 17.1.1) that for $A$ to be regular, it is necessary and sufficient that its maximal ideal $m$ is quasi-regular (or regular, which amounts to the same thing given that $A$ is Noetherian). For an affine Noetherian scheme $X$ to be regular, it is necessary and sufficient that for every closed point $x \in X$, the canonical injection $\text{Spec}(k(x)) \rightarrow X$ to be a regular immersion.

Proposition (16.9.13). — Let $X$ be a locally Noetherian prescheme, $Y$ a subprescheme of $X$, $Y'$ a subprescheme of $Y$, such that the canonical injection $j : Y' \rightarrow Y$ is regular. Then the sequence of $\mathcal{O}_{Y'}$-modules

\[
0 \rightarrow j^*(\mathcal{N}_{Y/X}) \rightarrow \mathcal{N}_{Y'/X} \rightarrow \mathcal{N}_{Y'/Y} \rightarrow 0
\]

is exact; furthermore, for every $x \in X$, there is an open neighborhood $U$ of $x$ such that the restrictions to $U$ of the homomorphisms of (16.9.13.1) form a split exact sequence.
Lemma (16.9.13.2). — Let $A$ be a ring, $\mathcal{J}$ an ideal of $A$, $A' = A/\mathcal{J}$, $(f_i)_{1 \leq i \leq r}$ a sequence of elements of $A$ which is $A'$-regular, $\mathfrak{r} = \sum f_i A$, $\mathfrak{r}' = \sum f_i A'$, so that $C = A/\mathfrak{r}$ is isomorphic to $A'/\mathfrak{r}'$. For every integer $n > 0$, and every integer $N \geq n$, we have the relation

\[(16.9.13.3) \quad \mathcal{J} \cap \mathfrak{r}^n = \mathcal{J}\mathfrak{r}^n + \mathcal{J}\mathfrak{r}^N.\]

Proof. It is clearly sufficient to prove that every element of the first is contained in the second, and, by induction on $n$, we reduce to the case $N = n + 1$. An element of the first member of (16.9.13.3), being in $\mathfrak{r}^n$, is written as $P(f_1, \ldots, f_n)$, where $P \in A[T_1, \ldots, T_r]$ is homogeneous of degree $n$. If $f'_i$ is the canonical image of $f_i$ in $A'$, the hypothesis $P(f_1, \ldots, f_n) \in \mathcal{J}$ means that $P(f'_1, \ldots, f'_n) = 0$. But $P(f'_1, \ldots, f'_n) \in \mathfrak{r}^n$, so the canonical image of $P(f'_1, \ldots, f'_n)$ in $\mathfrak{r}'/\mathfrak{r}'^{n+1}$ is zero. Now the hypothesis that $(f_i)$ is $A'$-regular implies that the canonical homomorphism $\mathcal{S}_n^L(\mathfrak{r}'/\mathfrak{r}'^{2}) \to \mathfrak{r}'/\mathfrak{r}'^{n+1}$ is bijective (0, 15.1.9); we conclude that the coefficients of $P$ are in $\mathcal{L} = \mathcal{J} + \mathfrak{r}$. It follows immediately that $P(f_1, \ldots, f_n) \in \mathcal{J}\mathfrak{r}^n + \mathfrak{r}^{n+1}$, and since $P(f_1, \ldots, f_n) \in \mathcal{J}$, we finally have $P(f_1, \ldots, f_n) \in \mathcal{J}\mathfrak{r}^n + \mathfrak{r}^{n+1}$ which proves the lemma. \hfill $\square$

By taking the quotient of both side of (16.9.13.3) by $\mathcal{J}\mathfrak{r}^n$, we see that the relations (16.9.13.3) for $N \geq n$ entail

\[(16.9.13.4) \quad (\mathcal{J} \cap \mathfrak{r}^n)/\mathcal{J}\mathfrak{r}^n \subset \bigcap_{N \geq n} \mathfrak{r}^N.(A/(\mathcal{J}\mathfrak{r}^n)).\]

We deduce the

Corollary (16.9.13.5). — Suppose that the hypothesis of (16.9.13.2) are verified and also that the ring $A$ is Noetherian and $\mathfrak{r}$ is contained in the radical of $A$. Then for every integer $n > 0$,

\[(16.9.13.6) \quad \mathcal{J} \cap \mathfrak{r}^n = \mathcal{J}\mathfrak{r}^n.\]

Proof. Indeed, the second member of (16.9.13.4) is then zero, given that $A/\mathcal{J}\mathfrak{r}^n$ is an $A$-module of finite type (Bourbaki, Alg., comm., chap. III, §3, n°3, prop. 6). \hfill $\square$

Taking in particular $n = 2$ in (16.9.13.6), and remarking that we have $\mathfrak{L}^2 = \mathfrak{J}^2 + \mathfrak{J}\mathfrak{r} + \mathfrak{r}^2 = \mathfrak{J}\mathfrak{L} + \mathfrak{r}^2$; since $\mathfrak{J}\mathfrak{L} \subset \mathfrak{L}^2$, we deduce that

\[(16.9.13.7) \quad \mathcal{J} \cap \mathfrak{L}^2 = \mathfrak{J}\mathfrak{L} + (\mathcal{J} \cap \mathfrak{r}^2) = \mathfrak{J}\mathfrak{L} + \mathcal{J}\mathfrak{r}^2 = \mathfrak{J}\mathfrak{L},\]

in other words,

\[(16.9.13.7) \quad \mathcal{J} \cap \mathfrak{L}^2 = \mathfrak{J}\mathfrak{L},\]

which we can also express in saying that the canonical homomorphism

\[\mathcal{J}/\mathfrak{J}\mathfrak{L} \longrightarrow (\mathcal{J} + \mathfrak{J}^2)/\mathfrak{J}^2\]

is bijective.

Proof (16.9.13). Having demonstrated the lemmas, let us prove the first claim of (16.9.13): It is clearly enough to prove that the sequence of stalks of the sheaves appearing in (16.9.13.1), in a point $x \in Y'$, is exact. Now, if we take $A = \mathcal{O}_X,x$, we can write $\mathcal{O}'_{Y,x} = A' = A/\mathcal{J}$, where $\mathcal{J}$ is an ideal contained in the maximal ideal of $A$, then $\mathcal{O}'_{Y,x} = A'/\mathfrak{r}'$, where $\mathfrak{r}'$ is generated by an $A'$-regular sequence of elements of $A'$, which themselves are images of elements from a $A'$-regular sequence of elements of $A$ belonging to the maximal ideal of $A$. If $\mathfrak{r}$ is the ideal generated by the latter and $\mathfrak{L} = \mathfrak{J} + \mathfrak{r}$, we have $\mathcal{O}'_{Y,x} = A/\mathfrak{L}$, and since we are in the situation of (16.9.13.5), the canonical homomorphism $\mathcal{J}/\mathfrak{J}\mathfrak{L} \longrightarrow (\mathcal{J} + \mathfrak{J}^2)/\mathfrak{J}^2$ is bijective. But this shows that the sequence

\[0 \longrightarrow \mathcal{J}/\mathfrak{J}\mathfrak{L} \longrightarrow \mathfrak{L}/\mathfrak{L}^2 \longrightarrow (\mathcal{J}/\mathfrak{L})/(\mathcal{J}/\mathfrak{L})^2 \longrightarrow 0\]

is exact (see the demonstration of (16.2.7)), and the modules making up this sequence are precisely the stalks in $x$ of the sheaves of (16.9.13.1). The second claim follows from the fact that $\mathcal{N}_{X/Y}$ is a locally free $\mathcal{O}_Y$-module (16.9.8) and Bourbaki, Alg., chap. II, 3rd ed., §1, n°11, prop. 21. \hfill $\square$
16.10. Differentiably smooth morphisms.

**Definition (16.10.1).** — We say that a morphism of preschemes \( f : X \to S \) is differentially smooth (or that \( X \) is differentially smooth over \( S \)) if it satisfies the following conditions:

(i) \( \Omega^1_{X/S} \) is a locally projective \( \mathcal{O}_X \)-module, which is to say that every point \( x \in X \) admits an open affine neighborhood \( U \) such that \( \Gamma(U, \Omega^1_{X/S}) \) is a projective \( \Gamma(U, \mathcal{O}_X) \)-module (not necessarily of finite type).

(ii) The canonical morphism \((16.3.1.1)\)

\[
S^\bullet \mathcal{O}_X(\Omega^1_{X/S}) \to \mathcal{G}_s(\mathcal{P}_{X/S})
\]

is bijective.

In particular, if \( \Omega^1_{X/S} \) is locally free of finite rank, the \( \mathcal{P}_{X/S} \) are locally free of finite rank \( \mathcal{O}_X \)-modules (being extensions of such modules).

We say that \( f \) is differentially smooth at a point \( x \in X \) if there is an open neighborhood \( U \) of \( x \) in \( X \) such that \( f|U \) is differentially smooth.

We will see later (17.12.4) that a smooth morphism is differentially smooth, which justifies the terminology; but the converse is not true; indeed, a monomorphism \( f : X \to S \) is not even necessarily flat, neither \( a \text{ fortiori} \) bijective, however a monomorphism is clearly bijective; however a monomorphism is not even necessarily flat, neither \( a \text{ fortiori} \) bijective.

Let us limit ourselves to proving the following proposition:

**Proposition (16.10.2).** — Let \( A \) be a ring, \( B \) a formally smooth \( A \)-algebra for the discrete topology (0, 19.3.1). Then \( \text{Spec}(B) \) is differentially smooth over \( \text{Spec}(A) \).

**Proof.** Indeed, \( B \otimes_A B \) is then (with the discrete topology) a formally smooth \( B \)-algebra (by on or the other canonical homomorphisms \( b \mapsto b \otimes 1, b \mapsto 1 \otimes b \) of \( B \) to \( B \otimes_A B \) ) (0, 19.3.5, (iii)); therefore \( B \otimes_A B \) is a formally smooth \( A \)-algebra with the discrete topology (0, 19.3.5, (iii)). Taking \( T = \mathcal{O}_{B/A} \), it follows that \( B \otimes_A B \) is a formally smooth \( A \)-algebra for the \( \mathcal{O} \)-preadic topology (0, 19.3.8); since by hypothesis \( B = (B \otimes_A B)/\mathcal{O} \) is a formally smooth \( A \)-algebra for the discrete topology, the proposition follows from the equivalence of (a) and (b) in (0, 19.5.4).

**Proposition (16.10.3).** — For a morphism \( f : X \to S \) to be differentially smooth, it is necessary and sufficient that for every \( x \in X \), there is an open affine neighborhood \( x \), of \( x \) in \( A \), of \( \text{Spec}(\mathcal{O}_{X/S}) \) is an augmented topological \( A \)-algebra isomorphic to the completion \( \hat{B} \), where \( B = S_A(V) \), \( V \) being a projective \( A \)-module and \( B \) being endowed with the \( B^+ \)-preadic topology (where \( B^+ \) is the augmentation ideal). If \( \Omega^1_{X/S} \) is locally free of finite rank, we can replace \( \hat{B} \) with the ring of formal series \( A[[T_1, \ldots, T_n]] \).

**Proof.** The notion of a differentially smooth morphism is clearly local on \( X \), so we reduce to the case where \( S = \text{Spec}(B) \), \( X = \text{Spec}(C) \). Consider \( C \otimes_B C \) as a \( C \)-algebra (by the first factor); take \( T = T_{C/B} \) and endow \( C \) with the \( \mathcal{O} \)-preadic topology; we can apply to the topological \( C \)-algebra \( C \otimes_B C \) to the ideal \( T \) of \( C \otimes_B C \) the equivalence of (b) and (c) of (0, 19.5.4), given that \( (C \otimes_B C)/T = C \) is clearly a formally smooth \( C \)-algebra for the discrete topologies. The topology of \( \Gamma(U, \mathcal{P}_{X/S}) \) is clearly the projective limit topology of this ring (16.1.11).

We note that the integer \( n \) of the proposition (16.10.3) is the rank of \( \Omega^1_{X/S} \) in the point \( x \). We shall see (17.13.5) that when \( f \) is differentially smooth and locally of finite type, \( n \) is equal to the dimension of the fiber \( f^{-1}(f(x)) \).

**Proposition (16.10.4).** — Let \( f : X \to S \), \( g : S' \to S \) be two morphisms, and take \( X' = X \times_S S' \), \( f' = f|S' : X' \to S' \).

(i) If \( f \) is differentially smooth, the same is true for \( f' \).

(ii) Conversely, if \( g \) is faithfully flat and quasi-compact, and if \( f' \) is differentially smooth and \( \Omega^1_{X'/S'} \) is a finite type \( \mathcal{O}_{X'} \)-module, \( f \) is differentially smooth and \( \Omega^1_{X/S} \) is a finite type \( \mathcal{O}_X \)-module.

**Proof.** Indeed, if \( f \) is differentially smooth, the \( \mathcal{G}_n(\mathcal{P}^n_{X/S}) \) are flat \( \mathcal{O}_X \)-modules; therefore by (16.4.6), the homomorphism \( \mathcal{G}_n(\mathcal{P}^n_{X/S}) \otimes_{\mathcal{O}_X} \mathcal{O}_X \to \mathcal{G}_n(\mathcal{P}^n_{X'/S'})) \) is bijective for every \( n \), because of the commutative of the diagram (16.2.1.3), it follows from the definition (16.10.1) that \( f' \) is...
differentially smooth. On the other hand, if \( g \) is faithfully flat and quasi-compact, it follows also from (16.4.6) that \( \mathcal{O}_n(\mathcal{P}^n_{/X/S}) \otimes_{\mathcal{O}_X} \mathcal{O}_{X'} \to \mathcal{O}_n(\mathcal{P}^n_{/X'/S'}) \) is bijective for every \( n \). Suppose also that \( f' \) is differentially smooth and \( \Omega^1_{X'/S'} \) is of finite rank. Since the canonical projection \( X' \to X \) is a faithfully flat and quasi-compact morphism, it results first from (2.5.2) that \( \Omega^1_{X/S} \) is an \( \mathcal{O}_X \)-module locally free of finite rank, then from (2.2.7) that the canonical homomorphism (16.3.1.1) is bijective, and therefore \( f \) is differentially smooth.

\( \square \)

**Proposition (16.10.5). —** For a morphism locally of finite type \( f : X \to S \) to be differentially smooth, it is necessary and sufficient that the diagonal immersion \( \Delta_f : X \to X \times_S X \) to be quasi-regular.

**Proof.** Being a local problem, we can reduce to the case where \( S \) and \( X \) are affines, and therefore the diagonal subschemes of \( X \times_S X \) is closed. The hypothesis that \( f \) is locally of finite type implies that \( \Delta_f \) is locally of finite presentation (1, 4.3.1), therefore the diagonal subschemes of \( X \times_S X \) is defined by an ideal \( \mathscr{I} \) of finite type, and \( \Omega^1_{X/S} = \mathscr{I}/\mathscr{I}^2 \) is an \( \mathcal{O}_X \)-module of finite type. The proposition is now immediate from the comparison of the conditions of (16.10.1) and (16.9.4).

**Remark (16.10.6).** — Let \( f : X \to S \) be a morphism such that the \( \mathcal{O}_X \)-module \( \Omega^1_{X/S} \) is locally free of finite rank. It results from (0, 20.4.7) that every \( x \in X \) has an open neighborhood such that there is a finite family \( (z_{\lambda})_{\lambda \in L} \) of sections of \( \mathcal{O}_X \) over \( U \) for which \( (dz_{\lambda})_{\lambda \in L} \) forms a basis of the \( \Gamma(U, \mathcal{O}_X) \)-module \( \Gamma(U, \Omega^1_{X/S}) \).

### 16.11. Differential operators on a differentially smooth \( S \)-prescheme

**Remark (16.10.7).** — Let \( f : X \to S \) be a morphism such that the \( \mathcal{O}_X \)-module \( \Omega^1_{X/S} \) is locally free of finite rank. It results from (0, 20.4.7) that every \( x \in X \) has an open neighborhood such that there is a finite family \( (z_{\lambda})_{\lambda \in L} \) of sections of \( \mathcal{O}_X \) over \( U \) for which \( (dz_{\lambda})_{\lambda \in L} \) forms a basis of the \( \Gamma(U, \mathcal{O}_X) \)-module \( \Gamma(U, \Omega^1_{X/S}) \).

**Proposition (16.10.8).** — For a morphism locally of finite type \( f : X \to S \) to be differentially smooth, it is necessary and sufficient that the diagonal immersion \( \Delta_f : X \to X \times_S X \) to be quasi-regular.

**Proof.** Being a local problem, we can reduce to the case where \( S \) and \( X \) are affines, and therefore the diagonal subschemes of \( X \times_S X \) is closed. The hypothesis that \( f \) is locally of finite type implies that \( \Delta_f \) is locally of finite presentation (1, 4.3.1), therefore the diagonal subschemes of \( X \times_S X \) is defined by an ideal \( \mathscr{I} \) of finite type, and \( \Omega^1_{X/S} = \mathscr{I}/\mathscr{I}^2 \) is an \( \mathcal{O}_X \)-module of finite type. The proposition is now immediate from the comparison of the conditions of (16.10.1) and (16.9.4).

**Remark (16.10.9).** — Let \( f : X \to S \) be a morphism such that the \( \mathcal{O}_X \)-module \( \Omega^1_{X/S} \) is locally free of finite rank. It results from (0, 20.4.7) that every \( x \in X \) has an open neighborhood such that there is a finite family \( (z_{\lambda})_{\lambda \in L} \) of sections of \( \mathcal{O}_X \) over \( U \) for which \( (dz_{\lambda})_{\lambda \in L} \) forms a basis of the \( \Gamma(U, \mathcal{O}_X) \)-module \( \Gamma(U, \Omega^1_{X/S}) \).

**Proposition (16.10.10).** — For a morphism locally of finite type \( f : X \to S \) to be differentially smooth, it is necessary and sufficient that the diagonal immersion \( \Delta_f : X \to X \times_S X \) to be quasi-regular.

**Proof.** Being a local problem, we can reduce to the case where \( S \) and \( X \) are affines, and therefore the diagonal subschemes of \( X \times_S X \) is closed. The hypothesis that \( f \) is locally of finite type implies that \( \Delta_f \) is locally of finite presentation (1, 4.3.1), therefore the diagonal subschemes of \( X \times_S X \) is defined by an ideal \( \mathscr{I} \) of finite type, and \( \Omega^1_{X/S} = \mathscr{I}/\mathscr{I}^2 \) is an \( \mathcal{O}_X \)-module of finite type. The proposition is now immediate from the comparison of the conditions of (16.10.1) and (16.9.4).

**Remark (16.10.11).** — Let \( f : X \to S \) be a morphism such that the \( \mathcal{O}_X \)-module \( \Omega^1_{X/S} \) is locally free of finite rank. It results from (0, 20.4.7) that every \( x \in X \) has an open neighborhood such that there is a finite family \( (z_{\lambda})_{\lambda \in L} \) of sections of \( \mathcal{O}_X \) over \( U \) for which \( (dz_{\lambda})_{\lambda \in L} \) forms a basis of the \( \Gamma(U, \mathcal{O}_X) \)-module \( \Gamma(U, \Omega^1_{X/S}) \).

**Proposition (16.10.12).** — For a morphism locally of finite type \( f : X \to S \) to be differentially smooth, it is necessary and sufficient that the diagonal immersion \( \Delta_f : X \to X \times_S X \) to be quasi-regular.

**Proof.** Being a local problem, we can reduce to the case where \( S \) and \( X \) are affines, and therefore the diagonal subschemes of \( X \times_S X \) is closed. The hypothesis that \( f \) is locally of finite type implies that \( \Delta_f \) is locally of finite presentation (1, 4.3.1), therefore the diagonal subschemes of \( X \times_S X \) is defined by an ideal \( \mathscr{I} \) of finite type, and \( \Omega^1_{X/S} = \mathscr{I}/\mathscr{I}^2 \) is an \( \mathcal{O}_X \)-module of finite type. The proposition is now immediate from the comparison of the conditions of (16.10.1) and (16.9.4).
Theorem (16.11.2). — Let \( f : X \to S \) be a morphism, \( U \) an open set of \( X \), \((z_\lambda)_{\lambda \in L}\) a family of sections of \( \mathcal{O}_X \) over \( U \) such that the family \((dz_\lambda)_{\lambda \in L}\) generates \( \Omega^1_X|U = \Omega^1_{U/S} \). The following conditions are equivalent:

(a) \( f'|U \) is differentially smooth and \((dz_\lambda)\) is a basis of the \( \mathcal{O}_U \)-module \( \Omega^1_{U/S} \).

(b) There is a family \((D_p)_{p \in N(U)}\) of differential operators of \( \mathcal{O}_U \) to itself verifying the conditions

\[
D_p(z^q) = \left(\frac{q}{p}\right)z^{q-p}, \quad (p, q \text{ in } N(U)).
\]

Also, when these conditions are verified, the family \((D_p)\) is uniquely determined by the conditions (16.11.2.1) and satisfies the relations

\[
D_q \circ D_p = D_p \circ D_q = \left(\frac{p+q}{p!q!}\right)D_{p+q} \quad (p, q \text{ in } N(U)).
\]

Finally, if \( L \) is finite, for every integer \( m \), the \( D_p \) such that \(|p| \leq m\) form a basis of the \( \mathcal{O}_U \)-module \( \text{Diff}^m_{U/S} \), in other words, every differential operator of order \( \leq m \) on \( U \) can be written uniquely as

\[
D = \sum_{|p| \leq m} a_p D_p
\]

where the \( a_p \) are sections of \( \mathcal{O}_X \) over \( U \).

PROOF. Note first that because of (16.11.1.6) and (16.11.1.5), we verify immediately that the conditions (16.11.2.1) are equivalent to

\[
\langle \zeta^p, D_q \rangle = \delta_{pq} \quad (\text{Kronecker's symbol}).
\]

The existence of the family \((D_p)\) verifying these conditions implies first (by taking \(|p| = 1\)) that the \( dz_\lambda \) are linearly independent, and therefore form a basis of the \( \mathcal{O}_U \)-module \( \Omega^1_{U/S} \). Then, for every integer \( m \geq 1 \), we deduce similarly from (16.11.2.3) that the \( \zeta^p \) such that \(|p| \leq m\) are linearly independent; it follows that the canonical homomorphism (16.3.1.1) is injective, and therefore bijective, which proves that (b) implies (a). The converse follows immediately from the definition (16.10.1), the fact that \( \zeta^p \) form a basis of \( \mathcal{P}^m_{U/S} \) for \(|p| \leq m\) implies the existence and uniqueness of a family of homomorphisms \( u_{q,m} : \mathcal{P}^m_{U/S} \to \mathcal{O}_U(|q| \leq m) \) such that \( \langle \zeta^p, u_{q,m} \rangle = \delta_{pq} \) for \(|p| \leq m\), \(|q| \leq m\). For a given value of \( q \), the differential operators corresponding to \( u_{q,m} \) for \( m \geq |q| \) are identified with the same operator \( D_q \). This proves that (a) implies (b), and also that the family \((D_q)\) is uniquely determined, and that, if \( L \) is finite, for \(|p| \leq m\), the \( D_p \) form a basis of the dual \( \text{Diff}^m_{U/S} \) of \( \mathcal{P}^m_{U/S} \). Finally, the relations (16.11.2.2) follows immediately from the expression of the values of the three operators considered on the \( z^r \), and of the fact that the \( \zeta^r \) for \(|r| \leq m\) generate \( \mathcal{P}^m_{U/S} \). \( \square \)

Remarks (16.11.3). — (i) The fact that, because of (16.11.2.2), the \( D_p \) are pairwise permutable naturally do not imply that the \( \mathcal{O}_U \)-algebra \( \text{Diff}^m_{U/S} \) is commutative, the \( D_p \) do not commute with the sections of \( \mathcal{O}_U \) unless \( n = 0 \).

(ii) The indices \( p \) such that \(|p| = 1\) are the \( e_\lambda = (\epsilon_{\lambda \mu})_{\mu \in L} \) where \( \epsilon_{\lambda \mu} = 0 \) if \( \mu \neq \lambda \) and \( \epsilon_{\lambda \lambda} = 1 \); when \( L \) is finite, the operators \( D_{e_\lambda} \) are exactly the \( S \)-derivations \( D_i \) introduced in (16.5.7).

We note that in general (contrary to what happens in classical analysis), it is not true that a differential operator of any order can be written as a linear combination of powers of \( D_i \) (cf. (16.12)).

(iii) For every integer \( r \geq 1 \), we can define the notion of differentially smooth up to order \( r \) by replacing in (16.10.1) the condition (ii) by the condition that the homomorphisms

\[
S^n_{\mathcal{O}_X}(\Omega^1_{X/S}) \to \mathcal{D}_m(\mathcal{P}^n_{X/S})
\]

are bijective for every \( m \leq r \). The argument of (16.11.2) proves also that if, in the condition (a), we replace “differentially smooth” by “differentially smooth up to order \( r \)”, this condition is equivalent to (b) by restricting ourselves to \( p \in N(L), q \in N(L) \) such that \(|p| \leq r, |q| \leq r \).

(16.12.1). We say that a prescheme $X$ is of characteristic $p$ ($p$ equal to zero or a prime number) if, for every affine open set $U$ of $X$, the ring $\Gamma(U, \mathcal{O}_X)$ is of characteristic $p$ (0, 21.1.1). It follows from (0, 21.1.3) that for $X$ to be of characteristic 0, it is necessary and sufficient that for every closed point $x$ of $X$, the residue field $k(x)$ is of characteristic 0, or even that $X$ can be given a structure of $Q$-prescheme (necessarily unique).

**Theorem (16.12.2).** — Let $X$ be a scheme of characteristic 0, $f : X \to S$ a morphism. If $\Omega^1_{X/S}$ is a locally free $\mathcal{O}_X$-module (not necessarily of finite type), $f$ is differentially smooth.

**Proof.** The problem being local on $X$, we can suppose there is a family $(z_\lambda)$ of sections of $\mathcal{O}_X$ over $X$ such that the $(dz_\lambda)$ is a basis for the $\mathcal{O}_X$-module $\Omega^1_{X/S}$. Applying the criterion (16.11.2), it is enough for the operators

$$D_p = (p!)^{-1} \prod_{\lambda} D_{\lambda}^p$$

(where the $D_\lambda$ are the coordinate forms corresponding to the basis $(dz_\lambda)$) to verify the relations (16.11.2.1), which is a consequence of the fact that the $D_\lambda$ are derivations. □

(16.12.3). The theorem above is not true if we discard the hypothesis that $X$ is of characteristic 0. For example, if $S = \text{Spec}(k)$, where $k$ is a field of characteristic $p > 0$, $X = \text{Spec}(K)$ where $K = k(\alpha)$ where $\alpha \notin k$, $\alpha^p \in k$, we verify immediately that $\Omega^1_{X/S}$ has rank 1, and that the morphism $X \to S$ has rank 1, and that the morphism $X \to S$ is differentially smooth up to order $p - 1$ (16.11.3, (iii)), but not of order $p$. However, the proof of (16.12.2) proves that if $\Omega^1_{X/S}$ is locally free, and if $n!\mathcal{O}_X$ is invertible in $\Gamma(X, \mathcal{O}_X)$, then $X$ is differentially smooth over $S$ up to order $n$.

§17. SMOOTH MORPHISMS, UNRAMIFIED (OR NET) MORPHISMS, AND ÉTALE MORPHISMS.

In this paragraph, we revisit the concepts studied in (0\text{III}, 9), expressed in the geometric language of schemes from a global point of view, for preschemes locally of finite presentation over a given base. Most of the results (except 17.7, 17.8, 17.9, 17.13, and 17.16) are reduced to various properties already encountered in (0\text{III}, 9).

For more specific results on étale morphisms, the reader should consult §18.

17.1. Formally smooth morphisms, formally unramified morphisms, formally étale morphisms.

**Definition (17.1.1).** — Let $f : X \to Y$ be a morphism of preschemes. We say that $f$ is formally smooth (resp. formally unramified, resp. formally étale) if, for all affine schemes $Y'$, all closed subschemes $Y'_0$ of $Y'$ defined by a nilpotent ideal $\mathcal{J}$ of $\mathcal{O}_{Y'}$, and every morphism $Y' \to Y$, the map

$$(17.1.1.1) \quad \text{Hom}_Y(Y', X) \longrightarrow \text{Hom}_Y(Y'_0, X)$$

induced by the canonical map $Y'_0 \to Y'$, is surjective (resp. injective, resp. bijective).

One also says that $X$ is formally smooth (resp. formally unramified, resp. formally étale) over $Y$.

It is clear that for $f$ to be formally étale, it is necessary and sufficient for $f$ to be formally smooth and formally unramified.

**Remark (17.1.2).** —

(i) Suppose that $Y = \text{Spec}(A)$ and $X = \text{Spec}(B)$ are affine, so that $f$ comes from a homomorphism of rings $\varphi : A \to B$. According to (0, 19.3.1) and (0, 19.10.1), saying that $f$ is formally smooth (resp. formally unramified, resp. formally étale) means that, via $\varphi$, $B$ is a formally smooth (resp. formally unramified, resp. formally étale) $A$-algebra, for the discrete topologies on $A$ and $B$.

(ii) To verify that $f$ is formally smooth (resp. formally unramified, resp. formally étale), we can, in Definition (17.1.1), restrict to the case where $\mathcal{J}^2 = 0$. To see this, if $f$ satisfies the corresponding condition of Definition (17.1.1) in the particular case $\mathcal{J}^2 = 0$, and if we have $\mathcal{J}^n = 0$, then we consider the closed subscheme $Y'_j$ of $Y'$ defined by the sheaf of ideals
\[ f^j \] for \( 0 \leq j \leq n - 1 \), so that \( Y'_j \) is a closed subscheme of \( Y'_{j+1} \) defined by a square-zero sheaf of ideals; the hypotheses imply that each of the maps

\[ \text{Hom}_Y(Y'_j, X) \rightarrow \text{Hom}_Y(Y'_j, X) \quad (0 \leq j \leq n - 1) \]

is surjective (resp. injective, resp. bijective); by composition, we conclude that the same holds for \((17.1.1.1)\).

(iii) Note that the properties of the morphism \( f \) defined in \((17.1.1)\) are properties of the representable functor \((0_{\text{III}}, 8.1.8)\)

\[ Y' \rightarrow \text{Hom}_Y(Y',X) \]

from the category of \( Y \)-preschemes to the category of sets; they keep a meaning for any contravariant functor with the same domain and codomain, representable or not.

(iv) Assume that the morphism \( f \) is formally unramified (resp. formally étale); consider an arbitrary \( Y \)-prescheme \( Z \) and a closed subscheme \( Z_0 \) of \( Z \) defined by a locally nilpotent sheaf of ideals \( \mathcal{I} \) of \( O_Z \). Then the map

\[ (17.1.2.1) \quad \text{Hom}_Y(Z,X) \rightarrow \text{Hom}_Y(Z_0,X) \]

induced by the canonical injection \( Z_0 \rightarrow Z \), is still injective (resp. bijective). To see this, let \((U_a)\) be an affine open covering of \( Z \) such that the sheaves of ideals \( \mathcal{I}|U_a \) are nilpotent, and for each \( a \), let \( U^0_a \) be the inverse image of \( U_a \) in \( Z_0 \), which is the closed subscheme of \( U^0_a \) defined by \( \mathcal{I}|U_a \). Let \( f_0 : Z_0 \rightarrow X \) be a \( Y \)-morphism; by hypothesis, for each \( a \), there is at most one (resp. one and only one) \( Y \)-morphism \( f_a : U^0_a \rightarrow X \) whose restriction to \( Z_0 \) coincides with \( f_0|U^0_a \). We immediately conclude that if \( f_a \) and \( f_{\beta} \) are defined, then, for each affine open \( V \subset U^0_a \cap U^0_{\beta} \), we have \( f_a|V = f_{\beta}|V \), as the restrictions of these morphisms to the inverse image \( V_0 \) of \( V \) in \( Z_0 \) coincide. There is therefore at most one (resp. one and only one) \( Y \)-morphism \( f : Z \rightarrow X \) whose restriction to \( Z_0 \) coincides with \( f_0 \).

**Proposition (17.1.3).**

(i) A monomorphism of preschemes is formally unramified; an open immersion is formally étale.

(ii) The composition of two formally smooth (resp. formally unramified, resp. formally étale) morphisms is formally smooth (resp. formally unramified, resp. formally étale).

(iii) If \( f : X \rightarrow Y \) is a formally smooth (resp. formally unramified, resp. formally étale) \( S \)-morphism, then so is \( f_{(S')} : X_{(S')} \rightarrow Y_{(S')} \) for any base extension \( S' \rightarrow S \).

(iv) If \( f : X \rightarrow X' \) and \( g : Y \rightarrow Y' \) are two formally smooth (resp. formally unramified, resp. formally étale) \( S \)-morphisms, then so is \( f \times_S g : X \times_S Y \rightarrow X' \times_S Y' \).

(v) Let \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) be two morphisms; if \( g \circ f \) is formally unramified, then so is \( f \).

(vi) If \( f : X \rightarrow Y \) is a formally unramified morphism, then so is \( f_{\text{red}} : X_{\text{red}} \rightarrow Y_{\text{red}} \).

**Proof.** According to \((I, 5.5.12)\), it suffices to prove (i), (ii), and (iii). The assertions in (i) are both trivial. To prove (ii), consider two morphisms \( f : X \rightarrow Y, g : Y \rightarrow Z, \) an affine scheme \( Z' \), a closed subscheme \( Z'_0 \) of \( Z \) defined by a nilpotent ideal and a morphism \( Z' \rightarrow Z \). Suppose that \( f \) and \( g \) formally smooth, and consider a \( Z \)-morphism \( u_0 : Z'_0 \rightarrow X; \) the hypothesis on \( g \) implies that there exists a \( Z \)-morphism \( v : Z' \rightarrow Y \) such that \( f \circ u_0 = v \circ j \) (where \( j : Z'_0 \rightarrow Z \) is the canonical injection); the hypothesis on \( f \) then implies that there exists a morphism \( u : Z' \rightarrow X \) such that \( f \circ u = v \) and \( u \circ j = u_0 \), therefore \( (g \circ f) \circ u \) is equal to the given morphism \( Z' \rightarrow Z \) and \( u \circ j = u_0 \), which proves that \( g \circ f \) is formally smooth; we argue the same way when we suppose that \( f \) and \( g \) are formally unramified.

Finally, to prove (iii), let \( X' = X_{S'}, Y' = Y_{S'}, f' = f_{S'} \); consider an affine scheme \( Y'' \), a closed subscheme \( Y''_0 \) defined by a nilpotent sheaf of ideals, and a morphism \( g : Y'' \rightarrow Y' \) making \( Y'' \) a \( Y' \)-prescheme; we then know by \((I, 3.3.8)\) that \( \text{Hom}_{Y'}(Y'', X) \) is canonically identified with \( \text{Hom}_Y(Y'', X) \), and \( \text{Hom}_{Y'}(Y''_0, X) \) with \( \text{Hom}_Y(Y''_0, X) \), and the conclusion follows immediately from Definition \((17.1.1)\). \( \square \)

We note that a closed immersion is not necessarily formally smooth.

**Proposition (17.1.4).** Let \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) be two morphisms, and suppose that \( g \) is formally unramified. Then, if \( g \circ f \) is formally smooth (resp. formally étale), so is \( f \).
**Proof.** Let \( Y' \) be an affine scheme, \( Y'_0 \) a closed subscheme of \( Y' \) defined by a nilpotent sheaf of ideals, \( h : Y' \to Y \) a morphism, \( f : Y'_0 \to Y' \) the canonical injection, \( u_0 : Y'_0 \to Y \) a \( Y \)-morphism, such that \( f \circ u_0 = h \circ j \). Suppose that \( g \circ f \) is formally smooth; then there exists a morphism \( u : Y' \to X \) such that \( u \circ f = u_0 \) and \( (g \circ f) \circ u = g \circ h \). But these two relations imply that \( f \circ u \) and \( h \) are \( Z \)-morphisms from \( Y' \) to \( Y \) such that \( (f \circ u) \circ j = h \circ j \); by virtue of the hypothesis that \( g \) is formally unramified, we get that \( f \circ u \) is a \( Y \)-morphism; thus \( f \) is formally smooth. Taking into account (17.1.3, (v)), this proves the proposition. \( \square \)

**Corollary (17.1.5).** — Suppose that \( g \) is formally étale; then, for \( g \circ f \) to be formally smooth (resp. formally unramified, resp. formally étale), it is necessary and sufficient that \( f \) is.

**Proof.** This follows from (17.1.4) and (17.1.3, (ii) and (iv)). \( \square \)

**Proposition (17.1.6).** — Let \( f : X \to Y \) be a morphism of preschemes.

(i) Let \( (U_\alpha) \) be an open covering of \( X \) and, for each \( \alpha \), let \( i_\alpha : U_\alpha \to X \) be the canonical injection. For \( f \) to be formally smooth (resp. formally unramified, resp. formally étale), it is necessary and sufficient that each \( f \circ i_\alpha \) is.

(ii) Let \( (V_\lambda) \) be an open covering of \( Y \). For \( f \) to be formally smooth (resp. formally unramified, resp. formally étale), it is necessary and sufficient that each of the restrictions \( f^{-1}(V_\lambda) \to V_\lambda \) of \( f \) is.

**Proof.** First note that (ii) is a consequence of (i): if \( j_\lambda : V_\lambda \to Y \) and \( i_\lambda : f^{-1}(V_\lambda) \to X \) are the canonical injections, then the restriction \( f_\lambda : f^{-1}(V_\lambda) \to V_\lambda \) of \( f \) is such that \( j_\lambda \circ f_\lambda = f \circ i_\lambda \); if \( f \) is formally smooth (resp. formally unramified), then so is \( f \circ i_\lambda \) since \( i_\lambda \) is formally étale (17.1.3); but since \( j_\lambda \) is formally étale, this means that \( f_\lambda \) is formally smooth (resp. formally unramified), by virtue of (17.1.5). Conversely, if all the \( f_\lambda \) are formally smooth (resp. formally unramified), the same applies to \( j_\lambda \circ f_\lambda \) (17.1.3), so also to \( f \) in virtue of (i).

If we take into account that the \( i_\alpha \) are formally étale, everything comes down to proving that if the \( f \circ i_\alpha \) are formally étale, everything comes down to proving that if \( f \circ i_\alpha \) are formally smooth (resp. formally unramified), then the same applies to \( f \).

Therefore let \( Y' \) be an affine scheme, \( Y'_0 \) a closed subscheme of \( Y' \) defined by a nilpotent ideal \( \mathcal{J} \), which we may assume to satisfy \( \mathcal{J}^2 = 0 \) (17.1.2, (ii)), and finally let \( g : Y' \to Y \) be a morphism. Suppose we are given a \( Y \)-morphism \( u_0 : Y'_0 \to X \); denote by \( W_a \) (resp. \( W'_a \)) the prescheme induced by \( Y_0' \) (resp. \( Y'_0 \)) on the open subset \( u_0^{-1}(U_a) \) (we recall that \( Y' \) and \( Y'_0 \) share the same underlying topological space). Let us first suppose that the \( f \circ i_\alpha \) are formally unramified, and show that, if \( u' \) and \( u'' \) are two \( Y \)-morphisms from \( Y' \) to \( X \) whose restrictions to \( Y'_0 \) coincide, then we have \( u' = u'' \). Indeed, taking into account (17.1.2, (iv)), the hypothesis that the \( f \circ i_\alpha \) are formally unramified implies that for all \( \alpha \), we have \( u'|W_a = u''|W_a \), since the restrictions of both \( Y \)-morphisms to \( W'_a \) coincide. Hence the conclusion follows.

Now suppose that the \( f \circ i_\alpha \) are formally smooth and prove the existence of a \( Y \)-morphism \( u : Y' \to X \) whose restriction to \( Y'_0 \) is \( u_0 \). Now, since \( Y' \) is an affine scheme, we can apply (16.5.17), the hypotheses of which are satisfied, and the conclusion of which precisely proves the existence of \( u \). \( \square \)

We can therefore say that the notions introduced in (17.1.1) are local on \( X \) and \( Y \), which always allows, in virtue of (17.1.2, (i)), to be reduced to the study of formally smooth (resp. formally unramified, resp. formally étale) algebras.
17.2. General properties of differentials

**Proposition (17.2.1).** — For a morphism \( f : X \to Y \) to be formally unramified, it is necessary and sufficient that \( \Omega^1_{X/Y} = 0 \) (what we still write \( \Omega^1_{X/Y} = 0 \) (16.3.1)).

**Proof.** Taking into account (17.1.6), we reduce to the case where \( Y = \text{Spec}(A) \) and \( X = \text{Spec}(B) \) are affine, and the conclusion then follows from (0, 20.7.4) and the interpretation of \( \Omega^1_{X/Y} \) in this case (16.3.7).

**Corollary (17.2.2).** — Let \( f : X \to Y \) and \( g : Y \to Z \) be two morphisms. For \( f \) being formally unramified, it is necessary and sufficient that the canonical morphism (16.4.19)

\[
f^*(\Omega^1_{Y/Z}) \longrightarrow \Omega^1_{X/Z}
\]

is surjective.

**Proof.** This is an immediate consequence of (17.2.1) and the exact sequence (16.4.19.1).

**Proposition (17.2.3).** — Let \( f : X \to Y \) be a formally smooth morphism.

(i) The \( \mathcal{O}_X \)-module \( \Omega^1_{X/Y} \) is locally projective (16.10.1). If \( f \) is locally of finite type, then \( \Omega^1_{X/Y} \) is locally free and of finite type.

(ii) For all morphisms \( g : Y \to Z \), the sequence (16.4.19) of \( \mathcal{O}_X \)-modules

\[
0 \longrightarrow f^*(\Omega^1_{Y/Z}) \longrightarrow \Omega^1_{X/Z} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0
\]

is exact; moreover, for each \( x \in X \), there exists an open neighborhood \( U \) of \( x \) such that the restrictions to \( U \) of the homomorphisms in (17.2.3.1) form a split exact sequence.

**Corollary (17.2.4).** — If \( f : X \to Y \) is formally étale, then, for all morphisms \( g : Y \to Z \), the canonical homomorphism of \( \mathcal{O}_X \)-modules

\[
f^*(\Omega^1_{Y/Z}) \longrightarrow \Omega^1_{X/Z}
\]

is bijective.

**Proof.** This follows from the exactness of the sequence (17.2.3.1) and from the fact that we then have \( \Omega^1_{X/Y} = 0 \) (17.2.1).

**Proposition (17.2.5).** — Let \( f : X \to Y \) be a morphism, \( X' \) a subprescheme of \( X \) such that the composite morphism \( X' \stackrel{j}{\to} X \stackrel{f}{\to} Y \) (where \( j \) is the canonical injection) is formally smooth. Then the sequence of \( \mathcal{O}_X \)-modules (16.4.21)

\[
0 \longrightarrow \mathcal{N}^1_{X'/X} \longrightarrow \Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'} \longrightarrow \Omega^1_{X'/Y} \longrightarrow 0
\]

is exact; moreover, for each \( x \in X \), there exists an open neighborhood \( U \) of \( x \) such that the restrictions to \( U \) of the homomorphisms in (17.2.5.1) form a split exact sequence.

**Proof.** By virtue of (17.1.6), we reduce to the case where \( Y = \text{Spec}(A) \) and \( X = \text{Spec}(B) \) are affine, and \( X' = \text{Spec}(B/\mathfrak{J}) \), where \( \mathfrak{J} \) is an ideal of \( B \). The conormal sheaf \( \mathcal{N}^1_{X'/X} \) then corresponds to the \( B \)-module \( \mathfrak{J}/\mathfrak{J}^2 \) (16.1.3), and the conclusion follows from (0, 20.5.14).

**Proposition (17.2.6).** — Let \( X \) and \( Y \) be two preschemes, \( f : X \to Y \) a morphism locally of finite type. The following conditions are equivalent:

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17. Smooth morphisms, unramified (or net) morphisms, and étale morphisms

Definition (17.3.1). — We say that a morphism \( f : X \to Y \) is smooth (resp. unramified, or net\(^5\) resp. étale) if it is locally of finite presentation and formally smooth (resp. formally unramified, resp. formally étale).

We then also say that \( X \) is smooth (resp. unramified, resp. étale) over \( Y \).

We will see later (17.5.2) that this definition of a smooth morphism coincides with the definition already given in (6.8.1); until then, we will exclusively use definition (17.3.1).

It is clear that saying that \( f \) is étale means that it is both smooth and unramified.

Remark (17.3.2). —

(i) Note that definition (17.3.1) can be phrased using only the functor 
\[
Y' \mapsto \text{Hom}_{\mathcal{Y}}(Y', X)
\]
considered in (17.1.2, (iii)) because to say that \( f \) is locally of finite presentation is equivalent to saying that the preceding functor commutes with projective limits of affine schemes (8.14.2).

(ii) Let \( A \) be a ring and \( B \) an \( A \)-algebra. We say that \( B \) is a smooth (resp. unramified, resp. étale) \( A \)-algebra if the corresponding morphism \( \text{Spec}(B) \to \text{Spec}(A) \) is smooth (resp. unramified, resp. étale). It is equivalent to say that \( B \) is an \( A \)-algebra of finite presentation (1.4.6) that is furthermore formally smooth (resp. formally unramified, resp. formally étale) for the discrete topologies.

(iii) It follows from (17.1.6) and the definition of a morphism locally of finite presentation (1.4.2) that the notion of a smooth (resp. unramified, resp. étale) morphism is local on \( X \) and on \( Y \).

Proposition (17.3.3). —

(i) An open immersion is étale. For an immersion to be unramified, it is necessary and sufficient to it be locally of finite presentation.

(ii) The composition of two smooth (resp. unramified, resp. étale) morphisms is smooth (resp. unramified, resp. étale).

(iii) If \( f : X \to Y \) is a smooth (resp. unramified, resp. étale) \( S \)-morphism, then so is \( f_{(S')} : X_{(S')} \to Y_{(S')} \) for any base extension \( S' \to S \).

(iv) If \( f : X \to X' \) and \( g : Y \to Y' \) are smooth (resp. unramified, resp. étale) \( S \)-morphisms, then so is \( f \times_S g : X \times_S Y \to X' \times_S Y' \).

\(^5\)The words “net” and “formally net” seem more preferable to the terminology used in “unramified” (resp. formally unramified”) and will be used almost exclusively in Chapter V. In this chapter, we have kept the old terminology so as not to conflict with 0.19.10.
(v) Let \( f : X \to Y \) and \( g : Y \to Z \) be two morphisms; if \( g \) is locally of finite type and if \( g \circ f \) is unramified, then \( f \) is unramified.

**Proof.** This follows from (1.4.3) and (17.1.3). □

**Proposition (17.3.4).** — Let \( f : X \to Y \) and \( g : Y \to Z \) be two morphisms, and suppose that \( g \) is unramified. Then, if \( g \circ f \) is smooth (resp. unramified, resp. étale), so is \( f \).

**Proof.** As \( g \) and \( g \circ f \) are locally of finite presentation, so is \( f \) (1.4.3, (v)); the conclusion thus follows from (17.1.4) and (17.1.3, (v)). □

**Corollary (17.3.5).** — Suppose that \( g \) is étale; then, for \( f \) to be smooth (resp. unramified, resp. étale) it is necessary and sufficient that \( g \circ f \) is.

**Proof.** This follows from (17.3.4) and (17.3.3, (ii)). □

**Proposition (17.3.6).** — Let \( g : Y \to S \) and \( h : X \to S \) be two morphisms locally of finite presentation. For an \( S \)-morphism \( f : X \to Y \) to be unramified, it is necessary and sufficient that the canonical homomorphism (16.4.19)

\[
f^{*}(\Omega^{1}_{Y/S}) \to \Omega^{1}_{X/S}
\]

is surjective.

**Proof.** As \( f \) is locally of finite presentation (1.4.3, (v)), the proposition follows from (17.2.2). □

**Definition (17.3.7).** — Let \( f : X \to Y \) be a morphism. We say that \( f \) is smooth (resp. unramified, resp. étale) at a point \( x \in X \), if there exists an open neighborhood \( U \) of \( x \) in \( X \) such that the restriction \( f|_{U} \) is a smooth (resp. unramified, resp. étale) morphism from \( U \) to \( Y \).

We then also say that \( X \) is smooth (resp. unramified, resp. étale) over \( Y \) at the point \( x \).

Taking into account remark (17.3.2, (iii)), it is equivalent to say that \( f \) is smooth (resp. unramified, resp. étale) and to say that \( f \) is smooth (resp. unramified, resp. étale) at all points of \( X \).

It is clear that the set of points of \( X \) at which the morphism \( f : X \to Y \) is smooth (resp. unramified, resp. étale) is open in \( X \).

**Proposition (17.3.8).** — For all preschemes \( Y \) and all locally free \( O_{Y} \)-modules \( \mathcal{E} \) of finite type, the vector bundle prescheme \( V(\mathcal{E}) \) (II, 1.7.8) associated to \( \mathcal{E} \) is a smooth \( Y \)-prescheme.

**Proof.** Indeed (17.3.2, (iii)), we can restrict ourselves to the case where \( Y = \text{Spec}(A) \) is affine and \( V(\mathcal{E}) = \text{Spec}(A[T_{1}, \ldots, T_{r}]) \); as \( A[T_{1}, \ldots, T_{r}] \) is a formally smooth \( A \)-algebra for the discrete topologies (0, 19.3.2), and of finite presentation, this proves the proposition (17.3.2, (ii)). □

**Corollary (17.3.9).** — Under the hypotheses of (17.3.8), the projective prescheme \( P(\mathcal{E}) \) (II, 4.1.1) is a smooth \( Y \)-prescheme.

**Proof.** We can still restrict to the case where \( Y = \text{Spec}(A) \) is affine and \( P(\mathcal{E}) = P^{r}_{Y} \). We then know (II, 2.3.14) that we have a finite open cover of \( P^{r}_{Y} \) by the \( D_{+}(T_{i}) \) (\( 0 \leq i \leq r \)) respectively equal to the spectrum of the ring \( S(f)_{i} \), where we wrote \( S \) for \( A[T_{1}, \ldots, T_{r}] \) and \( f \) for \( T_{i} \); but it follows immediately from the definition of \( S(f) \) (II, 2.2.1), that this ring, in this case, is isomorphic to \( A[T_{0}, \ldots, T_{i-1}, T_{i+1}, \ldots, T_{r}] \); hence the corollary follows by (17.3.8). □
17.4. Characterizations of unramified morphisms.

§18. Supplement on étale morphisms. Henselian local rings and strictly local rings

§19. Regular immersions and normal flatness

§20. Meromorphic functions and pseudo-morphisms

20.0. Introduction Most of the concepts and results of §§20 and 21 directly relate to Chapter I, and hardly depend on Chapters I and IV, except for the occasional usage of the notion of depth and of a local regular ring (in (20.6), (21.11), (21.13), and (21.15)), of Zariski’s “Main theorem” in (20.4) and (21.12), and the properties of transversely regular immersions in (20.6) and (21.15).

In §20, we introduce several variants of the concept of a rational map, already studied in (I, 7) from a point of view still fairly close to the classical point of view, and for this reason quite ill-suited to the case of not necessarily reduced preschemes. The notions and results of §20 are used in §21 (n°9(21.1) and (21.7)) to develop the general notion of a divisor and its most elementary properties. This notion is especially convenient when the local rings of the preschemes considered are Noetherian and integrally closed, and especially when they are also factorial ((21.6) and (21.7)), because of their identification in the latter case with the notion of a 1-codimensional cycle (a linear combination of irreducible subschemes of codimension 1). In (21.9), we determine the divisors on a Noetherian prescheme of dimension 1 but not necessarily normal, which is useful for various applications. The (21.11) and (21.12) give two important theorems, due respectively to Auslander–Buchsbaum and Van der Waerden, and relate the notion of a factorial ring (the n°9(21.9), (21.11), and (21.12) are independent of each other). In the n°9(21.13) and (21.14), also independent of the previous three, we study a useful variant of the notion of a local factorial ring, that of a local parafactorial ring, which is introduced in particular [?] in the development of comparison theorem of the Picard group of a projective prescheme X over a field k and a “hyperplane section”. We will see in (21.14.1) (Ramanujam–Samuel theorem) that the local parafactorial rings are much more numerous than one would have expected a priori.

In (20.5), (20.6), and (21.15), we review the previous notions but from a point of view “relative” to a fixed base prescheme. For the moment these notions are used only relatively rarely; in particular, the concept of a relative divisor is hardly used except when it comes to positive divisors, and in this case it is explained advantageously without the help of the notion of a relative meromorphic functions, using the notion of a transversely regular immersion of codimension 1. It will therefore be advantageous to omit these sections on a first reading.

20.1. Meromorphic functions

(20.1.1). Let \((X, \mathcal{O}_X)\) be a ringed space, and let \(\mathcal{S}\) be a subsheaf of sets of \(\mathcal{O}_X\). For every open \(U\) of \(X\), consider the ring of fractions \(\Gamma(U, \mathcal{O}_X)[\Gamma(U, \mathcal{S})^{-1}]\) (Bourbaki, Alg. comm., chap. II, §2, n°1). It is immediate that the map \(U \mapsto \Gamma(U, \mathcal{O}_X)[\Gamma(U, \mathcal{S})^{-1}]\) is a presheaf of rings ((0, 1.5.1) and (0, 1.5.7)). We denote by \(\mathcal{O}_X[\mathcal{S}^{-1}]\) the sheaf of rings associated to this presheaf and we say that this is the sheaf of rings of fractions of \(\mathcal{O}_X\) with denominators in \(\mathcal{S}\); this is a flat \(\mathcal{O}_X\)-module. It is immediate that for every \(x \in X\), we have a canonical isomorphism

\[
(\mathcal{O}_X[\mathcal{S}^{-1}])_x \simeq \mathcal{O}_x[\mathcal{S}^{-1}],
\]

since the reasoning of (0, 1.4.5) generalizes immediately in the case where we have an inductive system \((A_\alpha, \phi_{\alpha\beta})\) of rings, and for each index \(\alpha\) a subset \(S_\alpha\) of \(A_\alpha\) such that \(\phi_{\alpha\beta}(S_\alpha) \subset S_\beta\) for \(\alpha \leq \beta\); we then take for \(S\) the inductive limit in \(A = \lim_{\rightarrow} A_\alpha\) of the inductive limit of the subsets \((S_\alpha)\).

(20.1.2). Now let \(\mathcal{F}\) be an \(\mathcal{O}_X\)-module. We then set

\[
\mathcal{F}[\mathcal{S}^{-1}] = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X[\mathcal{S}^{-1}],
\]

and we say that this is the sheaf of modules of fractions of \(\mathcal{F}\) with denominators in \(\mathcal{S}\); it is immediate that it is associated to the presheaf of modules \(U \mapsto \Gamma(U, \mathcal{F})[\Gamma(U, \mathcal{S})^{-1}]\), and that for every \(x \in X\),...
we have a canonical isomorphism

\[(20.1.2.2) \quad (\mathcal{F}[\mathcal{J}^{-1}])_x \simeq \mathcal{F}_x[\mathcal{J}^{-1}].\]

(20.1.3). We will focus here on the case where \(\mathcal{J}\) is the subsheaf \(\mathcal{J}(\mathcal{O}_X)\) of \(\mathcal{O}_X\) such that for every open \(U, \Gamma(U, \mathcal{J})\) is the set of regular elements of the ring \(\Gamma(U, \mathcal{O}_X)\); it is immediate that it is a sheaf (and not only a presheaf), the regularity of a section of \(\mathcal{O}_X\) over \(U\) being verified “fibre by fibre” (i.e. meaning that the germ of the section in \(x\) is regular in \(\mathcal{O}_X\) for all \(x \in U\), in other words \(\mathcal{J}(\mathcal{O}_X)_x\) is none other then the set of regular elements of \(\mathcal{O}_{X,x}\). The corresponding sheaf of rings

\[\mathcal{M}_X = \mathcal{O}_X[\mathcal{J}^{-1}]\]

is called the sheaf of germs of meromorphic functions on \(X\), and the sections of \(\mathcal{M}_X\) over \(X\) are called the meromorphic functions on \(X\); they form a ring which we denote by \(M(X)\). For every \(\mathcal{O}_X\)-module \(\mathcal{F}\),

\[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}_X = \mathcal{F}[\mathcal{J}^{-1}]\]

is also denoted \(\mathcal{M}_X(\mathcal{F})\) and called the sheaf of germs of meromorphic sections of \(\mathcal{F}\); its sections over \(X\) form an \(M(X)\)-module denoted by \(M(X, \mathcal{F})\), whose elements are called meromorphic sections of \(\mathcal{F}\) over \(X\). These definitions imply that for every open \(U\) of \(X\), we have a canonical isomorphism \(\mathcal{M}_X(\mathcal{F})|U \simeq \mathcal{M}_U(\mathcal{F}|U)\), in particular \(\mathcal{M}_X|U \simeq \mathcal{M}_U\).

(20.1.3.1). If \(X\) is a reduced prescheme, then we note that if an element \(s \in \Gamma(U, \mathcal{O}_X)\) is such that \(s_x \neq 0\) for every maximal point \(\xi\) of \(U\), then \(s\) is regular. Indeed, if \(st = 0\) for a \(t \in \Gamma(U, \mathcal{O}_X)\), then we have \(s_\xi t_\xi = 0\), so \(t_\xi = 0\) since \(\mathcal{O}_{X,\xi}\) is a field, and say that \(t_\xi = 0\) for every maximal point \(\xi\) of \(X\) means that \(t = 0\): we are immediately reduced to the case where \(U\) is affine, and an element of a reduced ring which belongs to all the minimal prime ideals is zero by definition. The converse is true if the set of irreducible components of \(X\) is locally finite. We immediately reduce to the case where \(X = \text{Spec}(A)\) is affine; if \(p_i\) \((1 \leq i \leq n)\) are the minimal prime ideals of \(A\) and if \(s \in p_i\) for an index \(i\), then there exists \(t \in A\) such that \(t \in p_j\) for \(j \neq i\) and \(t \notin p_i\) (Bourbaki, Alg. comm., chap. II, §1. n°1, Prop. 1); so we have \(st \in p_i\) for all \(i\), and as a result \(st = 0\) since \(A\) is reduced; \(s\) is therefore nonregular.

(20.1.4). For every open \(U\) of \(X\), the homomorphism \(t \mapsto t/1\) from \(\Gamma(U, \mathcal{O}_X)\) to \(\Gamma(U, \mathcal{O}_X)\)\(\Gamma(U, \mathcal{J})^{-1}\) (which is none other than the total ring of fractions of \(\Gamma(U, \mathcal{O}_X)\)) is injective; these homomorphisms thus define a canonical injective homomorphism

\[(20.1.4.1) \quad i : \mathcal{O}_X \rightarrow \mathcal{M}_X\]

which allows us to identify \(\mathcal{O}_X\) with a subsheaf of \(\mathcal{M}_X\). Given a meromorphic function \(\phi \in M(X)\), we say that \(\phi\) is defined on an open \(U\) of \(X\) if \(\phi(U)\) is a section of \(\mathcal{O}_U\) over \(U\); the axioms of sheaves show that there is, for a given section \(\phi\), a largest open on which \(\phi\) is defined; we call this the domain of definition of \(\phi\) and denote it by \(\text{dom}(\phi)\).

(20.1.5). For every \(\mathcal{O}_X\)-module \(\mathcal{F}\), we obtain from (20.1.4.1) a di-homomorphism consisting of \(i\) and the homomorphism of sheaves of additive groups

\[(20.1.5.1) \quad 1: \mathcal{F} \otimes i : \mathcal{M}_X(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}_X.\]

We note that the latter is not injective in general; when it is injective, we say that \(\mathcal{F}\) is strictly torsion-free: this means that for every open \(U\) of \(X\) and every section \(s \in \Gamma(U, \mathcal{O}_X)\) which is a regular element in this ring the homothety \(z \mapsto sz\) of \(\Gamma(U, \mathcal{F})\) is injective; this condition is evidently satisfied if \(\mathcal{F}\) is locally free.

**Proposition (20.1.6). —** Let \(X\) be a locally Noetherian prescheme, \(\mathcal{F}\) a quasi-coherent \(\mathcal{O}_X\)-module. For \(\mathcal{F}\) to be strictly torsion-free, it is necessary and sufficient that \(\text{Ass}(\mathcal{F}) \subset \text{Ass}(\mathcal{O}_X)\).

**Proof.** We immediately reduce to the case where \(X = \text{Spec}(A)\) is affine, \(\mathcal{F} = \tilde{M}\), and we know that the elements \(s\) of \(A\) belonging to an ideal of \(\text{Ass}(M)\) are exactly those for which the homothety \(z \mapsto sz\) is not injective (Bourbaki, Alg. comm., chap. IV, §1, n°1, cor. 2 of prop. 2).

(20.1.7). If \(u\) is a section of \(\mathcal{M}_X(\mathcal{F})\) over \(X\), then we say that \(u\) is defined at a point \(x \in X\) if there exists an open neighborhood \(V\) of \(x\) in \(X\) such that \(u|V\) is the image of a section of \(\mathcal{F}\) over \(V\) under the di-homomorphism (20.1.5.1). We say that \(u\) is defined on an open \(U\) of \(X\) if it is defined at every point in \(U\); there is still a larger open in which \(u\) is defined, called the domain of definition of \(u\) and denoted
with a subsheaf of $M(\text{20.1.8})$. Let $\Gamma$ obtain from regular is a $A$ already in from the field of fractions implies its global (and unique) existence; moreover, the existence of the canonical image of $s$ is again bijective $O$ isomorphism from $M$ its restriction to $h$ defines a homomorphism sections of subsheaf of isomorphic to $O$ isomorphism on $M$ sections of $certain$ authors, who call “regular” meromorphic functions those which are the regular meromorphic functions (note that we are deviating here from the terminology followed by certain authors, who call “regular” meromorphic functions those which are sections of $O_X$, identified with a subsheaf of $M$).

Let $L$ be an invertible $O_X$-module (0t, 5.4.1); then it is clear that $M_X(L) = L \otimes_{O_X} M_X$ is an invertible $M_X$-module. Let $U$ be an open such that $L|U$ is isomorphic to $O_U$; as every automorphism of $M_U$ is multiplication by an invertible element of $\Gamma(U, M_X)$ (0t, 5.4.7), it is equivalent to say that a section $s \in \Gamma(U, M_X)$ has an invertible image in $\Gamma(U, M_X)$ under an isomorphism or by any isomorphism on $\Gamma(U, M_X)$; we say in this case that $s$ is a regular meromorphic section of $L$ over $U$; a section $s$ of $L$ over $X$ is called a regular meromorphic section of $L$ if, for every open $U$ such that $L|U$ is isomorphic to $O_U$, $s|U$ is a regular meromorphic section of $L$ over $U$. We denote by $(M_X(L))^*$ the subsheaf of $M_X(L)$ such that for every open $U$, $\Gamma(U, (M_X(L))^*)$ is the set of regular meromorphic sections of $L$ over $U$. Let $s$ be a meromorphic section of $L$ over $X$ (i.e. a section of $M_X(L)$); it defines a homomorphism $h_s : M_X \to M_X(L)$ which sends every section $t$ of $M_X$ over an open $U$ to $(s|U)t$. It follows immediately from the above that for $s$ to be regular, it is necessary and sufficient for $h_s$ to be injective, and in fact $h_s$ is then a bijective homomorphism from $M_X$ to $M_X(L)$, and its restriction to $M_X^*$ is a bijection to $(M_X(L))^*$. We conclude that the homothety $t \mapsto ts$ is an isomorphism from $M(X)$ to $M(X, L)$.

(20.1.9). Let $s$ be a regular meromorphic section of an invertible $O_X$-module $L$ over $X$, then for every $O_X$-module $F$, $s$ similarly defines a homomorphism $h_s \otimes 1_F : M_X(F) \to M_X(F \otimes_{O_X} L)$, which is again bijective.

(20.1.10). Let $s$ be a meromorphic section of an invertible $O_X$-module $L$ over $X$; for $s$ to be regular, it is necessary and sufficient for there to exist a meromorphic section $s'$ of $L^{-1}$ over $X$ such that the canonical image of $s \otimes s'$ in $M_X(0t, 5.4.3)$ is the unit section, and this section $s'$ is then unique: indeed, the necessity of the local existence of such a section is evident, and its local uniqueness implies its global (and unique) existence; moreover, the existence of $s'$ is trivially sufficient for $s$ to be regular. We will take $s' = s^{-1}$.

Finally, if $L'$ is a second invertible $O_X$-module, $s$ (resp. $s'$) a regular meromorphic section of $L$ (resp. $L'$) over $X$, then $s \otimes s'$ is evidently a regular meromorphic section of $L \otimes L'$ over $X$.

(20.1.11). If $f : X' \to X$ is a morphism of ringed spaces, then there is in general no natural map sending a meromorphic function on $X$ to a meromorphic function on $X'$. For example, if $X$ is the spectrum of a local integral domain $A$, $X'$ its residue field $k$, then there is no natural homomorphism from the field of fractions $K$ of $A$ to $k$, and we can only send an element of $K$ to an element of $k$ if it is already in $A$.

In general, if $f = (\psi, \theta)$, then for every open $U$ of $X$, denote by $J_f(U)$ the set of regular sections $s \in \Gamma(U, O_X)$ such that the image of $s$ under

$$\Gamma(\theta^n) : \Gamma(U, O_X) \to \Gamma(f^{-1}(U), O_{X'})$$

is a regular section. It is immediate that $U \mapsto J_f(U)$ is a subsheaf of the sheaf of sets $J(O_X)$, which we denote by $J_f$. We set $M_f = O_X[J_f^{-1}]$; this is a subsheaf of rings of $M_X$, and we canonically obtain from $\theta^n : \psi^*(O_X) \to O_X$, a homomorphism of sheaves of rings $\theta^n : \psi^*(M_f) \to M_X$, extending $\theta^n$ (Bourbaki, Alg. comm., chap. II, §2, n°1, prop. 2); hence, recalling that $f^*(M_f) = \psi^*(M_f) \otimes_{\psi^*(O_X)} O_{X'}$, we get a canonical homomorphism of $O_{X'}$-algebras

$$f^*(M_f) \to M_{X'}.$$
For every meromorphic function $\phi$ on $X$ that is a section of $M_f$, $\Gamma(\theta^\#)(\phi)$ is a meromorphic function on $X'$, called the inverse image of $\phi$ under $f$, and denoted by $\phi \circ f$ is there is no cause for confusion.

Similarly, if $\mathcal{F}$ is an $\mathcal{O}_X$-module, then we set $M_f(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} M_f$, and we immediately obtain from $\theta^\#$ a canonical homomorphism (which is also written as $u \mapsto u \circ f$)

$$\Gamma(X, M_f(\mathcal{F})) \longrightarrow \Gamma(X', \mathcal{M}_X(f^*(\mathcal{F}))).$$

In addition, if $u \in \Gamma(X, M_f(\mathcal{F}))$ is defined (20.1.7) at a point $x$, then $u$ coincides, on a neighborhood $U$ of $x$, with a section of the form $\sum h_i \otimes (t_i/s_i)$, where the $h_i$ belong to $\Gamma(U, \mathcal{F})$, the $t_i$ to $\Gamma(U, \mathcal{O}_X)$, and the $s_i$ to $\Gamma(U, \mathcal{F})$. As by hypothesis the images of the $s_i$ in $\Gamma(f^{-1}(U), \mathcal{O}_X)$ are regular, we see that $u \circ f$ is defined at every point of $f^{-1}(U)$; in other words, we have

$$(20.1.11.2) \quad f^{-1}(\text{dom}(u)) \subset \text{dom}(u \circ f).$$

We will see later (20.6.5, (i)) examples (with which will be defined below.

Consider in particular the case where $M_f = M_X$; then, if $\mathcal{F}$ is an invertible $\mathcal{O}_X$-module, the image in $\mathcal{M}_X(f^*(\mathcal{F}))$, under $\Gamma(\theta^\#)$, of a regular meromorphic section of $\mathcal{F}$ over $X$ (20.1.8) is a regular meromorphic section of $f^*(\mathcal{F})$ over $X'$, as it follows immediately from the definition of its sections, and from the fact that a homomorphism of rings sends an invertible element to an invertible element.

Let $f' : X'' \rightarrow X'$ be a second morphism of ringed spaces, and suppose that $M_f = M_X$ and $M_{f'} = M_{X'}$; then, if we set $f'' = f \circ f'$, we also have $M_{f''} = M_{X''}$, and we immediately see that for every meromorphic section $u$ of $\mathcal{F}$ over $X$, we have $u \circ f'' = (u \circ f) \circ f'$.

**Proposition (20.1.12).** — If the morphism $f : X' \rightarrow X$ is flat (04.6.7.1), then we have $M_f = M_X$, and the homomorphism $\phi \mapsto \phi \circ f$ is defined on all of $M(X)$. In addition, if $f$ is a (flat) morphism of locally ringed spaces, then we have $\text{dom}(\phi \circ f) = f^{-1}(\text{dom}(\phi))$; if in addition $f$ is surjective (thus faithfully flat), then the homomorphism $\phi \mapsto \phi \circ f$ is injective.

**Proof.** The first assertion follows from the fact that, if $B$ is an $A$-algebra which is a flat $A$-module, then every element of $A$ is a divisor of 0 in $A$ is not a divisor of 0 in $B$ (04.6.3.4). To prove the other assertions, note that, for every $x' \in X'$, if $x = f(x')$, then $\mathcal{O}_{X,x'}$ is a flat $\mathcal{O}_{X,x}$-module, and as the homomorphism $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X',x'}$ is local by hypothesis, it is injective ((04.6.5.1) and (04.6.6.2)); if we set $A = \mathcal{O}_{X,x', B = \mathcal{O}_{X',x'}$, such that $A$ identifies with a subring of $B$, then $(f^*(\mathcal{O}_X))_{x'}$ is equal to $S^{-1}A \otimes_A B = S^{-1}B$, where $S$ is the set of regular elements of $A$, $(\mathcal{O}_X')_{x'}$ is equal to $T^{-1}B$, where $T$ is the set of regular elements of $B$, and as we have seen that $S \subset T$, the homomorphism $S^{-1}B \rightarrow T^{-1}B$ is injective; in other words, this proves that the homomorphism (20.1.11.1) $f^*(\mathcal{O}_X) \rightarrow \mathcal{O}_{X'}$ is injective (hence the last assertion of the statement). The quotient $f^*(\mathcal{O}_X)/\mathcal{O}_{X'}$ identifies with a $\mathcal{O}_{X'}$-submodule of $\mathcal{O}_X'/\mathcal{O}_{X'}$; and $(f^*(\mathcal{O}_X))/\mathcal{O}_{X'}$ identifies with $\mathcal{O}_{X'}/\mathcal{O}_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'/x'}$. Then suppose that $x \notin \text{dom}(\phi)$; the image of $\phi_x$ in $(\mathcal{O}_X/\mathcal{O}_{X'})_x$ is therefore $\neq 0$; by faithful flatness, we deduce that it is the same for the image of $(\phi \circ f)_{x'}$, so $x' \notin \text{dom}(\phi \circ f)$, which finishes the proof. □

**Remark (20.1.13).** — Let $X$ be a reduced complex analytic space; then the notion of a meromorphic function on $X$ defined above coincides with the usual notion. Consider on the other hand a prescheme $Y$, locally of finite type over the field $\mathbf{C}$; we then know that we can associate to $Y$ an analytic space $Y^\text{an}$ having the same underlying topological space, and the canonical morphism $f : Y^\text{an} \rightarrow Y$ is flat [?]; by virtue of (20.1.12), the canonical homomorphism $u \mapsto u \circ f$ from $M(Y)$ to $M(Y^\text{an})$ is therefore always defined and is injective; but it is not surjective in general. For example, when $Y = \mathbf{V}_0^r$ (Err4, 14) is the affine space of dimension $r$ over $\mathbf{C}$, $M(Y)$ canonically identifies with the field $\mathcal{R}(Y)$ of rational functions on $Y$ (20.2.13, (i)), while $M(Y^\text{an})$ is the field of usual meromorphic functions on $\mathcal{C}'$. Because of this fact, it is often preferable, in algebraic geometry, to abstain from the terminology introduced in this section, and to use the equivalent terminology of “pseudo-function” which will be defined below.
20.2. Pseudo-morphisms and pseudo-functions

§21. DIVISORS
Bibliography


[Sam53b] P. Samuel, Commutative algebra (notes by D. Herzig), 1953.


