ELEMENTARY GLOBAL STUDY OF SOME CLASSES OF MORPHISMS (EGA II)


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SUMMARY

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The various classes of morphisms studied in this chapter are used extensively in cohomological methods; further study using these methods will be done in Chapter III, where we make particular use of §§2, 4, and 5 of Chapter II. On a first reading, §8 can be omitted: it supplements the formalism developed in §§1 and 3, reducing to easy applications of this formalism, and we will use it less consistently than the other results of this chapter.

§1. AFFINE MORPHISMS

1.1. S-preschemes and $\mathscr{O}_S$-algebras.

(1.1.1). Let S be a prescheme, X an S-prescheme, and $f: X \to S$ its structure morphism. We know (**0**, 4.2.4) that the direct image $f_*(\mathscr{O}_X)$ is an $\mathscr{O}_S$-algebra, which we denote $\mathscr{A}(X)$ when there is little **II | 6** chance of confusion; if U is an open subset of S, then we have

$$\mathscr{A}(f^{-1}(U)) = \mathscr{A}(X)|U.$$

Similarly, for every $\mathscr{O}_X$-module $\mathscr{F}$ (resp. every $\mathscr{O}_X$-algebra $\mathscr{B}$), we write $\mathscr{A}(\mathscr{F})$ (resp. $\mathscr{A}(\mathscr{B})$) for the direct image $f_*(\mathscr{F})$ (resp. $f_*(\mathscr{B})$) which is an $\mathscr{A}(X)$-module (resp. an $\mathscr{A}(X)$-algebra) and not only an $\mathscr{O}_S$-module (resp. an $\mathscr{O}_S$-algebra).

(1.1.2). Let Y be a second S-prescheme, $g: Y \to S$ its structure morphism, and $h: X \to Y$ an S-morphism; we then have the commutative diagram

(1.1.2.1)
$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & {\scriptstyle f}\searrow \qquad \swarrow{\scriptstyle g} & \\ & S. & \end{array}$$

We have by definition $h = (\psi, \theta)$, where $\theta: \mathscr{O}_Y \to h_*(\mathscr{O}_X) = \psi_*(\mathscr{O}_X)$ is a homomorphism of sheaves of rings; we induce (**0**, 4.2.2) a homomorphism of $\mathscr{O}_S$-algebras $g_*(\theta): g_*(\mathscr{O}_Y) \to g_*(h_*(\mathscr{O}_X)) = f_*(\mathscr{O}_X)$, in other words, a homomorphism of $\mathscr{O}_S$-algebras $\mathscr{A}(Y) \to \mathscr{A}(X)$, which we denote by $\mathscr{A}(h)$. If $h': Y \to Z$ is a second S-morphism, then it is immediate that $\mathscr{A}(h' \circ h) = \mathscr{A}(h) \circ \mathscr{A}(h')$. We havve thus define a *contravariant functor* $\mathscr{A}(X)$ from the category of S-preschemes to the category of $\mathscr{O}_S$-algebras.

Now let $\mathscr{F}$ be an $\mathscr{O}_X$-module, $\mathscr{G}$ an $\mathscr{O}_Y$-module, and $u: \mathscr{G} \to \mathscr{F}$ an h-morphism, that is (**0**, 4.4.1) a homomorphism of $\mathscr{O}_Y$-modules $\mathscr{G} \to h_*(\mathscr{F})$. Then $g_*(u): g_*(\mathscr{G}) \to g_*(h_*(\mathscr{F})) = f_*(\mathscr{F})$ is a homomorphism $\mathscr{A}(\mathscr{G}) \to \mathscr{A}(\mathscr{F})$ of $\mathscr{O}_S$-modules, which we denote by $\mathscr{A}(u)$; in addition, the pair $(\mathscr{A}(h), \mathscr{A}(u))$ form a *di-homomorphism* from the $\mathscr{A}(Y)$-module $\mathscr{A}(\mathscr{G})$ to the $\mathscr{A}(X)$-module $\mathscr{A}(\mathscr{F})$.

(1.1.3). If we fix the prescheme S, then we can consider the pairs $(X, \mathscr{F})$, where X is an S-prescheme and $\mathscr{F}$ is an $\mathscr{O}_X$-module, as forming a *category*, by defining a *morphism* $(X, \mathscr{F}) \to (Y, \mathscr{G})$ as a pair (h, u), where $h : X \to Y$ is an S-morphism and $u : \mathscr{G} \to \mathscr{F}$ is an h-morphism. We can then say that $(\mathscr{A}(X), \mathscr{A}(\mathscr{F}))$ is a *contravariant functor* with values in the category whose objects are pairs consisting of an $\mathscr{O}_S$-algebra and a module over that algebra, and the morphisms are the di-homomorphisms.

1.2. Affine preschemes over a prescheme.

Definition (1.2.1). — Let X be an S-prescheme, $f : X \to S$ its structure morphism. We say that X is *affine over* S if there exists a cover (S_α) of S by affine open sets such that for all α, the induced prescheme on X by the open set $f^{-1}(S_\alpha)$ is affine.

Example (1.2.2). — Every closed subprescheme of S is an affine S-prescheme over S ((**I**, 4.2.3) and (**I**, 4.2.4)).

Remark (1.2.3). — An affine prescheme X over S is not necessarily an affine scheme, as the example $X = S$ shows (1.2.2). On the other hand, if an affine scheme X is an S-prescheme, then X is not necessarily affine over S (see Example (1.3.3)). However, remember that if S is a *scheme*, then every **II** | 7
S-prescheme which is an affine scheme is affine over S (**I**, 5.5.10).

Proposition (1.2.4). — *Every S-prescheme which is affine over S is separated over S (in other words, it is an S-scheme).*

Proof. This follows immediately from (**I**, 5.5.5) and (**I**, 5.5.8). □

Proposition (1.2.5). — *Let X be an S-scheme affine over S, $f : X \to S$ its structure morphism. For every open $U \subset S$, $f^{-1}(U)$ is affine over U.*

Proof. By Definition (1.2.1), we can reduce to the case where $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine; then $f = ({}^a\phi, \widetilde{\phi})$, where $\phi : A \to B$ is a homomorphism. As the $D(g)$ for $g \in A$ form a basis for S, we reduce to the case where $U = D(g)$; but we then know that $f^{-1}(U) = D(\phi(g))$ (**I**, 1.2.2.2), hence the proposition. □

Proposition (1.2.6). — *Let X be an S-scheme affine over S, $f : X \to S$ its structure morphism. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $f_*(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_S$-module.*

Proof. Taking into account Proposition (1.2.4), this follows from (**I**, 9.2.2, a). □

In particular, the $\mathscr{O}_S$-algebra $\mathscr{A}(X) = f_*(\mathscr{O}_X)$ is *quasi-coherent*.

Proposition (1.2.7). — *Let X be an S-scheme affine over S. For every S-prescheme Y, the map $h \mapsto \mathscr{A}(h)$ from the set $\operatorname{Hom}_S(Y, X)$ to the set $\operatorname{Hom}(\mathscr{A}(X), \mathscr{A}(Y))$ (1.1.2) is bijective.*

Proof. Let $f : X \to S$ and $g : Y \to S$ be the structure morphisms. First, suppose that $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine; we must prove that for every homomorphism $\omega : f_*(\mathscr{O}_X) \to g_*(\mathscr{O}_Y)$ of $\mathscr{O}_S$-algebras, there exists a unique S-morphism $h : Y \to X$ such that $\mathscr{A}(h) = \omega$. By definition, for every open $U \subset S$, ω defines a homomorphism $\omega_U = \Gamma(U, \omega) : \Gamma(f^{-1}(U), \mathscr{O}_X) \to \Gamma(g^{-1}(U), \mathscr{O}_Y)$ of $\Gamma(U, \mathscr{O}_S)$-algebras. In particular, for $U = S$, this gives a homomorphism $\phi : \Gamma(X, \mathscr{O}_X) \to \Gamma(Y, \mathscr{O}_Y)$ of $\Gamma(S, \mathscr{O}_S)$-algebras, to which corresponds a well-defined S-morphism $h : Y \to X$, since X is affine (**I**, 2.2.4). It remains to prove that $\mathscr{A}(h) = \omega$, in other words, for every open set U of a basis for S, ω_U coincides with the homomorphism of algebras ϕ_U corresponding to the S-morphism $g^{-1}(U) \to f^{-1}(U)$, a restiction of h. We can reduce to the case where $U = D(\lambda)$, with $\lambda \in S$; then, if $f = ({}^a\rho, \widetilde{\rho})$, where $\rho : A \to B$ is a ring homomorphism, we have $f^{-1}(U) = D(\mu)$, where $\mu = \rho(\lambda)$, and $\Gamma(f^{-1}(U), \mathscr{O}_X)$ is the ring of fractions B_μ; the diagram

$$\begin{array}{ccc} B & \xrightarrow{\ \phi\ } & \Gamma(Y, \mathscr{O}_Y) \\ \downarrow & & \downarrow \\ B_\mu & \xrightarrow{\ \phi_U\ } & \Gamma(g^{-1}(U), \mathscr{O}_Y) \end{array}$$

is commutative, and so is the analogous diagram where ϕ_U is replaced by ω_U; the equality $\phi_U = \omega_U$ then follows from the universal property of rings of fractions (**0**, 1.2.4).

We now pass to the general case; let (S_α) be a cover of S by affine open sets such that the $f^{-1}(S_\alpha)$ **II** | 8
are affine. Then every homomorphism $\omega : \mathscr{A}(X) \to \mathscr{A}(Y)$ of $\mathscr{O}_S$-algebras gives by restriction a family of homomorphisms

$$\omega_\alpha : \mathscr{A}(f^{-1}(S_\alpha)) \longrightarrow \mathscr{A}(g^{-1}(S_\alpha))$$

of $\mathscr{O}_{S_\alpha}$-algebras, hence a family of S_α-morphisms $h_\alpha : g^{-1}(S_\alpha) \to f^{-1}(S_\alpha)$ by the above. It remains to see that for every affine open set of a basis for $S_\alpha \cap S_\beta$, the restriction of h_α and h_β to $g^{-1}(U)$ coincide, which is evident since by the above, these restrictions both correspond to the homomorphism $\mathscr{A}(X)|U \to \mathscr{A}(Y)|U$, a restriction of ω. □

Corollary (1.2.8). — *Let X and Y be two S-schemes which are affine over S. For an S-morphism $h : Y \to X$ to be an isomorphism, it is necessary and sufficient for $\mathscr{A}(h) : \mathscr{A}(X) \to \mathscr{A}(Y)$ to be an isomorphism.*

Proof. This follows immediately from Proposition (1.2.7) and from the functorial nature of $\mathscr{A}(X)$. □

1.3. Affine preschemes over S associated to an $\mathscr{O}_S$-algebra.

Proposition (1.3.1). — *Let S be a prescheme. For every quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$, there exists a prescheme X affine over S, defined up to unique S-isomorphism, such that $\mathscr{A}(X) = \mathscr{B}$.*

Proof. The uniqueness follows from Corollary (1.2.8); we prove the existence of X. For every affine open $U \subset S$, let X_U be the prescheme $\operatorname{Spec}(\Gamma(U, \mathscr{B}))$; as $\Gamma(U, \mathscr{B})$ is a $\Gamma(U, \mathscr{O}_S)$-algebras, X_U is an S-prescheme (**I**, 1.6.1). In addition, as $\mathscr{B}$ is quasi-coherent, the $\mathscr{O}_S$-algebra $\mathscr{A}(X_U)$ canonically identifies with $\mathscr{B}|U$ ((**I**, 1.3.7), (**I**, 1.3.13), (**I**, 1.6.3)). Let V be a second affine open subset of S, and let $X_{U,V}$ be the prescheme induced by X_U on $f_U^{-1}(U \cap V)$, where f_U denotes the structure morphism $X_U \to S$; $X_{U,V}$ and $X_{V,U}$ are affine over $U \cap V$ (1.2.5), and by definition $\mathscr{A}(X_{U,V})$ and $\mathscr{A}(X_{V,U})$ canonically identify with $\mathscr{B}|(U \cap V)$. Hence there is (1.2.8) a canonical S-isomorphism $\theta_{U,V} : X_{U,V} \to X_{U,V}$; in addition, if W is a third affine open subset of S, and if $\theta'_{U,V}$, $\theta'_{V,W}$, and $\theta'_{U,W}$ are the restrictions of $\theta_{U,V}$, $\theta_{V,W}$, and $\theta_{U,W}$ to the inverse images of $U \cap V \cap W$ in X_V, X_W, and X_W respectively under the structure morphisms, then we have $\theta'_{U,V} \circ \theta'_{V,W} = \theta'_{U,W}$. As a result, there exists a prescheme X, a cover (T_U) of X by affine open sets, and for every U an isomorphism $\phi_U : X_U \to T_U$, such that ϕ_U maps $f_U^{-1}(U \cap V)$ to $T_U \cap T_V$, and we have $\theta_{U,V} = \phi_U^{-1} \circ \phi_V$ (**I**, 2.3.1). The morphism $g_U = f_U \circ \phi_U^{-1}$ makes T_U an S-prescheme, and the morphisms g_U and g_V coincide on $T_U \cap T_V$, hence X is an S-prescheme. In addition, it is clear by definition that X is affine over S and that $\mathscr{A}(T_U) = \mathscr{B}|U$, hence $\mathscr{A}(X) = \mathscr{B}$. □

We say that the S-scheme X defined in this way is *associated to the $\mathscr{O}_S$-algebra $\mathscr{B}$*, or is the *spectrum of $\mathscr{B}$*, and we denote it by $\operatorname{Spec}(\mathscr{B})$.

Corollary (1.3.2). — *Let X be a prescheme affine over S, $f : X \to S$ the structure morphism. For every affine open $U \subset S$, the induced prescheme on $f^{-1}(U)$ is the affine scheme with ring $\Gamma(U, \mathscr{A}(X))$.* **II** | 9

Proof. As we can suppose that X is associated to an $\mathscr{O}_S$-algebra by Propositions (1.2.6) and (1.3.1), the corollary follows from the construction of X described in Proposition (1.3.1). □

Example (1.3.3). — Let S be the affine plane over a field K, where the point 0 has been doubled (**I**, 5.5.11); with the notation of (**I**, 5.5.11), S is the union of two affine open sets Y_1 and Y_2; if f is the open immersion $Y_1 \to S$, then $f^{-1}(Y_2) = Y_1 \cap Y_2$ is not an affine open set in Y_1 (*loc. cit.*), hence we have an example of an affine scheme which is not affine over S.

Corollary (1.3.4). — *Let S be an affine scheme; for an S-prescheme X to be affine over S, it is necessary and sufficient for X to be an affine scheme.*

Corollary (1.3.5). — *Let X be a prescheme affine over a prescheme S, and let Y be an X-prescheme. For Y to be affine over S, it is necessary and sufficient for Y to be affine over X.*

Proof. We immediately reduce to the case where S is an affine scheme, and then we can reduce to the case where X is an affine scheme (1.3.4); the two conditions of the statement then give that Y is an affine scheme (1.3.4). □

(1.3.6). Let X be a prescheme affine over S. To define a prescheme Y affine *over* X, it is equivalent, by Corollary (1.3.5), to give a prescheme Y affine *over* S, and an S-morphism $g : Y \to X$; in other words (Proposition (1.3.1) and (1.2.7)), it is equivalent to give a quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$ and a homomorphism $\mathscr{A}(X) \to \mathscr{B}$ of $\mathscr{O}_S$-algebras (which can be considered as defining on $\mathscr{B}$ an $\mathscr{A}(X)$-algebra structure). If $f : X \to S$ is the structure morphism, then we have $\mathscr{B} = f_*(g_*(\mathscr{O}_Y))$.

Corollary (1.3.7). — *Let X be a prescheme affine over S; for X to be of finite type over S, it is necessary and sufficient for the quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{A}(X)$ to be of finite type* (**I**, 9.6.2)*.*

Proof. By definition (**I**, 9.6.2), we can reduce to the case where S is affine; then X is an affine scheme (1.3.4), and if $S = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$, then $\mathscr{A}(X)$ is the $\mathscr{O}_S$-algebra $\widetilde{B}$; as $\Gamma(U, \widetilde{B}) = B$, the corollary follows from (**I**, 9.6.2) and (**I**, 6.3.3). □

Corollary (1.3.8). — *Let X be a prescheme affine over S; for X to be reduced, it is necessary and sufficient for the quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{A}(X)$ to be reduced* (**0**, 4.1.4)*.*

Proof. The question is local on S; by Corollary (1.3.2), the corollary follows from (**I**, 5.1.1) and (**I**, 5.1.4). □

1.4. Quasi-coherent sheaves over a prescheme affine over S.

Proposition (1.4.1). — *Let X be a prescheme affine over S, Y an S-prescheme, and $\mathscr{F}$ (resp. $\mathscr{G}$) a quasi-coherent $\mathscr{O}_X$-module (resp. an $\mathscr{O}_Y$-module). Then the map $(h, u) \mapsto (\mathscr{A}(h), \mathscr{A}(u))$ from the set of morphism $(Y, \mathscr{G}) \to (X, \mathscr{F})$ to the set of di-homomorphisms $(\mathscr{A}(X), \mathscr{A}(\mathscr{F})) \to (\mathscr{A}(Y), \mathscr{A}(\mathscr{G}))$ ((1.1.2) and (1.1.3)) is bijective.*

Proof. The proof follows exactly as that of Proposition (1.2.7) by using (**I**, 2.2.5) and (**I**, 2.2.4), and the details are left to the reader. □

Corollary (1.4.2). — *If, in addition to the hypotheses of Proposition (1.4.1), we suppose that Y is affine over S, then for (h, u) to be an isomorphism, it is necessary and sufficient for $(\mathscr{A}(h), \mathscr{A}(u))$ to be a di-isomorphism.*

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Proposition (1.4.3). — *For every pair $(\mathscr{B}, \mathscr{M})$ consisting of a quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$ and a quasi-coherent $\mathscr{B}$-module $\mathscr{M}$* (considered as an $\mathscr{O}_S$-module or as a $\mathscr{B}$-module, which are equivalent (**I**, 9.6.1))*, there exists a pair $(X, \mathscr{F})$ consisting of a prescheme X affine over S and of a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, such that $\mathscr{A}(X) = \mathscr{B}$ and $\mathscr{A}(\mathscr{F}) = \mathscr{M}$; in addition, this couple is determined up to unique isomorphism.*

Proof. The uniqueness follows from Proposition (1.4.1) and Corollary (1.4.2); the existence is proved as in Proposition (1.3.1), and we leave the details to the reader. □

We denote by $\widetilde{\mathscr{M}}$ the $\mathscr{O}_X$-module $\mathscr{F}$, and we say that it is *associated* to the quasi-coherent $\mathscr{B}$-module $\mathscr{M}$; for every affine open $U \subset S$, $\mathscr{M}|p^{-1}(U)$ (where p is the structure morphism $X \to S$) is canonically isomorphic to $(\Gamma(U, \mathscr{M}))^{\sim}$.

Corollary (1.4.4). — *On the category of quasi-coherent $\mathscr{B}$-modules, $\widetilde{\mathscr{M}}$ is an additive covariant exact functor in $\mathscr{M}$, which commutes with inductive limitd and direct sums.*

Proof. We immediately reduce to the case where S is affine, and the corollary then follows from (**I**, 1.3.5), (**I**, 1.3.9), and (**I**, 1.3.11). □

Corollary (1.4.5). — *Under the hypotheses of Proposition (1.4.3), for $\widetilde{\mathscr{M}}$ to be an $\mathscr{O}_X$-module of finite type, it is necessary and sufficient for $\mathscr{M}$ to be a $\mathscr{B}$-module of finite type.*

Proof. We immediately reduce to the case where $S = \operatorname{Spec}(A)$ is an affine scheme. Then $\mathscr{B} = \widetilde{B}$, where B is an A-algebra of finite type (**I**, 9.6.2), and $\mathscr{M} = \widetilde{M}$, where M is a B-module (**I**, 1.3.13); *over the prescheme* X, $\mathscr{O}_X$ is associated to the ring B and $\widetilde{\mathscr{M}}$ to the B-module M; for $\widetilde{\mathscr{M}}$ to be of finite type, it is therefore necessary and sufficient for M to be of finite type (**I**, 1.3.13), hence our assertion. □

Proposition (1.4.6). — *Let Y be a prescheme affine over S, X and X' two preschemes affine over Y* (hence also over S (1.3.5))*. Let $\mathscr{B} = \mathscr{A}(Y)$, $\mathscr{A} = \mathscr{A}(X)$, and $\mathscr{A}' = \mathscr{A}(X')$. Then $X \times_Y X'$ is affine over Y* (thus also over S)*, and $\mathscr{A}(X \times_Y X')$ canonically identifies with $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$.*

Proof. By (**I**, 9.6.1), $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ is a quasi-coherent $\mathscr{B}$-algebra, thus also a quasi-coherent $\mathscr{O}_S$-algebra (**I**, 9.6.1); let Z be the spectrum of $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ (1.3.1). The canonical $\mathscr{B}$-homomorphisms $\mathscr{A} \to \mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ and $\mathscr{A}' \to \mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ correspond (1.2.7) to Y-morphisms $Z \to X$ and $p' : Z \to X'$. To see that the triple (Z, p, p') is a product $X \times_Y X'$, we can reduce to the case where S is an affine scheme with ring C (**I**, 3.2.6.4). But then Y, X, and X' are affine schemes (1.3.4) whose rings B, A, and A' are C-algebras such that $\mathscr{B} = \widetilde{B}$, $\mathscr{A} = \widetilde{A}$, and $\mathscr{A}' = \widetilde{A'}$. We then know (**I**, 1.3.13) that $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ canonically identifies with the $\mathscr{O}_S$-algebra $(A \otimes_B A')^\sim$, hence the ring $A(Z)$ identifies with $A \otimes_B A'$ and the morphisms p and p' correspond to the canonical homomorphisms $A \to A \otimes_B A'$ and $A' \to A \otimes_B A'$. The proposition then follows from (**I**, 3.2.2). □

Corollary (1.4.7). — *Let $\mathscr{F}$ (resp. $\mathscr{F}'$) be a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_{X'}$-module); then $\mathscr{A}(\mathscr{F} \otimes_Y \mathscr{F}')$ canonically identifies with $\mathscr{A}(\mathscr{F}) \otimes_{\mathscr{A}(Y)} \mathscr{A}(\mathscr{F}')$.*

Proof. We know that $\mathscr{F} \otimes_Y \mathscr{F}'$ is quasi-coherent over $X \times_Y X'$ (**I**, 9.1.2). Let $g : Y \to S$, $f : X \to Y$, and $f' : X' \to Y$ be the structure morphisms, such that the structure morphism $h : Z \to S$ is equal to **II** | 11
$g \circ f \circ p$ and to $g \circ f' \circ p'$. We define a canonical homomorphism

$$\mathscr{A}(\mathscr{F}) \otimes_{\mathscr{A}(Y)} \mathscr{A}(\mathscr{F}') \longrightarrow \mathscr{A}(\mathscr{F} \otimes_Y \mathscr{F}')$$

in the following way: for every open $U \subset S$, we have canonical homomorphisms $\Gamma(f^{-1}(g^{-1}(U)), \mathscr{F}) \to \Gamma(h^{-1}(U), p^*(\mathscr{F}))$ and $\Gamma(f'^{-1}(g^{-1}(U)), \mathscr{F}') \to \Gamma(h^{-1}(U), p'^*(\mathscr{F}'))$ (**0**, 4.4.3), thus we obtain a canonical homomorphism

$$\Gamma(f^{-1}(g^{-1}(U)), \mathscr{F}) \otimes_{\Gamma(g^{-1}(U), \mathscr{O}_Y)} \Gamma(f'^{-1}(g^{-1}(U)), \mathscr{F}') \longrightarrow \Gamma(h^{-1}(U), p^*(\mathscr{F})) \otimes_{\Gamma(h^{-1}(U), \mathscr{O}_Z)} \Gamma(h^{-1}(U), p'^*(\mathscr{F}')).$$

To see that we have defined an isomorphism of $\mathscr{A}(Z)$-modules, we can reduce to the case where S (and as a result X, X', Y, and $X \times_Y X'$) are affine scheme, and (with the notation of Proposition (1.4.6)), $\mathscr{F} = \widetilde{M}$, $\mathscr{F}' = \widetilde{M'}$, where M (resp. M') is an A-module (resp. an A'-module). Then $\mathscr{F} \otimes_Y \mathscr{F}'$ identifies with the sheaf on $X \times_Y X'$ associated to the $(A \otimes_B A')$-module $M \otimes_B M'$ (**I**, 9.1.3), and the corollary follows from the canonical identification of the $\mathscr{O}_S$-modules $(M \otimes_B M')^\sim$ and $\widetilde{M} \otimes_{\widetilde{B}} \widetilde{M'}$ (where M, M', and B are considered as C-modules) ((**I**, 1.3.12) and (**I**, 1.6.3)). □

If we apply Corollary (1.4.7) in particular to the case where $X = Y$ and $\mathscr{F}' = \mathscr{O}_{X'}$, then we see that the $\mathscr{A}'$-module $\mathscr{A}(f'^*(\mathscr{F}))$ identifies with $\mathscr{A}(\mathscr{F}) \otimes_{\mathscr{B}} \mathscr{A}'$.

(1.4.8). In particular, when $X = X' = Y$ (X being affine over S), we see that if $\mathscr{F}$ and $\mathscr{G}$ are two quasi-coherent $\mathscr{O}_X$-modules, then we have

(1.4.8.1) $$\mathscr{A}(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) = \mathscr{A}(\mathscr{F}) \otimes_{\mathscr{A}(X)} \mathscr{A}(\mathscr{G})$$

up to canonical functorial isomorphism. If in addition $\mathscr{F}$ admits a finite presentation, then it follows from (**I**, 1.6.3) and (**I**, 1.3.12) that

(1.4.8.2) $$\mathscr{A}(\mathscr{H}om_X(\mathscr{F}, \mathscr{G})) = \mathscr{H}om_{\mathscr{A}(X)}(\mathscr{A}(\mathscr{F}), \mathscr{A}(\mathscr{G}))$$

up to canonical isomorphism.

Remark (1.4.9). — If X and X' are two preschemes affine over S, then the sum $X \sqcup X'$ is also affine over S, as the sum of two affine schemes is an affine scheme.

Proposition (1.4.10). — *Let S be a prescheme, $\mathscr{B}$ a quasi-coherent $\mathscr{O}_S$-algebra, and $X = \operatorname{Spec}(\mathscr{B})$. For a quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{B}$, $\widetilde{\mathscr{J}}$ is quasi-coherent sheaf of ideals of $\mathscr{O}_X$, and the closed subprescheme Y of X defined by $\widetilde{\mathscr{J}}$ is canonically isomorphic to* $\operatorname{Spec}(\mathscr{B}/\mathscr{J})$.

Proof. It follows immediately from (**I**, 4.2.3) that Y is affine over S; by Proposition (1.3.1), we reduce to the case where S is affine, and the proposition then follows immediately from (**I**, 4.1.2). □

We can also express the result of Proposition (1.4.10) by saying that if $h : \mathscr{B} \to \mathscr{B}'$ is a *surjective* homomorphism of quasi-coherent $\mathscr{O}_S$-algebras, $\mathscr{A}(h) : \operatorname{Spec}(\mathscr{B}') \to \operatorname{Spec}(\mathscr{B})$ is a *closed immersion*. **II** | 12

Proposition (1.4.11). — *Let S be a prescheme, $\mathscr{B}$ a quasi-coherent $\mathscr{O}_S$-algebra, and $X = \operatorname{Spec}(\mathscr{B})$. For every quasi-coherent sheaf of ideals $\mathscr{K}$ of $\mathscr{O}_S$, we have (denoting by f the structure morphism $X \to S$) $f^*(\mathscr{K})\mathscr{O}_X = (\mathscr{K}\mathscr{B})^\sim$ up to canonical isomorphism.*

Proof. The question being local on S, we can reduce to the case where $S = \operatorname{Spec}(A)$ is affine, and in this case the proposition is none other than (**I**, 1.6.9). □

1.5. Change of base prescheme.

Proposition (1.5.1). — *Let X be a prescheme affine over S. For every extension $g : S' \to S$ of the base prescheme, $X' = X_{(S')} = X \times_S S'$ is affine over S'.*

Proof. If f' is the projection $X' \to S'$, then it suffices to prove that $f'^{-1}(U')$ is an affine open set for every affine open subset U' of S' such that $g(U')$ is contained in an affine open subset U of S (1.2.1); we can thus reduce to the case where S and S' are affine, and it suffices to prove that X' is then an affine scheme (1.3.4). But then (1.3.4) X is an affine scheme, and if A, A', and B are the rings of S, S', and X respectively, then we know that X' is the affine scheme with ring $A' \otimes_A B$ (**I**, 3.2.2). □

Corollary (1.5.2). — *Under the hypotheses of Proposition (1.5.1), let $f : X \to S$ be the structure morphism, $f' : X' \to S'$ and $g' : X' \to X$ the projections, such that the diagram*

$$\begin{array}{ccc} X & \xleftarrow{\;g'\;} & X' \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ S & \xleftarrow{\;g\;} & S' \end{array}$$

is commutative. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, there exists a canonical isomorphism of $\mathscr{O}_{S'}$-modules

(1.5.2.1) $$u : g^*(f_*(\mathscr{F})) \simeq f'_*(g'^*(\mathscr{F})).$$

In particular, there exists a canonical isomorphism from $\mathscr{A}(X')$ to $g^*(\mathscr{A}(X))$.

Proof. To define u, it suffices to define a homomorphism

$$v : f_*(\mathscr{F}) \longrightarrow g_*(f'_*(g'^*(\mathscr{F}))) = f_*(g'_*(g'^*(\mathscr{F})))$$

and to set $u = v^\sharp$ (**0**, 4.4.3). We take $v = f_*(\rho)$, where ρ is the canonical homomorphism $\mathscr{F} \to g'_*(g'^*(\mathscr{F}))$ (**0**, 4.4.3). To prove that u is an isomorphism, we can reduce to the case where S and S', hence X and X', are affine; with the notation of Proposition (1.5.1), we then have $\mathscr{F} = \widetilde{M}$, where M is a B-module. We then note immediately that $g^*(f_*(\mathscr{F}))$ and $f'_*(g'^*(\mathscr{F}))$ are both equal to the $\mathscr{O}_{S'}$-module associated to the A'-module $A' \otimes_A M$ (where M is considered as an A-module), and that u is the homomorphism associated to the identity ((**I**, 1.6.3), (**I**, 1.6.5), (**I**, 1.6.7)). □

Remark (1.5.3). — We do not have that Corollary (1.5.2) remains true when X is not assumed affine over S, even when $S' = \operatorname{Spec}(k(s))$ ($s \in S$) and $S' \to S$ is the canonical morphism (**I**, 2.4.5)—in which case X' is none other than the *fibre* $f^{-1}(s)$ (**I**, 3.6.2). In other words, when X is not affine over S, the operation "direct image of quasi-coherent sheaves" does not commute with the operation **II** | 13
of "passing to fibres". However, we will see in Chapter III (**III**, 4.2.4) a result in this sense, of an "asymptotic" nature, valid for *coherent* sheaves on X when f is proper (5.4) and S is Noetherian.

Corollary (1.5.4). — *For every prescheme X affine over S and every $s \in S$, the fibre $f^{-1}(s)$ (where f denoted the structure morphism $X \to S$) is an affine scheme.*

Proof. It suffices to apply Proposition (1.5.1) with $S' = \operatorname{Spec}(k(s))$ and to use Corollary (1.3.4). □

Corollary (1.5.5). — *Let X be an S-prescheme, S' a prescheme affine over S; then $X' = X_{(S')}$ is a prescheme affine over X. In addition, if $f : X \to S$ is the structure morphism, then there is a canonical isomorphism of $\mathscr{O}_X$-algebras $\mathscr{A}(X') \simeq f^*(\mathscr{A}(S'))$, and for every quasi-coherent $\mathscr{A}(S')$-module $\mathscr{M}$, a canonical di-isomorphism $f^*(\mathscr{M}) \simeq \mathscr{A}(f'^*(\widetilde{\mathscr{M}}))$, denoting by $f' = f_{(S')}$ the structure morphism $X' \to S'$.*

Proof. It suffices to swap the roles of X and S' in (1.5.1) and (1.5.2). □

(1.5.6). Now let S, S' be two preschemes, $q : S' \to S$ a morphism, $\mathscr{B}$ (resp. $\mathscr{B}'$) a quasi-coherent $\mathscr{O}_S$-algebra (resp. $\mathscr{O}_{S'}$-algebra), $u : \mathscr{B} \to \mathscr{B}'$ a q-morphism (that is, a homomorphism $\mathscr{B} \to q_*(\mathscr{B}')$ of $\mathscr{O}_S$-algebras). If $X = \operatorname{Spec}(\mathscr{B})$, $X' = \operatorname{Spec}(\mathscr{B}')$, then we canonically obtain a morphism

$$v = \operatorname{Spec}(u) : X' \longrightarrow X$$

such that the diagram

(1.5.6.1)
$$\begin{array}{ccc} X' & \xrightarrow{v'} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{q} & S \end{array}$$

is commutative (the vertical arrows being the structure morphisms). Indeed, the data of u is equivalent to that of a homomorphism of quasi-coheren $\mathscr{O}_{S'}$-algebras $u^\sharp : q^*(\mathscr{B}) \to \mathscr{B}'$ (**0**, 4.4.3); this thus canonically defines an S'-morphism

$$w : \operatorname{Spec}(\mathscr{B}') \longrightarrow \operatorname{Spec}(q^*(\mathscr{B}))$$

such that $\mathscr{A}(w) = u^\sharp$ (1.2.7). On the other hand, it follows from (1.5.2) that $\operatorname{Spec}(q^*(\mathscr{B}))$ canonically identifies with $X \times_S S'$; the morphism v is the composition $X' \xrightarrow{w} X \times_S S' \xrightarrow{p_1} X$ of w with the first projection, and the commutativity of (1.5.6.1) follows from the definitions. Let U (resp. U') be an affine open of S (resp. S') such that $q(U') \subset U$, $A = \Gamma(U, \mathscr{O}_S)$, $A' = \Gamma(U', \mathscr{O}_{S'})$ their rings, $B = \Gamma(U, \mathscr{B})$, $B' = \Gamma(U', \mathscr{B}')$; the restriction of u to a $(q|U')$-morphism: $\mathscr{B}|U \to \mathscr{B}'|U'$ corresponds to a di-homomorphism of algebras $B \to B'$; if V, V' are the inverse images of U, U' in X, X' respectively, under the structure morphisms, then the morphism $V' \to V$, the restriction of v, corresponds (**I**, 1.7.3) to the above di-homomorphism.

(1.5.7). Under the same hypotheses as in (1.5.6), let $\mathscr{M}$ be a quasi-coherent $\mathscr{B}$-module; there is then a canonical isomorphism of $\mathscr{O}_{X'}$-modules

(1.5.7.1)
$$v^*(\widetilde{\mathscr{M}}) \simeq (q^*(\mathscr{M}) \otimes_{q^*(\mathscr{B})} \mathscr{B}')^\sim.$$

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Indeed, the canonical isomorphism (1.5.2.1) gives a canonical isomorphism from $p_1^*(\widetilde{\mathscr{M}})$ to the sheaf on $\operatorname{Spec}(q^*(\mathscr{B}))$ associated to the $q^*(\mathscr{B})$-module $q^*(\mathscr{M})$, and it then suffices to apply (1.4.7).

1.6. Affine morphisms.

(1.6.1). We say that a morphism $f : X \to Y$ of preschemes is *affine* if it defines X as a prescheme affine over Y. The properties of preschemes affine over another translates as follows in this language:

Proposition (1.6.2). —

(i) *A closed immersion is affine.*
(ii) *The composition of two affine morphisms is affine.*
(iii) *If $f : X \to Y$ is an affine S-morphism, then $f_{(S')} : X_{(S')} \to Y_{(Y')}$ is affine for every base change $S' \to S$.*
(iv) *If $f : X \to Y$ and $f' : X' \to Y'$ are two affine S-morphisms, then*
$$f \times_S f' : X \times_S X' \longrightarrow Y \times_S Y'$$
is affine.
(v) *If $f : X \to Y$ and $g : Y \to Z$ are two morphisms such that $g \circ f$ is affine and g is separated, then f is affine.*
(vi) *If f is affine, then so if f_{red}.*

Proof. By (**I**, 5.5.12), it suffices to prove (i), (ii), and (iii). But (i) is none other than Example (1.2.2), and (ii) is none other than Corollary (1.3.5); finally, (iii) follows from Proposition (1.5.1), since $X_{(S')}$ identifies with the product $X \times_Y Y_{(S')}$ (**I**, 3.3.11). □

Corollary (1.6.3). — *If X is an affine scheme and Y is a scheme, then every morphism $f : X \to Y$ is affine.*

Proposition (1.6.4). — *Let Y be a locally Noetherian prescheme, $f : X \to Y$ a morphism of finite type. For f to be affine, it is necessary and sufficient for f_{red} to be.*

Proof. By (1.6.2, vi), we see only need to prove the sufficiency of the condition. It suffices to prove that if Y is affine and Noetherian, then X is affine; but Y_{red} is then affine, so the same is true for X_{red} by hypothesis. Now X is Noetherian, so the conclusion follows from (**I**, 6.1.7). □

1.7. Vector bundle associated to a sheaf of modules.

(1.7.1). Let A be a ring, E an A-module. Recall that we call the *symmetric algebra* on E and denote by $\mathbf{S}(E)$ (or $\mathbf{S}_A(E)$) the quotient algebra of the tensor algebra $\mathbf{T}(E)$ by the two-sided ideal generated by the elements $x \otimes y - y \otimes x$, where x and y vary over E. The algebra $\mathbf{S}(E)$ is characterized by the following universal property: if σ is the canonical map $E \to \mathbf{S}(E)$ (obtained by composing $E \to \mathbf{T}(E)$ with the canonical map $\mathbf{T}(E) \to \mathbf{S}(E)$), then every A-linear map $E \to B$, where B is a *commutative* A-algebra, factors uniquely as $E \xrightarrow{\sigma} \mathbf{S}(E) \xrightarrow{g} B$, where g is an A-homomorphism *of algebras*. We immediately deduce from this characterization that for two A-modules E and F, we have

$$\mathbf{S}(E \oplus F) = \mathbf{S}(E) \oplus \mathbf{S}(F)$$

up to canonical isomorphism; in addition, $\mathbf{S}(E)$ is a covariant functor in E, from the category of **II** | 15
A-modules to that of commutative A-algebras; finally, the above characterization also shows that if $E = \varinjlim E_\lambda$, then we have $\mathbf{S}(E) = \varinjlim \mathbf{S}(E_\lambda)$ up to canonical isomorphism. By abuse of language, a product $\sigma(x_1)\sigma(x_2)\cdots\sigma(x_n)$, where $x_i \in E$, is often denoted by $x_1 x_2 \cdots x_n$ if no confusion follows. The algebra $\mathbf{S}(E)$ is *graded*, $\mathbf{S}_n(E)$ being the set of linear combinations of n elements of E ($n \geqslant 0$); the algebra $\mathbf{S}(A)$ is canonically isomorphic to the polynomial algebra $A[T]$ is an indeterminate, and the algebra $\mathbf{S}(A^n)$ with the polynomial algebra in n indeterminates $A[T_1, \ldots, T_n]$.

(1.7.2). Let ϕ be a ring homomorphism $A \to B$. If F is a B-module, then the canonical map $F \to \mathbf{S}(F)$ gives a canonical map $F_{[\phi]} \to \mathbf{S}(F)_{[\phi]}$, which thus factors as $F_{[\phi]} \to \mathbf{S}(F_{[\phi]}) \to \mathbf{S}(F)_{[\phi]}$; the canonical homomorphism $\mathbf{S}(F_{[\phi]}) \to \mathbf{S}(F)_{[\phi]}$ is surjective, but not necessarily bijective. If E is an A-module, then every di-homomorphism $E \to F$ (that is to say, every A-homomorphism $E \to F_{[\phi]}$) thus canonically gives an A-homomorphism of algebras $\mathbf{S}(E) \to \mathbf{S}(F_{[\phi]}) \to \mathbf{S}(F)_{[\phi]}$, that is to say a di-homomorphism of algebras $\mathbf{S}(E) \to \mathbf{S}(F)$.

With the same notations, for every A-module E, $\mathbf{S}(E \otimes_A B)$ canonically identifies with the algebra $\mathbf{S}(E) \otimes_A B$; this follows immediately from the universal property of $\mathbf{S}(E)$ (1.7.1).

(1.7.3). Let R be a multiplicative subset of the ring A; apply (1.7.2) to the ring $B = R^{-1}A$, and remembering that $R^{-1}E = E \otimes_A R^{-1}A$, we see that we have $\mathbf{S}(R^{-1}E) = R^{-1}\mathbf{S}(E)$ up to canonical isomorphism. In addition, if $R' \supset R$ is a second multiplicative subset of A, then the diagram

$$\begin{array}{ccc} R^{-1}E & \longrightarrow & R'^{-1}E \\ \downarrow & & \downarrow \\ \mathbf{S}(R^{-1}E) & \longrightarrow & \mathbf{S}(R'^{-1}E) \end{array}$$

is commutative.

(1.7.4). Now let $(S, \mathscr{A})$ be a ringed space, and let $\mathscr{E}$ be a $\mathscr{A}$-module over S. If to any open $U \subset S$ we associate the $\Gamma(U, \mathscr{A})$-module $\mathbf{S}(\Gamma(U, \mathscr{E}))$, then we define (see the functorial nature of $\mathbf{S}(E)$ (1.7.2)) a presheaf of algebras; we say that the associated sheaf, which we denote by $\mathbf{S}(\mathscr{E})$ or $\mathbf{S}_{\mathscr{A}}(\mathscr{E})$ is the *symmetric $\mathscr{A}$-algebra* on the $\mathscr{A}$-module $\mathscr{E}$. It follows immediately from (1.7.1) that $\mathbf{S}(\mathscr{E})$ is a solution to a universal problem: every homomorphism of $\mathscr{A}$-modules $\mathscr{E} \to \mathscr{B}$, where $\mathscr{B}$ is an $\mathscr{A}$-algebra, factors uniquely as $\mathscr{E} \to \mathbf{S}(\mathscr{E}) \to \mathscr{B}$, the second arrow being a homomorphism of $\mathscr{A}$-algebras. There is thus a bijective correspondence between homomorphisms $\mathscr{E} \to \mathscr{B}$ of $\mathscr{A}$-modules and homomorphisms $\mathbf{S}(\mathscr{E}) \to \mathscr{B}$ of $\mathscr{A}$-algebras. In particular, every homomorphism $u : \mathscr{E} \to \mathscr{F}$ of $\mathscr{A}$-modules defines a homomorphism $\mathbf{S}(u) : \mathbf{S}(\mathscr{E}) \to \mathbf{S}(\mathscr{F})$ of $\mathscr{A}$-algebras, and $\mathbf{S}(\mathscr{E})$ is thus a
covariant functor in $\mathscr{E}$. **II** | 16

By (1.7.2) and the commutativity of $\mathbf{S}$ with inductive limts, we have $(\mathbf{S}(\mathscr{E}))_x = \mathbf{S}(\mathscr{E}_x)$ for every point $x \in S$. If $\mathscr{E}, \mathscr{F}$ are two $\mathscr{A}$-modules, then $\mathbf{S}(\mathscr{E} \oplus \mathscr{F})$ canonically identifies with $\mathbf{S}(\mathscr{E}) \otimes_A \mathbf{S}(\mathscr{F})$, as we see for the corresponding presheaves.

We also note that $\mathbf{S}(\mathscr{E})$ is a graded $\mathscr{A}$-algebra, the infinite direct sum of the $\mathbf{S}_n(\mathscr{E})$, where the $\mathscr{A}$-module $\mathbf{S}_n(\mathscr{E})$ is the sheaf associated to the presheaf $U \mapsto \mathbf{S}_n(\Gamma(U, \mathscr{E}))$. If we take in particular $\mathscr{E} = \mathscr{A}$, then we see that $\mathbf{S}_{\mathscr{A}}(\mathscr{A})$ identifies with $\mathscr{A}[T] = \mathscr{A} \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ (T indeterminate, $\mathbf{Z}$ being considered as a simple sheaf).

(1.7.5). Let $(T, \mathscr{B})$ be a second ringed space, f a morphism $(S, \mathscr{A}) \to (T, \mathscr{B})$. If $\mathscr{F}$ is a $\mathscr{B}$-module, then $\mathbf{S}(f^*(\mathscr{F}))$ canonically identifies with $f^*(\mathbf{S}(\mathscr{F}))$; indeed, if $f = (\psi, \theta)$, then by definition (**0**, 4.3.1),

$$\mathbf{S}(f^*(\mathscr{F})) = \mathbf{S}(\psi^*(\mathscr{F}) \otimes_{\psi^*(\mathscr{B})} \mathscr{A}) = \mathbf{S}(\psi^*(\mathscr{F})) \otimes_{\psi^*(\mathscr{B})} \mathscr{A}$$

(1.7.2); for every open U of S and every section h of $\mathbf{S}(\psi^*(\mathscr{F}))$ over U, h coincides, in a neighborhood V of every point $s \in U$, with an element of $\mathbf{S}(\Gamma(V, \psi^*(\mathscr{F})))$; if we refer to the definition of $\psi^*(\mathscr{F})$ (**0**, 3.7.1) and take into account that every element of $\mathbf{S}(E)$ for a module E is a linear combination of a finite number of products of elements of E, then we see that there is a neighborhood W of $\psi(s)$ in T, a section h' of $\mathbf{S}(\mathscr{F})$ over W, and a neighborhood $V' \subset V \cap \psi^{-1}(W)$ of s such that h coincises with $t \mapsto h'(\psi(t))$ over V'; hence out assertion.

Proposition (1.7.6). — *Let A be a ring, $S = \operatorname{Spec}(A)$ its prime spectrum, $\mathscr{E} = \widetilde{M}$ the $\mathscr{O}_S$-module associated to an A-module M; then the $\mathscr{O}_S$-algebra $\mathbf{S}(\mathscr{E})$ is associated to the A-algebra $\mathbf{S}(M)$.*

Proof. For every $f \in A$, $\mathbf{S}(M_f) = (\mathbf{S}(M))_f$ (1.7.3), and the proposition thus folloes from the definition (**I**, 1.3.4). □

Corollary (1.7.7). — *If S is a prescheme, $\mathscr{E}$ a quasi-coherent $\mathscr{O}_S$-module, then the $\mathscr{O}_S$-algebra $\mathbf{S}(\mathscr{E})$ is quasi-coherent. If in addition $\mathscr{E}$ is of finite type, then each of the $\mathscr{O}_S$-modules $\mathbf{S}_n(\mathscr{E})$ is of finite type.*

Proof. The first assertion is an immediate consequence of (1.7.6) and of (**I**, 1.4.1); the second follows from the fact that if E is an A-module of finite type, then $\mathbf{S}_n(E)$ is an A-module of finite type; we then apply (**I**, 1.3.13). □

Definition (1.7.8). — Let $\mathscr{E}$ be a quasi-coherent $\mathscr{O}_S$-module. We call the *vector bundle over S defined by $\mathscr{E}$* and denote by $\mathbf{V}(\mathscr{E})$ the spectrum (1.3.1) of the quasi-coherent $\mathscr{O}_S$-algebra $\mathbf{S}(\mathscr{E})$.

By (1.2.7), for every S-prescheme X, there is a canonical bijective correspondence between the S-morphisms $X \to \mathbf{V}(\mathscr{E})$ and the homomorphisms of $\mathscr{O}_S$*-algebras* $\mathbf{S}(\mathscr{E}) \to \mathscr{A}(X)$, thus also between these S-morphisms and the homomorphisms of $\mathscr{O}_S$*-modules* $\mathscr{E} \to \mathscr{A}(X) = f_*(\mathscr{O}_X)$ (where f is the structure morphism $X \to S$). In particular:

(1.7.9). Take for X a subprescheme induced by S on an *open* $U \subset S$. Then the S-morphisms $U \to \mathbf{V}(\mathscr{E})$ are none other that the U-sections (**I**, 2.5.5) of the U-prescheme induces by $\mathbf{V}(\mathscr{E})$ on the open $p^{-1}(U)$ (where p is the structure morphism $\mathbf{V}(\mathscr{E}) \to S$). From what we have just seen, these U-sections bijectively correspond to homomorphisms of $\mathscr{O}_S$-modules $\mathscr{E} \to j_*(\mathscr{O}_S|U)$ (where j is the canonical injection $U \to S$), or equivalently (**0**, 4.4.3) with the $(\mathscr{O}_S|U)$-homomorphisms $j^*(\mathscr{E}) = \mathscr{E}|U \to \mathscr{O}_S|U$. **II** | 17
In addition, it is immediate that the restriction to an open $U' \subset U$ of an S-morphism $U \to \mathbf{V}(\mathscr{E})$ corresponds to the restriction to U' of the corresponding homomorphism $\mathscr{E}|U \to \mathscr{O}_S|U$. We conclude that *the sheaf of germs of S-sections* of $\mathbf{V}(\mathscr{E})$ canonically identifies with the *dual* $\mathscr{E}^\vee$ of $\mathscr{E}$.

In particular, if we set $X = U = S$, then the *zero* homomorphism $\mathscr{E} \to \mathscr{O}_S$ corrresponds to a canonical S-section of $\mathbf{V}(\mathscr{E})$, called the *zero S-section* (cf. (8.3.3)).

(1.7.10). Now take X to be the spectrum $\{\xi\}$ of a field K; the structure morphism $f : X \to S$ then corresponfs to a monomorphism $k(s) \to K$, where $s = f(\xi)$ (**I**, 2.4.6); the S-morphisms $\{\xi\} \to \mathbf{V}(\mathscr{E})$ are none other than the *geometric points of $\mathbf{V}(\mathscr{E})$ with values in the extension K of $k(s)$* (**I**, 3.4.5), points which are localized at the points of $p^{-1}(s)$. The set of these points, which we can call *the rational geometric fibre over K* of $\mathbf{V}(\mathscr{E})$ *over the point s*, is identified by (1.7.8) with the set of homomorphisms of $\mathscr{O}_S$-modules $\mathscr{E} \to f_*(\mathscr{O}_X)$, or, equivalently (**0**, 4.4.3) with the set of homomorphisms of $\mathscr{O}_X$-modules $f^*(\mathscr{E}) \to \mathscr{O}_X = K$. But we have by definition (**0**, 4.3.1) $f^*(\mathscr{E}) = \mathscr{E}_s \otimes_{\mathscr{O}_s} K = \mathscr{E}^s \otimes_{k(s)} K$, setting $\mathscr{E}^s = \mathscr{E}_s / \mathfrak{m}_s \mathscr{E}_s$; the geometric fibre of $\mathbf{V}(\mathscr{E})$ rational over K over s thus identifies with the *dual* of the *K-vector space $\mathscr{E}^s \otimes_{k(s)} K$*; if $\mathscr{E}^s$ or K is of finite dimension over $k(s)$, then this dual also identifies with $(\mathscr{E}^s)^\vee \otimes_{k(s)} K$, denoting by $(\mathscr{E}^s)^\vee$ the dual of the $k(s)$-vector space $\mathscr{E}^s$.

Proposition (1.7.11). —

(i) $\mathbf{V}(\mathscr{E})$ *is a contravariant functor in $\mathscr{E}$ from the category of quasi-coherent $\mathscr{O}_S$-modules to the category of affine S-schemes.*
(ii) *If $\mathscr{E}$ is an $\mathscr{O}_S$-module of finite type, then $\mathbf{V}(\mathscr{E})$ is of finite type over S.*

(iii) *If $\mathscr{E}$ and $\mathscr{F}$ are two quasi-coherent $\mathscr{O}_S$-modules, then $\mathbf{V}(\mathscr{E} \oplus \mathscr{F})$ canonically identifies with $\mathbf{V}(\mathscr{E}) \times_S \mathbf{V}(\mathscr{F})$.*

(iv) *Let $g : S' \to S$ be a morphism; for every quasi-coherent $\mathscr{O}_S$-module $\mathscr{E}$, $\mathbf{V}(g^*(\mathscr{E}))$ canonically identifies with $\mathbf{V}(\mathscr{E})_{(S')} = \mathbf{V}(\mathscr{E}) \times_S S'$.*

(v) *A surjective homomorphism $\mathscr{E} \to \mathscr{F}$ of quasi-coherent $\mathscr{O}_S$-modules corresponds to a closed immersion $\mathbf{V}(\mathscr{F}) \to \mathbf{V}(\mathscr{E})$.*

Proof. (i) is an immediate consequence of (1.2.7), taking into account that every homomorphism of $\mathscr{O}_S$-modules $\mathscr{E} \to \mathscr{F}$ canonically defines a homomorphism of $\mathscr{O}_S$-algebras $\mathbf{S}(\mathscr{E}) \to \mathbf{S}(\mathscr{F})$. (ii) follows immediately from the definition (**I**, 6.3.1) and the fact that if E is an A-module of finite type, then $\mathbf{S}(E)$ is an A-algebra of finite type. To prove (iii), it suffices to start with the canonical isomorphism $\mathbf{S}(\mathscr{E} \oplus \mathscr{F}) \simeq \mathbf{S}(\mathscr{E}) \otimes_{\mathscr{O}_S} \mathbf{S}(\mathscr{F})$ (1.7.4) and to apply (1.4.6). Similarly, to prove (iv), it suffices to start with the canonical isomorphism $\mathbf{S}(g^*(\mathscr{E})) \simeq g^*(\mathbf{S}(\mathscr{E}))$ (1.7.5) and to apply (1.5.2). Finally, to establish (v), it suffices to remark that if the homomorphism $\mathscr{E} \to \mathscr{F}$ is surjective, then so is the corresponding homomorphism $\mathbf{S}(\mathscr{E}) \to \mathbf{S}(\mathscr{F})$ of $\mathscr{O}_S$-algebras, and the conclusion follows from (1.4.10). ☐

(1.7.12). Take in particular $\mathscr{E} = \mathscr{O}_S$; the prescheme $\mathbf{V}(\mathscr{O}_S)$ is the affine S-scheme, spectrum of the $\mathscr{O}_S$-algebra $\mathbf{S}(\mathscr{O}_S)$ which identifies with the $\mathscr{O}_S$-algebra $\mathscr{O}_S[T] = \mathscr{O}_S \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ (T indeterminate); this **II** | 18
is evident when $S = \operatorname{Spec}(\mathbf{Z})$, by virtue of (1.7.6), and we pass from there to the general case by considering the structure morphism $S \to \operatorname{Spec}(\mathbf{Z})$ and using (1.7.11, iv). Because of this result, we set $\mathbf{V}(\mathscr{O}_S) = S[T]$, and we thus have the formula

(1.7.12.1) $$S[T] = S \otimes_{\mathbf{Z}} \mathbf{Z}[T].$$

The identification of the sheaf of germs of S-sections of $S[T]$ with $\mathscr{O}_S$, already seen in (**I**, 3.3.15), here in a more general context, as a special case of (1.7.9).

(1.7.13). For every S-prescheme X, we have seen (1.7.8) that $\operatorname{Hom}_S(X, S[T])$ canonically identifies with $\operatorname{Hom}_{\mathscr{O}_S}(\mathscr{O}_S, \mathscr{A}(X))$, which is canonically isomorphic to $\Gamma(S, \mathscr{A}(X))$, and as a result is equipped with the structure of a ring; in addition, to every S-morphism $h : X \to Y$ there corresponds a morphism $\Gamma(\mathscr{A}(h)) : \Gamma(S, \mathscr{A}(Y)) \to \Gamma(S, \mathscr{A}(X))$ for the ring structures (1.1.2). When we equip $\operatorname{Hom}_S(X, S[T])$ with a ring structure as defined, then we can see that $\operatorname{Hom}(X, S[T])$ can be considered as a *contravariant functor* in X, from the category of S-preschemes to that of rings. On the other hand, $\operatorname{Hom}_S(X, \mathbf{V}(\mathscr{E}))$ is likewise identified with $\operatorname{Hom}_{\mathscr{O}_S}(\mathscr{E}, \mathscr{A}(X))$ (where $\mathscr{A}(X)$ is considered as an $\mathscr{O}_S$*-module*); as a result, we can canonically give a *module* structure on the ring $\operatorname{Hom}_S(X, S[T])$, and we see as above that the pair

$$(\operatorname{Hom}_S(X, S[T]), \operatorname{Hom}(X, \mathbf{V}(\mathscr{E})))$$

is a contravariant functor in X, with values in the category whose elements are the pairs (A, M) consisting of a ring A and an A-module M, the morphisms being di-homomorphisms.

We will interpret these facts by saying that $S[T]$ is an *S-scheme of rings* and that $\mathbf{V}(\mathscr{E})$ is an *S-scheme of modules* on the S-scheme of rings $S[T]$ (cf. Chapter 0, §8).

(1.7.14). We will see that the structure of an S-scheme of modules defined on the S-scheme $\mathbf{V}(\mathscr{E})$ allows us to reconstruct the $\mathscr{O}_S$-module $\mathscr{E}$ up to unique isomorphism: for this, we will show that $\mathscr{E}$ is canonically isomorphic to a $\mathscr{O}_S$-submodule of $\mathbf{S}(\mathscr{E}) = \mathscr{A}(\mathbf{V}(\mathscr{E}))$, defined by means of this structure. Indeed (1.7.4) the set $\operatorname{Hom}_{\mathscr{O}_S}(\mathbf{S}(\mathscr{E}), \mathscr{A}(X))$ of homomorphisms of $\mathscr{O}_S$*-algebras* canonical identifies with $\operatorname{Hom}_{\mathscr{O}_S}(\mathscr{E}, \mathscr{A}(X))$, the set of homomorphisms of $\mathscr{O}_S$*-modules*: if h and h' are two elements of this latter set, s_i ($1 \leqslant i \leqslant n$) sections of $\mathscr{E}$ over an open $U \subset S$, t a section of $\mathscr{A}(X)$ over U, then we have by definition

$$(h + h')(s_1 s_2 \cdots s_n) = \prod_{i=1}^{n} (h(s_i) + h'(s_i))$$

and

$$(t \cdot h)(s_1 s_2 \cdots s_n) = t^n \prod_{i=1}^{n} h(s_i).$$

This being so, if z is a section of $\mathbf{S}(\mathscr{E})$ over U, then $h \mapsto h(z)$ is a map from $\operatorname{Hom}_S(X, \mathbf{V}(\mathscr{E})) = \operatorname{Hom}_{\mathscr{O}_S}(\mathbf{S}(\mathscr{E}), \mathscr{A}(X))$ to $\Gamma(U, \mathscr{A}(X))$. We will show that $\mathscr{E}$ identifies with a submodule of $\mathbf{S}(\mathscr{E})$ such **II** | 19 that, *for every open $U \subset S$, every section z of this $\mathscr{O}_S$-submodule of U, and every S-prescheme X, the map $h \mapsto h(z)$ from $\operatorname{Hom}_{\mathscr{O}_S}(\mathbf{S}(\mathscr{E})|U, \mathscr{A}(X)|U)$ to $\Gamma(U, \mathscr{A}(X))$ is a homomorphism of $\Gamma(U, \mathscr{A}(X))$-modules.*

It is immediate that $\mathscr{E}$ has this property; to show the converse, we can reduce to proving that when $S = \operatorname{Spec}(A)$, $\mathscr{E} = \widetilde{M}$, a section z of $\mathbf{S}(\mathscr{E})$ over S that (for $U = S$) has the property stated above is necessarily a section of $\mathscr{E}$; we then have $z = \sum_{n=0}^{\infty} z_n$, where $z_n \in \mathbf{S}_n(M)$, and it is a question of proving that $z_n = 0$ for $n \neq 1$. Set $B = \mathbf{S}(M)$ and take for X the prescheme $\operatorname{Spec}(B[T])$, where T is an indeterminate. The set $\operatorname{Hom}_{\mathscr{O}_S}(\mathbf{S}(\mathscr{E}), \mathscr{A}(X))$ identifies with the set of ring homomorphisms $h : B \to B[T]$ (**I**, 1.3.13), and from what we saw above, we have $(T \cdot h)(z) = \sum_{n=0}^{\infty} T^n h(z_n)$: the hypothesis on z implies that we have $\sum_{n=0}^{\infty} T^n h(z_n) = T \cdot \sum_{n=0}^{\infty} h(z_n)$ for every homomorphism h. In in particular we take for h the canonical injection, then $\sum_{n=0}^{\infty} T^n z_n = T \cdot \sum_{n=0}^{\infty} z_n$, which implies the conclusion $z_n = 0$ for $n \neq 1$.

Proposition (1.7.15). — *Let Y be a prescheme whose underlying space is Noetherian, or a quasi-compact scheme. Every affine Y-scheme X of finite type over Y is Y-isomorphic to a closed Y-subscheme of the form $\mathbf{V}(\mathscr{E})$, where $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type.*

Proof. The quasi-coherent $\mathscr{O}_Y$-algebra $\mathscr{A}(X)$ is of finite type (1.3.7). The hypotheses imply that $\mathscr{A}(X)$ is generated by a quasi-coherent $\mathscr{O}_Y$-submodule of finite type $\mathscr{E}$ (**I**, 9.6.5); by definition, this implies that the canonical homomorphism $\mathbf{S}(\mathscr{E}) \to \mathscr{A}(X)$ canonically extending the injection $\mathscr{E} \to \mathscr{A}(X)$ is *surjective*; the conclusion then follows from (1.4.10). ☐

§2. Homogeneous prime spectra

2.1. Generalities on graded rings and modules.

Notation (2.1.1). — Given a ring S *graded* in positive degrees, we denote by S_n the subset of S consisting of homogeneous elements of degree n ($n \geqslant 0$), by S_+ the (direct) sum of the S_n for $n > 0$; we have $1 \in S_0$, S_0 is a subring of S, S_+ is a graded ideal of S, and S is the direct sum of S_0 and S_+. If M is a *graded* module over S (with positive or negative degrees), we similarly denote by M_n the S_0-module consisting of homogeneous elements of M of degree n (with $n \in \mathbf{Z}$).

For every integer $d > 0$, we denote by $S^{(d)}$ the direct sum of the S_{nd}; by considering the elements of S_{nd} as homogeneous of degree n, the S_{nd} define on $S^{(d)}$ a graded ring structure.

For every integer k such that $0 \leqslant k \leqslant d-1$, we denote by $M^{(d,k)}$ the direct sum of the M_{nd+k} **II** | 20
($n \in \mathbf{Z}$); this is a graded $S^{(d)}$-module when we consider the elements of M_{nd+k} as homogeneous of degree n. We write $M^{(d)}$ in place of $M^{(d,0)}$.

With the above notation, for every integer n (positive or negative), we denote by $M(n)$ the graded S-module defined by $(M(n))_k = M_{n+k}$ for every $k \in \mathbf{Z}$. In particular, $S(n)$ will be a graded S-module such that $(S(n))_k = S_{n+k}$, by agreeing to set $S_n = 0$ for $n < 0$. We say that a graded S-module M is *free* if it is isomorphic, considered as a *graded* module, to a direct sum of modules of the form $S(n)$; as $S(n)$ is a monogeneous S-module, generated by the element 1 of S considered as an element of degree $-n$, it is equivalent to say that M admits a *basis* over S consisting of *homogeneous* elements.

We say that a graded S-module M *admits a finite presentation* if there exists an exact sequence $P \to Q \to M \to 0$, where P and Q are finite direct sums of modules of the form $S(n)$ and the homomorphisms are of degree 0 (cf. (2.1.2)).

(2.1.2). Let M and N be two graded S-modules; we define on $M \otimes_S N$ a *graded* S-module structure in the following way. On the tensor product $M \otimes_{\mathbf{Z}} N$, we can define a graded $\mathbf{Z}$-module structure (where $\mathbf{Z}$ is graded by $\mathbf{Z}_0 = \mathbf{Z}$, $\mathbf{Z}_n = 0$ for $n \neq 0$) by setting $(M \otimes_{\mathbf{Z}} N)_q = \bigoplus_{m+n=q} M_m \otimes_{\mathbf{Z}} N_n$ (as M and N are respectively direct sums of the M_m and the N_n, we know that we can canonically identify $M \otimes_{\mathbf{Z}} N$ with the direct sum of all the $M_m \otimes_{\mathbf{Z}} N_n$). This being so, we have $M \otimes_S N = (M \otimes_{\mathbf{Z}} N)/P$, where P is the $\mathbf{Z}$-submodule of $M \otimes_{\mathbf{Z}} N$ generated by the elements $(xs) \otimes y - x \otimes (sy)$ for $x \in M$, $y \in N$, $s \in S$; it is clear that P is a *graded* $\mathbf{Z}$-submodule of $M \otimes_{\mathbf{Z}} N$, and we see immediately that we obtain a graded S-module structure on $M \otimes_S N$ by passing to the quotient.

For two graded S-modules M and N, recall that a homomorphism $u : M \to N$ of S-modules is said to be *of degree k* if $u(M_j) \subset N_{j+k}$ for all $j \in \mathbf{Z}$. If H_n denotes the set of all the homomorphisms of degree n from M to N, then we denote by $\operatorname{Hom}_S(M, N)$ the (direct) *sum* of the H_n ($n \in \mathbf{Z}$) in the

S-module H of all the homomorphisms (of S-modules) from M to N; in general, $\mathrm{Hom}_S(M,N)$ is not equal to the later. However, we have $H = \mathrm{Hom}_S(M,N)$ when M is *of finite type*; indeed, we can then suppose that M is generated by a finite number of homogeneous elements x_i ($1 \leqslant i \leqslant n$), and every homomorphism $u \in H$ can be written in a unique way as $\sum_{k\in\mathbf{Z}} u_k$, where for each k, $u_k(x_i)$ is equal to the homogeneous component of degree $k + \deg(x_i)$ of $u(x_i)$ ($1 \leqslant i \leqslant n$), which implies that $u_k = 0$ except for a finite number of indices; we have by definition that $u_k \in H_k$, hence the conclusion.

We say that the elements of degree 0 of $\mathrm{Hom}_S(M,N)$ are the *homomorphisms of graded S-modules*. It is clear that $S_m H_n \subset H_{m+n}$, so the H_n define on $\mathrm{Hom}_S(M,N)$ a graded S-module structure.

It follows immediately from these definitions that we have

(2.1.2.1) $$M(m) \otimes_S N(n) = (M \otimes_S N)(m+n),$$

(2.1.2.2) $$\mathrm{Hom}_S(M(m), N(n)) = (\mathrm{Hom}_S(M,N))(n-m),$$

for two graded S-modules M and N. **II** | 21

Let S and S' be two graded rings; a homomorphism of *graded rings* $\phi : S \to S'$ is a homomorphism of rings such that $\phi(S_n) \subset S'_n$ for all $n \in \mathbf{Z}$ (in other words, ϕ must be a homomorphism *of degree* 0 of graded $\mathbf{Z}$-modules). The data of such a homomorphism defines on S' a *graded S'*-module structure; equipped with this structure and its graded ring structure, we say that S' is a *graded S'-algebra*.

If M is also a graded S-module, then the tensor product $M \otimes_S S'$ of *graded* S-modules is equipped in a natural way with a *graded S'*-module structure, the grading being defined as above.

Lemma (2.1.3). — *Let S be a ring graded in positive degrees. For a subset E of S_+ consisting of homogeneous elements to generate S_+ as an S-module, it is necessary and sufficient for E to generate S an an S_0-algebra.*

Proof. The condition is evidently sufficient; we show that it is necessary. Let E_n (resp. E^n) be the set of elements of E equal to n (resp. $\leqslant n$); it suffices to show, by induction on $n > 0$, that S_n is the S_0-module generated by the elements of degree n which are products of elements of E^n. This is evident for $n = 1$ by virtue of the hypothesis; the latter also shows that $S_n = \sum_{p=0}^{n-1} S_p E_{n-p}$, and the induction argument is then immediate. □

Corollary (2.1.4). — *For S_+ to be an ideal of finite type, it is necessary and sufficient for S to be an S_0-algebra of finite type.*

Proof. We can always assume that a finite system of generators of the S_0-algebra S (resp. of the S-ideal S_+) consists of homogeneous elements, by replacing each of the generators considered by its homogeneous components. □

Corollary (2.1.5). — *For S to be Noetherian, it is necessary and sufficient for S_0 to be Noetherian and for S to be an S_0-algebra of finite type.*

Proof. The condition is evidently sufficient; it is necessary, since S_0 is isomorphic to S/S_+ and S_+ must be an ideal of finite type (2.1.4). □

Lemma (2.1.6). — *Let S be a ring graded in positive degrees, which is an S_0-algebra of finite type. Let M be a graded S-module of finite type. Then:*

(i) *The M_n are S_0-modules of finite type, and there exists an integer n_0 such that $M_n = 0$ for $n \leqslant n_0$.*
(ii) *There exists an integer n_1 and an integer $h > 0$ such that, for every integer $n \geqslant n_1$, we have $M_{n+h} = S_h M_n$.*
(iii) *For every pair of integers (d,k) such that $d > 0$, $0 \leqslant k \leqslant d-1$, $M^{(d,k)}$ is an $S^{(d)}$-module of finite type.*
(iv) *For every integer $d > 0$, $S^{(d)}$ is an S_0-algebra of finite type.*
(v) *There exists an integer $h > 0$ such that $S_{mh} = (S_h)^m$ for all $m > 0$.*
(vi) *For every integer $n > 0$, there exists an integer m_0 such that $S_m \subset S_+^n$ for all $m \geqslant m_0$.*

Proof. We can assume that S is generated (as an S_0-algebra) by homogeneous elements f_i, of degrees h_i ($1 \leqslant i \leqslant r$), and M is generated (as an S-module) by homogeneous elements x_j of degrees k_j ($1 \leqslant j \leqslant s$). It is clear that M_n is formed by linear combinations, with coefficients in S_0, of elements **II** | 22
$f_1^{\alpha_1} \cdots f_r^{\alpha_r} x_j$ such that the α_i are integers $\geqslant 0$ satisfying $k_j + \sum_i \alpha_i h_i = n$; for each j, there are only

finitly many systems (α_i) satisfying this equation, since the h_i are > 0, hence the first assertion of (i); the second is evident. On the other hand, let h be the l.c.m. of the h_i and set $g_i = f_i^{h/h_i}$ $(1 \leqslant i \leqslant r)$ such that all the g_i are of degree h; let z_μ be the elements of M of the form $f_1^{\alpha_1} \cdots f_r^{\alpha_r} x_j$ with $0 \leqslant \alpha_i < h/h_i$ for $1 \leqslant i \leqslant r$; there are finitely many of these elements, so let n_1 be the largest of their degrees. It is clear that for $n \geqslant n_1$, every element of M_{n+h} is a linear combination of the z_μ whose cofficients are monomials of degree > 0 with respect to the g_i, so we have $M_{n+h} = S_h M_n$, which establishes (ii). In a similar way, we see (for all $d > 0$) that an element of $M^{(d,k)}$ is a linear combinations, with coeffients in S_0, of elements of the form $g^d f_1^{\alpha_1} \cdots f_r^{\alpha_r} x_j$ with $0 \leqslant \alpha_i < d$, g being a homogeneous element of S; hence (iii); (iv) then follows from (iii) and from Lemma (2.1.3), by taking $M = S_+$, since $(S_+)^{(d)} = (S^{(d)})_+$. The assertion of (v) is deduced from (ii) by taking $M = S$. Finally, for a given n, there are finitely many systems (α_i) such that $\alpha_i \geqslant 0$ and $\sum_i \alpha_i < n$, so if m_0 is the largest value of the sum $\sum_i \alpha_i h_i$ of these systems, then we have $S_m \subset S_+^n$ for $m > m_0$, which proves (vi). □

Corollary (2.1.7). — *If S is Noetherian, then so is $S^{(d)}$ for every integer $d > 0$.*

Proof. This follows from (2.1.5) and (2.1.6, iv). □

(2.1.8). Let $\mathfrak{p}$ be a *graded* prime ideal of the graded ring S; $\mathfrak{p}$ is thus a direct sum of the subgroups $\mathfrak{p}_n = \mathfrak{p} \cap S_n$. Suppose that $\mathfrak{p}$ *does not contain* S_+. Then if $f \in S_+$ is not in $\mathfrak{p}$, the relation $f^n x \in \mathfrak{p}$ is equivalent to $x \in \mathfrak{p}$; in particular, if $f \in S_d$ $(d > 0)$, for all $x \in S_{m-nd}$, then the relation $f^n x \in \mathfrak{p}_m$ is equivalent to $x \in \mathfrak{p}_{m-nd}$.

Proposition (2.1.9). — *Let n_0 be an integer > 0; for all $n \geqslant n_0$, let $\mathfrak{p}_n$ be a subgroup of S_n. For there to exist a graded prime ideal $\mathfrak{p}$ of S not containing S_+ and such that $\mathfrak{p} \cap S_n = \mathfrak{p}_n$ for all $n \geqslant n_0$, it is necessary and sufficient for the following coniditions to be satisfied:*

(1st) *$S_m \mathfrak{p}_n \subset \mathfrak{p}_{m+n}$ for all $m \geqslant 0$ and all $n \geqslant n_0$.*
(2nd) *For $m \geqslant n_0$, $n \geqslant n_0$, $f \in S_m$, $g \in S_n$, the relation $fg \in \mathfrak{p}_{m+n}$ implies $f \in \mathfrak{p}_m$ or $g \in \mathfrak{p}_n$.*
(3rd) *$\mathfrak{p}_n \neq S_n$ for at least one $n \geqslant n_0$.*

In addition, the graded prime ideal $\mathfrak{p}$ is then unique.

Proof. It is evident that the conditions (1st) and (2nd) are necessary. In addition, if $\mathfrak{p} \not\supset S_+$, then there exists at least one $k > 0$ such that $\mathfrak{p} \cap S_k \neq S_k$; if $f \in S_k$ is not in $\mathfrak{p}$, the relation $\mathfrak{p} \cap S_n = S_n$ implies $\mathfrak{p} \cap S_{n-mk} = S_{n-mk}$ according to (2.1.8); therefore, if $\mathfrak{p} \cap S_n = S_n$ for a certain value of n, we would have $\mathfrak{p} \supset S_+$ contrary to the hypothesis, which proves that (3rd) is necessary. Conversely, suppose that the conditions (1st), (2nd), and (3rd) are satisfied. Note that if for an integer $d \geqslant n_0$, $f \in S_d$ is not in $\mathfrak{p}_d$, then, if $\mathfrak{p}$ exists, $\mathfrak{p}_m$, for $m < n_0$, is necessarily equal to the set of the $x \in S_m$ such that $f^r x \in \mathfrak{p}_{m+rd}$, except for a finite number of values of r. This already proves that if $\mathfrak{p}$ exists, then it is unique. It remains to show that if we define the $\mathfrak{p}_m$ for $m < n_0$ by the previous condition, then $\mathfrak{p} = \sum_{n=0}^{\infty} \mathfrak{p}_n$ is a prime ideal. First, note that by virtue of (2nd), for $m \geqslant n_0$, $\mathfrak{p}_m$ is also defined as the set of the $x \in S_m$ such that $f^r x \in \mathfrak{p}_{m+rd}$ except for a finite number of values of r. This being **II** | 23
so, if $g \in S_m$, $x \in \mathfrak{p}_n$, then we have $f^r gx \in \mathfrak{p}_{m+n+rd}$ except for a finite number of values of r, so $gx \in \mathfrak{p}_{m+n}$, which proves that $\mathfrak{p}$ is an ideal of S. To establish that this ideal is prime, in other words that the ring $S/\mathfrak{p}$, graded by the subgroups $S_n/\mathfrak{p}_n$, is an integral domain, it suffices (by considering the components of higher degree of two elements of $S/\mathfrak{p}$) to prove that if $x \in S_m$ and $y \in S_n$ are such that $x \notin \mathfrak{p}_m$ and $y \notin \mathfrak{p}_n$, then $xy \notin \mathfrak{p}_{m+n}$. If not, for r large enough, we would have $f^{2r} xy \in \mathfrak{p}_{m+n+2rd}$; but we have $f^r y \notin \mathfrak{p}_{n+rd}$ for all $r > 0$; it then follows from (2nd) that, except for a finite number of values of r, we have $f^r x \in \mathfrak{p}_{m+rd}$, and we conclude that $x \in \mathfrak{p}_m$ contrary to the hypothesis. □

(2.1.10). We say that a subset $\mathfrak{J}$ of S_+ is an *ideal of* S_+ if it is an ideal of S, and $\mathfrak{J}$ is a *graded prime ideal of* S_+ if it is the intersection of S_+ and a graded prime ideal of S *not containing* S_+ (this prime ideal is also unique according to Proposition (2.1.9)). If $\mathfrak{J}$ is an ideal of S_+, the *radical of* $\mathfrak{J}$ *in* S_+ is the set of elements of S_+ which have a power in $\mathfrak{J}$, in other words the set $\mathfrak{r}_+(\mathfrak{J}) = \mathfrak{r}(\mathfrak{J}) \cap S_+$; in particular, the radical of 0 in S_+ is then called the *nilradical* of S_+ and denoted by $\mathfrak{N}_+$: this is the set of nilpotent elements of S_+. If $\mathfrak{J}$ is an *graded* ideal of S_+, then its radical $\mathfrak{r}_+(\mathfrak{J})$ is a *graded* ideal: by passing to the quotient ring $S/\mathfrak{J}$, we can reduce to the case $\mathfrak{J} = 0$, and it remains to see that if $x = x_h + x_{h+1} + \cdots + x_k$ is nilpotent, then so are the $x_i \in S_i$ $(1 \leqslant h \leqslant i \leqslant k)$; we can assume $x_k \neq 0$

and the component of highest degree of x^n is then x_k^n, hence x_k is nilpotent, and we then argue by induction on k. We say that the graded ring S is *essentially reduced* if $\mathfrak{N}_+ = 0$, in other words, if S_+ does not contain nilpotent elements $\neq 0$.

(2.1.11). We note that if, in the graded ring S, an element x is a zero-divisor, then so is its component of highest degree. We say that a ring S is *essentially integral* if the ring S_+ (*without the unit element*) does not contain a zero-divisor and is $\neq 0$; it suffices that a homogeneous element $\neq 0$ in S_+ is not a zero-divisor in this ring. It is clear that if $\mathfrak{p}$ is a graded prime ideal of S_+, then $S/\mathfrak{p}$ is essentially integral.

Let S be an essentially integral graded ring, and let $x_0 \in S_0$: if there then exists *a* homogeneous element $f \neq 0$ of S_+ such that $x_0 f = 0$, then we have $x_0 S_+ = 0$, since we have $(x_0 g) f = (x_0 f) g = 0$ for all $g \in S_+$, and the hypothesis thus implies $x_0 g = 0$. For S to be integral, it is necessary and sufficient for S_0 to be integral and the annihilator of S_+ in S_0 to be 0.

2.2. Rings of fractions of a graded ring.

(2.2.1). Let S be a graded ring, in positive degrees, f a *homogeneous* element of S, of degree $d > 0$; then the ring of fractions $S' = S_f$ is graded, taking for S'_n the set of the x/f^k, where $x \in S_{n+kd}$ with $k \geqslant 0$ (we observe here that n can take arbitrary negative values); we denote the subring $S'_0 = (S_f)_0$ of S' consisting of elements *of degree* 0 by the notation $S_{(f)}$.

If $f \in S_d$, then the monomials $(f/1)^h$ in S_f (h a positive or negative integer) form a *free system* over the ring $S_{(f)}$, and the set of their linear combinations is none other than the ring $(S^{(d)})_f$, which II | 24
is thus *isomorphic to* $S_{(f)}[T, T^{-1}] = S_{(f)} \otimes_{\mathbf{Z}} \mathbf{Z}[T, T^{-1}]$ (where T is an indeterminate). Indeed, if we have a relation $\sum_{h=-a}^{b} z_h (f/1)^h = 0$ with $z_h = x_h/f^m$, where the x_h are in S_{md}, then this relation is equivalent by definition to the existence of a $k > -a$ such that $\sum_{h=-a}^{b} f^{h+k} x_h = 0$, and as the degrees of the terms of this sum are distinct, we have $f^{h+k} x_h = 0$ for all h, hence $z_h = 0$ for all h.

If M is a graded S-module, then $M' = M_f$ is a graded S_f-module, M'_n being the set of the z/f^k with $z \in M_{n+kd}$ ($k \geqslant 0$); we denote by $M_{(f)}$ the set of the homomogenous elements of degree 0 of M'; it is immediate that $M_{(f)}$ is an $S_{(f)}$-module and that we have $(M^{(d)})_f = M_{(f)} \otimes_{S_{(f)}} (S^{(d)})_f$.

Lemma (2.2.2). — *Let d and e be integers > 0, $f \in S_d$, $g \in S_e$. There exists a canonical ring isomorphism*

$$S_{(fg)} \simeq (S_{(f)})_{g^d/f^e};$$

if we canonically identify these two rings, then there exists a canonical module isomorphism

$$M_{(fg)} \simeq (M_{(f)})_{g^d/f^e}.$$

Proof. Indeed, fg divides $f^e g^d$, and this latter element divides $(fg)^{de}$, so the graded rings S_{fg} and $S_{f^e g^d}$ are canonically identified; on the other hand, $S_{f^e g^d}$ also identifies with $(S_{f^e})_{g^d/1}$ (**0**, 1.4.6), and as $f^e/1$ is invertible in S_{f^e}, $S_{f^e g^d}$ also identifies with $(S_{f^e})_{g^d/f^e}$. The element g^d/f^e is of degree 0 in S_{f^e}; we immediately conclude that the subring of $(S_{f^e})_{g^d/f^e}$ consisting of elements of degree 0 is $(S_{(f^e)})_{g^d/f^e}$, and as we evidently have $S_{(f^e)} = S_{(f)}$, this proves the first part of the proposition; the second is established in a similar way. □

(2.2.3). Under the hypotheses of (2.2.2), it is clear that the canonical homomorphism $S_f \to S_{fg}$ (**0**, 1.4.1), which sends x/f^k to $g^k x/(fg)^k$, is of degree 0, thus gives by restriction a *canonical homomorphism* $S_{(f)} \to S_{(fg)}$, such that the diagram

$$\begin{array}{ccc} & S_{(f)} & \\ \swarrow & & \searrow \\ S_{(fg)} & \xrightarrow{\sim} & (S_{(f)})_{g^d/f^e} \end{array}$$

is commutative. We similarly define a canonical homomorphism $M_{(f)} \to M_{(fg)}$.

Lemma (2.2.4). — *If f and g are two homogeneous elements of S_+, then the ring $S_{(fg)}$ is generated by the union of the canonical images of $S_{(f)}$ and $S_{(g)}$.*

Proof. By virtue of Lemma (2.2.2), it suffices to see that $1/(g^d/f^e) = f^{d+e}/(fg)^d$ belongs to the canonical image of $S_{(g)}$ in $S_{(fg)}$, which is evident by definition. □

Proposition (2.2.5). — *Let d be an integer > 0 and let $f \in S_d$. Then there exists a canonical ring isomorphisms $S_{(f)} \simeq S^{(d)}/(f-1)S^{(d)}$; if we identify these two rings by this isomorphism, then there exists a canonical module isomorphism $M_{(f)} \simeq M^{(d)}/(f-1)M^{(d)}$.*

Proof. The first of these isomorphisms is defined by sending x/f^n, where $x \in S_{nd}$, to the element $\overline{x}$, the class of x mod. $(f-1)S^{(d)}$; this map is well-defined, because we have the congruence $f^h x \equiv x \pmod{(f-1)S^{(d)}}$ for all $x \in S^{(d)}$, so if $f^h x = 0$ for an $h > 0$, then we have $\overline{x} = 0$. On the other hand, if $x \in S_{nd}$ is such that $x = (f-1)y$ with $y = y_{hd} + y_{(h+1)d} + \cdots + y_{kd}$ with $y_{jd} \in S_{jd}$ and $y_{hd} \neq 0$, then we necessarily have $h = n$ and $x = -y_{hd}$, as well as the relations $y_{(j+1)d} = f y_{jd}$ for $h \leqslant j \leqslant k-1$, $f y_{kd} = 0$, which ultimately gives $f^{k-n}x = 0$; we send every class $\overline{x}$ mod. $(f-1)S^{(d)}$ of an element $x \in S_{nd}$ to the element x/f^n of $S_{(f)}$, since the preceding remark shows that this map is well-defined. It is immediate that these two maps thus defined are ring homomorphisms, each the reciprocal of the other. We proceed exactly the same way for M. □ **II | 25**

Corollary (2.2.6). — *If S is Noetherian, then so is $S_{(f)}$ for f homogeneous of degree > 0.*

Proof. This follows immediately from Corollary (2.1.7) and Proposition (2.2.5). □

(2.2.7). Let T be a multiplicative subset of S_+ consisting of *homogeneous* elements; $T_0 = T \cup \{1\}$ is then a multiplicative subset of S; as the elements of T_0 are homogeneous, the ring $T_0^{-1}S$ is still graded in the evident way; we denote by $S_{(T)}$ the subring of $T_0^{-1}S$ consisting of elements of order 0, that is to say, the elements of the form x/h, where $h \in T$ and x is homogeneous of degree equal to that of h. We know (**0**, 1.4.5) that $T_0^{-1}S$ is canonically identified with the inductive limit of the rings S_f, where f varies over T (with respect to the canonical homomorphisms $S_f \to S_{fg}$); as this identification respects the degrees, it identifies $S_{(T)}$ with the *inductive limit* of the $S_{(f)}$ for $f \in T$. For every graded S-module M, we similarly define the module $M_{(T)}$ (over the ring $S_{(T)}$) consisting of elements of degree 0 of $T_0^{-1}M$, and we see that this module is the inductive limit of the $M_{(f)}$ for $f \in T$.

If $\mathfrak{p}$ is a graded prime ideal of S_+, then we denote by $S_{(\mathfrak{p})}$ and $M_{(\mathfrak{p})}$ the ring $S_{(T)}$ and the module $M_{(T)}$ respectively, where T is the set of *homogeneous* elements of S_+ which do not belong to $\mathfrak{p}$.

2.3. Homogeneous prime spectrum of a graded ring.

(2.3.1). Given a graded ring S, in positive degrees, we call the *homogeneous prime spectrum* of S and denote it by $\operatorname{Proj}(S)$ the set of graded prime ideals of S_+ (2.1.10), or equivalently the set of graded prime ideals of S *not containing* S_+; we will define a *scheme* structure having $\operatorname{Proj}(S)$ as the underlying set.

(2.3.2). For every subset E of S, let $V_+(E)$ be the set of graded prime ideals of S containing S and not containing S_+; this is thus the subset $V(E) \cap \operatorname{Proj}(S)$ of $\operatorname{Spec}(S)$. From (**I**, 1.1.2) we deduce:

(2.3.2.1) $$V_+(0) = \operatorname{Proj}(S),\ V_+(S) = V_+(S_+) = \varnothing,$$

(2.3.2.2) $$V_+\big(\textstyle\bigcup_\lambda E_\lambda\big) = \textstyle\bigcap_\lambda V_+(E_\lambda),$$

(2.3.2.3) $$V_+(EE') = V_+(E) \cup V_+(E').$$

We do not change $V_+(E)$ by replacing E with the graded ideal generated by E; in addition, if $\mathfrak{J}$ is a graded ideal of S, then we have

(2.3.2.4) $$V_+(\mathfrak{J}) = V_+\big(\textstyle\bigcup_{q \geqslant n}(\mathfrak{J} \cap S_q)\big)$$

for all $n > 0$: indeed, if $\mathfrak{p} \in \operatorname{Proj}(S)$ contains the homogeneous elements of $\mathfrak{J}$ of degree $\geqslant n$, then as by hypothesis there exists a homogeneous element $f \in S_d$ not contained in $\mathfrak{p}$, for every $m \geqslant 0$ and every $x \in S_m \cap \mathfrak{J}$, we have $f^r x \in \mathfrak{J} \cap S_{m+rd}$ for all but finitely many values of r, so $f^r x \in \mathfrak{p} \cap S_{m+rd}$, which implies that $x \in \mathfrak{p} \cap S_m$ (2.1.9). **II | 26**

Finally, we have, for every graded ideal $\mathfrak{J}$ of S,

(2.3.2.5) $$V_+(\mathfrak{J}) = V_+(\mathfrak{r}_+(\mathfrak{J})).$$

§3. Homogeneous spectrum of a sheaf of graded algebras

3.1. Homogeneous spectrum of a quasi-coherent graded $\mathscr{O}_Y$-algebra.

§4. Projective bundles; ample sheaves

4.1. Definition of projective bundles.

§5. Quasi-affine morphisms; quasi-projective morphisms; proper morphisms; projective morphisms

5.1. Quasi-affine morphisms.

Definition (5.1.1). — We define a quasi-affine scheme to be a scheme isomorphic to some subscheme induced on some quasi-compact open subset of an affine scheme. We say that a morphism $f : X \to Y$ is quasi-affine, or that X (considered as a Y-prescheme via f) is a quasi-affine Y-scheme, if there exists a cover (U_α) of Y by affine open subsets such that the $f^{-1}(U_\alpha)$ are quasi-affine schemes.

It is clear that a quasi-affine morphism is *separated* ((**I**, 5.5.5) and (**I**, 5.5.8)) and *quasi-compact* (**I**, 6.6.1); every affine morphisms is evidently quasi-affine.

Recall that, for any prescheme X, setting $A = \Gamma(X, \mathscr{O}_X)$, the identity homomorphism $A \to A = \Gamma(X, \mathscr{O}_X)$ defines a morphism $X \to \operatorname{Spec}(A)$, said to be *canonical* (**I**, 2.2.4); this is nothing but the canonical morphism defined in (4.5.1) for the specific case where $\mathscr{L} = \mathscr{O}_X$, if we remember that $\operatorname{Proj}(A[T])$ is canonically identified with $\operatorname{Spec}(A)$ (3.1.7).

Proposition (5.1.2). — *Let X be a quasi-compact scheme or a prescheme whose underlying space is Noetherian, and A the ring $\Gamma(X, \mathscr{O}_X)$. The following conditions are equivalent.*

(a) *X is a quasi-affine scheme.*
(b) *The canonical morphism $u : X \to \operatorname{Spec}(A)$ is an open immersion.*
(b′) *The canonical morphism $u : X \to \operatorname{Spec}(A)$ is a homeomorphism from X to some subspace of the underlying space of* $\operatorname{Spec}(A)$.
(c) *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is very ample relative to u* (4.4.2).
(c′) *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is ample* (4.5.1).
(d) *When f ranges over A, the X_f form a basis for the topology of X.*
(d′) *When f varies over A, the X_f that are affine form a cover of X.* **II** | 95
(e) *Every quasi-coherent $\mathscr{O}_X$-module is generated by its sections over X.*
(e′) *Every quasi-coherent sheaf of ideals of $\mathscr{O}_X$ of finite type is generated by its sections over X.*

Proof. It is clear that (b) implies (a), and (a) implies (c) by (4.4.4, b) applied to the identity morphism (taking into account the remark preceding this proposition); Furthermore, (c) implies (c′) (4.5.10, i), and (c′), (b), and (b′) are all equivalent by (4.5.2, b) and (4.5.2, b′). Finally, (c′) is the same as each of (d), (d′), (e), and (e′) by (4.5.2, a), (4.5.2, a′), (4.5.2, c), and (4.5.5, d″). ☐

We further observe that, with the previous notation, the X_f that are affine form a *basis* for the topology of X, and that the canonical morphism u is *dominant* (4.5.2).

Corollary (5.1.3). — *Let X be a quasi-compact prescheme. If there exists a morphism $v : X \to Y$ from X to some affine scheme Y (which would be a homeomorphism from X to some open subspace of Y), then X is quasi-affine.*

Proof. There exists a family (g_α) of sections of $\mathscr{O}_Y$ over Y such that the $D(g_\alpha)$ form a basis for the topology of $v(X)$; if $v = (\psi, \theta)$ and we set $f_\alpha = \theta(g_\alpha)$, then we have $X_{f_\alpha} = \psi^{-1}(D(g_\alpha))$ (**I**, 2.2.4.1), so the X_{f_α} form a basis for the topology of X, and the criterion (5.1.2, d) is satisfied. ☐

Corollary (5.1.4). — *If X is a quasi-affine scheme, then* every *invertible $\mathscr{O}_X$-module is very ample (relative to the canonical morphism), and* a fortiori *ample.*

Proof. Such a module $\mathscr{L}$ is generated by its sections over X (5.1.2, e), so $\mathscr{L} \otimes \mathscr{O}_X = \mathscr{L}$ is very ample (4.4.8). ☐

Corollary (5.1.5). — *Let X be a quasi-compact prescheme. If there exists an invertible $\mathscr{O}_X$-module $\mathscr{L}$ such that $\mathscr{L}$ and $\mathscr{L}^{-1}$ are ample, then X is a quasi-affine scheme.*

Proof. Indeed, $\mathscr{O}_X = \mathscr{L} \otimes \mathscr{L}^{-1}$ is then ample (4.5.7). □

Proposition (5.1.6). — *Let $f : X \to Y$ be a quasi-compact morphism. Then the following conditions are equivalent.*

(a) *The morphism f is quasi-affine.*
(b) *The $\mathscr{O}_Y$-algebra $f_*(\mathscr{O}_X) = \mathscr{A}(X)$ is quasi-coherent, and the canonical morphism $X \to \operatorname{Spec}(\mathscr{A}(X))$ corresponding to the identity morphism $\mathscr{A}(X) \to \mathscr{A}(X)$ (1.2.7) is an open immersion.*
(b′) *The $\mathscr{O}_Y$-algebra $\mathscr{A}(X)$ is quasi-coherent, and the canonical morphism $X \to \operatorname{Spec}(\mathscr{A}(X))$ is a homeomorphism from X to some subspace of $\operatorname{Spec}(\mathscr{A}(X))$.*
(c) *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is very ample for f.*
(c′) *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is ample for f.*
(d) *The morphism f is separated, and, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\sigma : f^*(f_*(\mathscr{F})) \to \mathscr{F}$ (**0**, 4.4.3) is surjective.*

Furthermore, whenever f is quasi-affine, every invertible $\mathscr{O}_X$-module $\mathscr{L}$ is very ample relative to f.

Proof. The equivalence between (a) and (c′) follows from the local (on Y) character of the f-ampleness (4.6.4), Definition (5.1.1), and the criterion (5.1.2, c′). The other properties are local on Y and thus **II** | 96
follow immediately from (5.1.2) and (5.1.4), taking into account the fact that $f_*(\mathscr{F})$ is quasi-coherent whenever f is separated (**I**, 9.2.2, a). □

Corollary (5.1.7). — *Let $f : X \to Y$ be a quasi-affine morphism. For every open subset U of Y, the restriction $f^{-1}(U) \to U$ of f is quasi-affine.*

Corollary (5.1.8). — *Let Y be an affine scheme, and $f : X \to Y$ a quasi-compact morphism. For f to be quasi-affine, it is necessary and sufficient for X to be a quasi-affine scheme.*

Proof. This is an immediate consequence of (5.1.6) and (4.6.6). □

Corollary (5.1.9). — *Let Y be a quasi-compact scheme or a prescheme whose underlying space is Noetherian, and $f : X \to Y$ a morphism of* finite type. *If f is quasi-affine, then there exists a quasi-coherent $\mathscr{O}_Y$-subalgebra $\mathscr{B}$ of $\mathscr{A}(X) = f_*(\mathscr{O}_X)$ of* finite type (**I**, 9.6.2) *such that the morphism $X \to \operatorname{Spec}(\mathscr{B})$ corresponding to the canonical injection $\mathscr{B} \to \mathscr{A}(X)$ is an immersion. Further, every quasi-coherent $\mathscr{O}_Y$-subalgebra $\mathscr{B}'$ of finite type over $\mathscr{A}(X)$ containing $\mathscr{B}$ has the same property.*

Proof. Indeed, $\mathscr{A}(X)$ is the inductive limit of its quasi-coherent $\mathscr{O}_Y$-subalgebras of finite type (**I**, 9.6.5); the result is then a particular case of (3.8.4), taking into account the identification of $\operatorname{Spec}(\mathscr{A}(X))$ with $\operatorname{Proj}(\mathscr{A}(X)[T])$ (3.1.7). □

Proposition (5.1.10). —

(i) *A quasi-compact morphism $X \to Y$ that is a homeomorphism from the underlying space of X to some subspace of the underlying space of Y (so, in particular, any closed immersion) is quasi-affine.*
(ii) *The composition of any two quasi-affine morphisms is quasi-affine.*
(iii) *If $f : X \to Y$ is a quasi-affine S-morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is a quasi-affine morphism for any extension $S' \to S$ of the base prescheme.*
(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are quasi-affine S-morphisms, then $f \times_S g$ is quasi-affine.*
(v) *If $f : X \to Y$ and $g : Y \to Z$ are morphisms such that $g \circ f$ is quasi-affine, and if g is separated or the underlying space of X is locally Noetherian, then f is quasi-affine.*
(vi) *If f is a quasi-affine morphism, then so is f_{red}.*

Proof. Taking into account the criterion (5.1.6, c′), all of (i), (iii), (iv), (v), and (vi) follow immediately from (4.6.13, i *bis*), (4.6.13, iii), (4.6.13, iv), (4.6.13, v), and (4.6.13, vi) (respectively). To prove (ii), we can restrict to the case where Z is affine, and then the claim follows directly from applying (4.6.13, ii) to $\mathscr{L} = \mathscr{O}_X$ and $\mathscr{K} = \mathscr{O}_Y$. □

Remark (5.1.11). — Let $f : X \to Y$ and $g : Y \to Z$ be morphisms such that $X \times_Z Y$ is locally Noetherian. Then the graph immersion $\Gamma_f : X \to X \times_Z Y$ is quasi-affine, since it is quasi-compact (**I**, 6.3.5), and since (**I**, 5.5.12) shows that, in (v), the conclusion still holds true if we remove the hypothesis that g is separated.

Proposition (5.1.12). — *Let $f : X \to Y$ be a quasi-compact morphism, and $g : X' \to X$ a quasi-affine morphism. If $\mathscr{L}$ is an ample (for f) $\mathscr{O}_X$-module, then $g^*(\mathscr{L})$ is an ample (for $f \circ g$) $\mathscr{O}_{X'}$-module.*

Proof. Since $\mathscr{O}_{X'}$ is very ample for g, and the question is local on Y (4.6.4), it follows from (4.6.13, ii) that there exists (for Y affine) an integer n such that

$$g^*(\mathscr{L}^{\otimes n}) = (g^*(\mathscr{L}))^{\otimes n}$$

is ample for $f \circ g$, and so $g^*(\mathscr{L})$ is ample for $f \circ g$ (4.6.9) □

5.2. Serre's criterion.

Theorem (5.2.1). — (Serre's criterion). *Let X be a quasi-compact scheme or a prescheme whose underlying space is Noetherian. The following conditions are equivalent.*

(a) *X is an affine scheme.*

(b) *There exists a family of elements $f_\alpha \in A = \Gamma(X, \mathscr{O}_X)$ such that the X_{f_α} are affine, and such that the ideal generated by the f_α in A is equal to A itself.*

(c) *The functor $\Gamma(X, \mathscr{F})$ is exact in $\mathscr{F}$ on the category of quasi-coherent $\mathscr{O}_X$-modules, or, in other words, if*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0 \qquad (*)$$

is an exact sequence of quasi-coherent $\mathscr{O}_X$-modules, then the sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}') \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{F}'') \longrightarrow 0$$

is also exact.

(c′) *Condition* (c) *holds for every exact sequence $(*)$ of quasi-coherent $\mathscr{O}_X$-modules such that $\mathscr{F}$ is isomorphic to a $\mathscr{O}_X$-submodule of $\mathscr{O}_X^n$ for some finite n.*

(d) $\mathrm{H}^1(X, \mathscr{F}) = 0$ *for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$.*

(d′) $\mathrm{H}^1(X, \mathscr{J}) = 0$ *for every quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X$.*

Proof. It is evident that (a) implies (b); furthermore, (b) implies that the X_{f_α} cover X, because, by hypothesis, the section 1 is a linear combination of the f_α, and the $D(f_\alpha)$ thus cover $\operatorname{Spec}(A)$. The final claim of (4.5.2) thus implies that $X \to \operatorname{Spec}(A)$ is an isomorphism.

We know tha (a) implies (c) (**I**, 1.3.11), and (c) trivially implies (c′). We now prove that (c′) implies (b). First of all, (c′) implies that, for every *closed* point $x \in X$ and every open neighbourhood U of x, there exists some $f \in A$ such that $x \in X_f \subset X - U$. Let $\mathscr{J}$ (resp. $\mathscr{J}'$) be the quasi-coherent sheaf of ideals of $\mathscr{O}_X$ defining the reduced closed subprescheme of X that has $X - U$ (resp. $(X - U) \cup \{x\}$) as its underlying space (**I**, 5.2.1); it is clear that we have $\mathscr{J}' \subset \mathscr{J}$, and that $\mathscr{J}'' = \mathscr{J} / \mathscr{J}'$ is a quasi-coherent $\mathscr{O}_X$ module that has support equal to $\{x\}$, and such that $\mathscr{J}''_x = k(x)$. Hypothesis (c′) applied to the exact sequence $0 \to \mathscr{J}' \to \mathscr{J} \to \mathscr{J}'' \to 0$ shows that $\Gamma(X, \mathscr{J}) \to \Gamma(X, \mathscr{J}'')$ is surjective. The section of $\mathscr{J}''$ whose germ at x is 1_x is thus the image of some section $f \in \Gamma(X, \mathscr{J}) \subset \Gamma(X, \mathscr{O}_X)$, and we have, by definition, that $f(x) = 1_x$ and $f(y) = 0$ in $X - U$, which establishes our claim. Now, if U is affine, then so is X_f (**I**, 1.3.6), so the union of the X_f that are affine ($f \in A$) is an open set Z that contains *all the closed points* of X; since X is a quasi-compact Kolmogoroff space, we necessarily have $Z = X$ (**0**, 2.1.3). Because X is quasi-compact, there are a *finite* number of elements $f_i \in A$ ($1 \leqslant i \leqslant n$) such that the X_{f_i} are affine and cover X. So consider the homomorphism $\mathscr{O}_X^n \to \mathscr{O}_X$ defined by the sections f_i (**0**, 5.1.1); since, for all $x \in X$, at least one of the $(f_i)_x$ is invertible, this homomorphism is *surjective*, and we thus have an exact sequence $0 \to \mathscr{R} \to \mathscr{O}_X^n \to \mathscr{O}_X \to 0$, where $\mathscr{R}$ is a quasi-coherent $\mathscr{O}_X$-submodule of $\mathscr{O}_X$. It then follows from (c′) that the corresponding homomorphism $\Gamma(X, \mathscr{O}_X^n) \to \Gamma(X, \mathscr{O}_X)$ is surjective, which **II** | 98 proves (b).

Finally, (a) implies (d) (**I**, 5.1.9.2), and (d) trivially implies (d′). It remains to show that (d′) implies (c′). But if $\mathscr{F}'$ is a quasi-coherent $\mathscr{O}_X$-submodule of $\mathscr{O}_X^n$, then the filtration $0 \subset \mathscr{O}_X \subset \mathscr{O}_X^2 \subset \ldots \subset \mathscr{O}_X^n$ defines a filtration of $\mathscr{F}'$ given by the $\mathscr{F}'_k = \mathscr{F} \cap \mathscr{O}_X^k$ ($0 \leqslant k \leqslant n$), which are quasi-coherent $\mathscr{O}_X$-modules (**I**, 4.1.1), and $\mathscr{F}'_{k+1} / \mathscr{F}'_k$ is isomorphic to a quasi-coherent $\mathscr{O}_X$-submodule of $\mathscr{O}_X^{k+1} / \mathscr{O}_X^k = \mathscr{O}_X$, which is to say, a quasi-coherent sheaf of ideals of $\mathscr{O}_X$. Hypothesis (d′) thus implies that $\mathrm{H}^1(X, \mathscr{F}'_{k+1} / \mathscr{F}'_k) = 0$; the exact cohomology sequence $\mathrm{H}^1(X, \mathscr{F}'_k) \to \mathrm{H}^1(X, \mathscr{F}'_{k+1}) \to \mathrm{H}^1(X, \mathscr{F}'_{k+1} / \mathscr{F}'_k) = 0$ then lets us prove by induction on k that $H^1(X, \mathscr{F}'_k) = 0$ for all k. □

Remark (5.2.1.1). — When X is a *Noetherian* prescheme, we can replace "quasi-coherent" by "coherent" in the statements of (c′) and (d′). Indeed, in the proof of the fact that (c′) implies (b), $\mathscr{J}$ and $\mathscr{J}'$ are then *coherent* sheaves of ideals, and, furthermore, every quasi-coherent submodule of a coherent module is coherent (**I**, 6.1.1); whence the conclusion.

Corollary (5.2.2). — *Let $f : X \to Y$ be a separated quasi-compact morphism. The following conditions are equivalent.*

(a) *The morphism f is an affine morphism.*
(b) *The functor f_* is exact on the category of quasi-coherent $\mathscr{O}_X$-modules.*
(c) *For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have $\mathrm{R}^1 f_*(\mathscr{F}) = 0$.*
(c′) *for every quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X$, we have $\mathrm{R}^1 f_*(\mathscr{J}) = 0$.*

Proof. All these conditions are local on Y, by definition of the functor $\mathrm{R}^1 f_*$ (T, 3.7.3), and so we can assume that Y is affine. If f is affine, then X is affine, and property (b) is nothing more than (**I**, 1.6.4). Conversely, we now show that (b) implies (a): for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have that $f_*(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_Y$-module (**I**, 9.2.2, a). By hypothesis, the functor $f_*(\mathscr{F})$ is exact in $\mathscr{F}$, and the functor $\Gamma(Y, \mathscr{G})$ is exact in $\mathscr{G}$ (in the category of quasi-coherent $\mathscr{O}_Y$-modules) because Y is affine (**I**, 1.3.11); so $\Gamma(Y, f_*(\mathscr{F})) = \Gamma(X, \mathscr{F})$ is exact in $\mathscr{F}$, which proves our claim, by (5.2.1, c).

If f is affine, then $f^{-1}(U)$ is affine for every affine open subset U of Y (1.3.2), and so $\mathrm{H}^1(f^{-1}(U), \mathscr{F}) = 0$ (5.2.1, d), which, by definition, implies that $\mathrm{R}^1 f_*(\mathscr{F}) = 0$. Finally, suppose that condition (c′) is satisfied; the exact sequence of terms of low degree in the Leray spectral sequence (G, II, 4.17.1 and I, 4.5.1) give, in particular, the exact sequence

$$0 \longrightarrow \mathrm{H}^1(Y, f_*(\mathscr{J})) \longrightarrow \mathrm{H}^1(X, \mathscr{J}) \longrightarrow \mathrm{H}^0(Y, \mathrm{R}^1 f_*(\mathscr{J})).$$

Since Y is affine, and $f_*(\mathscr{J})$ quasi-coherent (**I**, 9.2.2, a), we have that $\mathrm{H}^1(Y, f_*(\mathscr{J})) = 0$ (5.2.1); hypothesis (c′) thus implies that $\mathrm{H}^1(X, \mathscr{J}) = 0$, and we conclude, by (5.2.1), that X is an affine scheme. □

Corollary (5.2.3). — *If $f : X \to Y$ is an affine morphism, then, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\mathrm{H}^1(Y, f_*(\mathscr{F})) \to \mathrm{H}^1(X, \mathscr{F})$ is bijective.* **II** | 99

Proof. We have the exact sequence

$$0 \longrightarrow \mathrm{H}^1(Y, f_*(\mathscr{F})) \longrightarrow \mathrm{H}^1(X, \mathscr{F}) \longrightarrow \mathrm{H}^0(Y, \mathrm{R}^1 f_*(\mathscr{F}))$$

of terms of low degree in the Leray spectral sequence, and the conclusion follows from (5.2.2). □

Remark (5.2.4). — In Chapter III, §1, we prove that, if X is affine, then we have $\mathrm{H}^i(X, \mathscr{F}) = 0$ for all $i > 0$ and all quasi-coherent $\mathscr{O}_X$-modules $\mathscr{F}$.

5.3. Quasi-projective morphisms.

Definition (5.3.1). — We say that a morphism $f : X \to Y$ is *quasi-projective*, or that X (considered as a Y-prescheme via f) is *quasi-projective over Y*, or that X is a *quasi-projective Y-scheme*, if f is of finite type and there exists an invertible f-ample $\mathscr{O}_X$-module.

We note that this notion *is not local on Y*: the counterexamples of Nagata [Nag58] and Hironaka show that, even if X and Y are non-singular algebraic schemes over an algebraically closed field, every point of Y can have an affine neighbourhood U such that $f^{-1}(U)$ is quasi-projective over U, without f being quasi-projective.

We note that a quasi-projective morphism is necessarily *separated* (4.6.1). When Y is quasi-compact, it is equivalent to say either that f is quasi-projective, or that f is of finite type and there exists a *very ample* (relative to f) $\mathscr{O}_X$-module ((4.6.2) and (4.6.11)). Further:

Proposition (5.3.2). — *Let Y be a quasi-compact scheme or a prescheme whose underlying space is Noetherian, and let X be a Y-prescheme. The following conditions are equivalent.*

(a) *X is a quasi-projective Y-scheme.*
(b) *X is of finite type over Y, and there exists some quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ of finite type such that X is Y-isomorphic to a subprescheme of $\mathbf{P}(\mathscr{E})$.*

(c) *X is of finite type over Y, and there exists some quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ such that $\mathscr{S}_1$ is of finite type and generates $\mathscr{S}$, and such that X is Y-isomorphic to a induced subprescheme on some everywhere-dense open subset of* $\operatorname{Proj}(\mathscr{S})$.

Proof. This follows immediately from the previous remark and from (4.4.3), (4.4.6), and (4.4.7). ☐

We note that, whenever Y is a *Noetherian* prescheme, we can, in conditions (b) and (c) of (5.3.2), remove the hypothesis that X is of finite type over Y, since this automatically satisfied (**I**, 6.3.5).

Corollary (5.3.3). — *Let Y be a quasi-compact scheme such that there exists an ample $\mathscr{O}_Y$-module $\mathscr{L}$* (4.5.3)*. For a Y-scheme X to be quasi-projective, it is necessary and sufficient for it to be of finite type over Y and also isomorphic to a Y-subscheme of a projective bundle of the form $\mathbf{P}^r_Y$.*

Proof. If $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type, then $\mathscr{E}$ is isomorphic to a quotient of an $\mathscr{O}_Y$-module $\mathscr{L}^{\otimes(-n)} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y^k$ (4.5.5), and so $\mathbf{P}(\mathscr{E})$ is isomorphic to a closed subscheme of $\mathbf{P}^{k-1}_Y$ ((4.1.2) and (4.1.4)). ☐

Proposition (5.3.4). —

(i) *A quasi-affine morphism of finite type (and, in particular, a quasi-compact immersion, or an affine morphism of finite type) is quasi-projective.*
(ii) *If $f : X \to Y$ and $g : Y \to Z$ are quasi-projective, and if Z is quasi-compact, then $g \circ f$ is quasi-projective.* **II** | 100
(iii) *If $f : X \to Y$ is a quasi-projective S-morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is quasi-projective for every extension $S' \to S$ of the base prescheme.*
(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are quasi-projective S-morphisms, then $f \times_S g$ is quasi-projective.*
(v) *If $f : X \to Y$ and $g : Y \to Z$ are morphisms such that $g \circ f$ is quasi-projective, and if g is separated or X locally Noetherian, then f is quasi-projective.*
(vi) *If f is a quasi-projective morphism, then so is f_{red}.*

Proof. (i) follows from (5.1.6) and (5.1.10, i). The other claims are immediate consequences of Definition (5.3.1), of the properties of morphisms of finite type (**I**, 6.3.4), and of (4.6.13). ☐

Remark (5.3.5). — We note that we can have f_{red} being quasi-projective without f being quasi-projective, even if we assume that Y is the spectrum of an algebra of finite rank over $\mathbf{C}$ and that f is proper.

Corollary (5.3.6). — *If X and X' are quasi-projective Y-schemes, then $X \sqcup X'$ is a quasi-projective Y-scheme.*

Proof. This follows from (4.6.18). ☐

5.4. Proper morphisms and universally closed morphisms.

Definition (5.4.1). — We say that a morphism of preschemes $f : X \to Y$ is *proper* if it satisfies the following two conditions:

(a) f is separated and of finite type; and
(b) for every prescheme Y' and every morphism $Y' \to Y$, the projection $f_{(Y')} : X \times_Y Y' \to Y'$ is a closed morphism (**I**, 2.2.6).

When this is the case, we also say that X (considered as a Y-prescheme with structure morphism f) is proper over Y.

It is immediate that conditions (a) and (b) are *local* on Y. To show that the image of a closed subset Z of $X \times_Y Y'$ under the projection $q : X \times_Y Y' \to Y'$ is closed in Y, it suffices to see that $q(Z) \cap U'$ is closed in U' for every affine open subset U' of Y'; since $q(Z) \cap U' = q(Z \cap q^{-1}(U'))$, and since $q^{-1}(U')$ can be identified with $X \times_Y U'$ (**I**, 4.4.1), we see that to satisfy condition (b) of Definition (5.4.1), we can restrict to the case where Y is an *affine* scheme. We further see (5.3.6) that, if Y is locally Noetherian, then we can even restrict to proving (b) in the case where Y' is of finite type over Y.

It is clear that every proper morphism is *closed*.

Proposition (5.4.2). —

(i) *A closed immersion is a proper morphism.*
(ii) *The composition of two proper morphisms is proper.*
(iii) *If X and Y are S-preschemes, and $f : X \to Y$ a proper S-morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is proper for every extension $S' \to S$ of the base prescheme.*
(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are proper S-morphisms, then $f \times_S g : X \times_S Y \to X' \times_S Y'$ is a proper S-morphism.*

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Proof. It suffices to prove (i), (ii), and (iii) (**I**, 3.5.1). In each of the three cases, verifying condition (a) of (5.4.1) follows from previous results ((**I**, 5.5.1) and (6.4.3)); it remains to verify condition (b). It is immediate in case (i), because if $X \to Y$ is a closed immersion, then so is $X \times_Y Y' \to Y \times_Y Y' = Y'$ ((**I**, 4.3.2) and (3.3.3)). To prove (ii), consider two proper morphisms $X \to Y$ and $Y \to Z$, and a morphism $Z' \to Z$. We can write $X \times_Z Z' = X \times_Y (Y \times_Z Z')$ (**I**, 3.3.9.1), and so the projection $X \times_Z Z' \to Z'$ factors as $X \times_Y (Y \times_Z Z') \to Y \times_Z Z' \to Z'$. Taking the initial remark into account, (ii) follows from the fact that the composition of two closed morphisms is closed. Finally, for every morphism $S' \to S$, we can identify $X_{(S')}$ with $X \times_Y Y_{(S')}$ (**I**, 3.3.11); for every morphism $Z \to Y_{(S')}$, we can write

$$X_{(S')} \times_{Y_{(S')}} Z = (X \times_Y Y_{(S')}) \times_{Y_{(S')}} Z = X \times_Y Z;$$

since by hypothesis $X \times_Y Z \to Z$ is closed, this proves (iii). □

Corollary (5.4.3). — *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms such that $g \circ f$ is proper.*

(i) *If g is separated, then f is proper.*
(ii) *If g is separated and of finite type, and if f is surjective, then g is proper.*

Proof. (i) follows from (5.4.2) by the general procedure (**I**, 5.5.12). To prove (ii), we need only verify that condition (b) of Definition (5.4.1) is satisfied. For every morphism $Z' \to Z$, the diagram

$$\begin{array}{ccc} X \times_Z Z' & \xrightarrow{f \times 1_{Z'}} & Y \times_Z Z' \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle p'} \\ & & Z' \end{array}$$

(where p and p' are the projections) commutes (**I**, 3.2.1); furthermore, $f \times 1_{Z'}$ is surjective because f is surjective (**I**, 3.5.2), and p is a closed morphism by hypothesis. Every closed subset F of $Y \times_Z Z'$ is thus the image under $f \times 1_{Z'}$ of some closed subset E of $X \times_Z Z'$, so $p'(F) = p(E)$ is closed in Z' by hypothesis, whence the corollary. □

Corollary (5.4.4). — *If X is a proper prescheme over Y, and $\mathscr{S}$ a quasi-coherent $\mathscr{O}_Y$-algebra, then every Y-morphism $f : X \to \mathrm{Proj}(\mathscr{S})$ is proper (and* a fortiori *closed).*

Proof. The structure morphism $p : \mathrm{Proj}(\mathscr{S}) \to Y$ is separated, and $p \circ f$ is proper by hypothesis. □

Corollary (5.4.5). — *Let $f : X \to Y$ be a separated morphism of finite type. Let $(X_i)_{1 \leqslant i \leqslant n}$ (resp. $(Y_i)_{1 \leqslant i \leqslant n}$) be a finite family of closed subpreschemes of X (resp. Y), and j_i (resp. h_i) the canonical injection $X_i \to X$ (resp. $Y_i \to Y$). Suppose that the underlying space of X is the union of the X_i, and that, for all i, there is a morphism $f_i : X_i \to Y_i$, such that the diagram*

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_i \\ {\scriptstyle j_i}\downarrow & & \downarrow{\scriptstyle h_i} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Then, for f to be proper, it is necessary and sufficient for all of the f_i to be proper.

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Proof. If f is proper, then so is $f \circ j_i$, because j_i is a closed immersion (5.4.2); since h_i is a closed immersion, and thus a separated morphism, f_i is proper, by (5.4.3). Conversely, suppose that all of the f_i are proper, and consider the prescheme Z given by the *sum* of the X_i; let u be the morphism $Z \to X$ which reduces to j_i on each X_i. The restriction of $f \circ u$ to each X_i is equal to $f \circ j_i = h_i \circ f_i$, and is thus proper, because both the h_i and the f_i are (5.4.2); it then follows immediately from

Definition (5.4.1) that u is proper. But since by hypothesis u is surjective, we conclude that f is proper by (5.4.3). □

Corollary (5.4.6). — *Let $f : X \to Y$ be a separated morphism of finite type; for f to be proper, it is necessary and sufficient for $f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}}$ to be proper.*

Proof. This is a particular case of (5.4.5), with $n = 1$, $X_1 = X_{\text{red}}$, and $Y_1 = Y_{\text{red}}$ (**I**, 5.1.5). □

(5.4.7). If X and Y are Noetherian preschemes, and $f : X \to Y$ a separated morphism of finite type, then we can, to show that f is proper, restrict to the the case of *dominant* morphisms and *integral* preschemes. Indeed, let X_i ($1 \leqslant i \leqslant n$) be the (finitely many) irreducible components of X, and consider, for each i, the unique reduced closed subprescheme of X that has X_i as its underlying space, which we again denote by X_i (**I**, 5.2.1). Let Y_i be the unique reduced closed subprescheme of Y that has $\overline{f(X_i)}$ as its underlying space. If g_i (resp. h_i) is the injection morphism $X_i \to X$ (resp. $Y_i \to Y$), then we conclude that $f \circ g_i = h_i \circ f_i$, where f_i is a dominant morphism $X_i \to Y_i$ (**I**, 5.2.2); we are then under the right conditions to apply (5.4.5), and for f to be proper, it is necessary and sufficient for all the f_i to be proper.

Corollary (5.4.8). — *Let X and Y be separated S-preschemes of finite type over S, and $f : X \to Y$ an S-morphism. For f to be proper, it is necessary and sufficient that, for every S-prescheme S', the morphism $f \times_S 1_{S'} : X \times_S S' \to Y \times_S S'$ be closed.*

Proof. First note that, if $g : X \to S$ and $h : Y \to S$ are the structure morphisms, then we have, by definition, $g = h \circ f$, and so f is separated and of finite type ((**I**, 5.5.1) and (6.3.4)). If f is proper, then so is $f \times_S 1_{S'}$ (5.4.2); *a fortiori*, $f \times_S 1_{S'}$ is closed. Conversely, suppose that the conditions of the statement are satisfied, and let Y' be a Y-prescheme; Y' can also be considered as an S-prescheme, and since $Y \to S$ is separated, $X \times_Y Y'$ can be identified with a closed subprescheme of $X \times_S Y'$ (**I**, 5.4.2). In the commutative diagram

$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{f \times 1_{Y'}} & Y \times_Y Y' = Y' \\ \downarrow & & \downarrow \\ X \times_S Y' & \xrightarrow{f \times 1_{S'}} & Y \times_S Y', \end{array}$$

the vertical arrows are closed immersions; it thus immediately follows that if $f \times 1_{S'}$ is a closed morphism, then so is $f \times 1_{Y'}$ □

Remark (5.4.9). — We say that a morphism $f : X \to Y$ is *universally closed* if it satisfies condition (b) of Definition (5.4.1). The reader will observe that, in (5.4.2) to (5.4.8), we can replace every occurrence II | 103
of "proper" with "universally closed" without changing the validity of the results (and in the hypotheses of (5.4.3), (5.4.5), (5.4.6), and (5.4.8), we can omit the finiteness conditions).

(5.4.10). Let $f : X \to Y$ be a morphism of finite type. We say that a closed subset Z of X is *proper on Y* (or *Y-proper*, or *proper for f*) if the restriction of f to a closed subprescheme of X, with underlying space Z (**1**, 5.2.1), is *proper*. Since this restriction is then separated, it follows from (5.4.6) and (**I**, 5.5.1, vi) that the preceding property *does not depend* on the closed subprescheme of X that has Z as its underlying space. If $g : X' \to X$ is a *proper* morphism, then $g^{-1}(Z)$ is a *proper* subset of X': if T is a subprescheme of X that has Z as its underlying space, it suffices to note that the restriction of g to the closed subprescheme $g^{-1}(T)$ of X' is a proper morphism $g^{-1}(T) \to T$, by (5.4.2, iii), and to then apply (5.4.2, ii). Further, if X'' is a *Y-scheme* of finite type, and $u : X \to X''$ a Y-morphism, then $u(Z)$ is a *proper* subset of X''; indeed, let us take T to be the reduced closed subprescheme of X having Z as its underlying space; then the restriction of f to T is proper, and thus so is the restriction of u to T (5.4.3, i), thus $u(Z)$ is closed in X''; let T'' be a closed subprescheme of X'' having $u(Z)$ as its underlying space (**I**, 5.2.1), such that $u|T$ factors as $T \xrightarrow{v} T'' \xrightarrow{j} X''$, where j is the canonical injection (**I**, 5.2.2), and v is thus proper and surjective (5.4.5); if g is the restriction to T'' of the structure morphism $X'' \to Y$, then g is separated and of finite type, and we have that $f|T = g \circ v$; it thus follows from (5.4.3, ii) that g is proper, whence our assertion.

It follows, in particular, from these remarks that, if Z is a Y-proper subset of X, then

(1) for every closed subprescheme X' of X, $Z \cap X'$ is a Y-proper subset of X'; and
(2) if X is a subprescheme of a Y-scheme of finite type X'', then Z is also a Y-proper subset of X'' (and so, in particular, is *closed in* X'').

5.5. Projective morphisms.

Proposition (5.5.1). — *Let X be a Y-prescheme. The following conditions are equivalent.*

(a) *X is Y-isomorphic to a* closed *subprescheme of a projective bundle $\mathbf{P}(\mathscr{E})$, where $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type.*
(b) *There exists a quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ such that $\mathscr{S}_1$ is of finite type and generates $\mathscr{S}$, and such that X is Y-isomorphic to* $\operatorname{Proj}(\mathscr{S})$.

Proof. Condition (a) implies (b), by (3.6.2, ii): if $\mathscr{J}$ is a quasi-coherent graded sheaf of ideals of $\mathbf{S}(\mathscr{E})$, then the quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S} = \mathbf{S}(\mathscr{E})/\mathscr{J}$ is generated by $\mathscr{S}_1$, and $\mathscr{S}_1$, the canonical image of $\mathscr{E}$, is an $\mathscr{O}_Y$-module of finite type. Condition (b) implies (a) by (3.6.2) applied to the case where $\mathscr{M} \to \mathscr{S}_1$ is the identity map. □

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Definition (5.5.2). — We say that a Y-prescheme X is *projective* on Y, or is a projective Y-scheme, if it satisfies either of the (equivalent) conditions (a) and (b) of (5.5.1). We say that a morphism $f : X \to Y$ is projective if it makes X a projective Y-scheme.

It is clear that if $f : X \to Y$ is projective, then there exists a *very ample* (relative to f) $\mathscr{O}_X$-module (4.4.2).

Theorem (5.5.3). —

(i) *Every projective morphism is quasi-projective and proper.*
(ii) *Conversely, let Y be a quasi-compact scheme or a prescheme whose underlying space is Noetherian; then every morphism $f : X \to Y$ that is quasi-projective and proper is projective.*

Proof.

(i) It is clear that if $f : X \to Y$ is projective, then it is of finite type and quasi-projective (thus, in particular, separated); furthermore, it follows immediately from (5.5.1, b) and (3.5.3) that if f is projective, then so is $f \times_Y 1_{Y'} : X \times_Y Y' \to Y'$ for every morphism $Y' \to Y$. To show that f is universally closed, it is thus enough to show that a projective morphism f is *closed*. Since the question is local on Y, we can suppose that $Y = \operatorname{Spec}(A)$, thus (5.5.1) $X = \operatorname{Proj}(S)$, where S is a graded A-algebra generated by a finite number of elements of S_1. For all $y \in Y$, the fibre $f^{-1}(y)$ can be identified with $\operatorname{Proj}(S) \times_Y \operatorname{Spec}(k(y))$ (**I**, 3.6.1), and so also with $\operatorname{Proj}(S \otimes_A k(y))$ (2.8.10); so $f^{-1}(y)$ is empty if and only if $S \otimes_A k(y)$ satisfies condition (TN) (2.7.4), or, in other words, if $S_n \otimes_A k(y) = 0$ for sufficiently large n. But since $(S_n)_y$ is an $\mathscr{O}_y$-module of finite type, the preceding condition implies that $(S_n)_y = 0$ for sufficiently large N, by Nakayama's lemma. If $\mathfrak{a}_n$ is the annihilator in A of the A-module S_n, then the preceding condition also implies that $\mathfrak{a}_n \subset \mathfrak{j}_n$ for sufficiently large n (**0**, 1.7.4). But since $S_n S_1 = S_{n+1}$, by hypothesis, we have that $\mathfrak{a}_n \subset \mathfrak{a}_{n+1}$, and if $\mathfrak{a}$ is the union of the $\mathfrak{a}_n$, then we see that $f(X) = V(\mathfrak{a})$, which proves that $f(X)$ is closed in Y. If now X' is an arbitrary closed subset of X, then there exists a closed subprescheme of X that has X' as its underlying space (**I**, 5.2.1), and it is clear (5.5.1, a) that the morphism $X' \to X \xrightarrow{f} Y$ is projective, and so $f(X')$ is closed in Y.
(ii) The hypothesis on Y and the fact that f is quasi-projective implies the existence of a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ of finite type, as well as a Y-immersion $j : X \to \mathbf{P}(\mathscr{E})$ (5.3.2). But since f is proper, j is *closed*, by (5.4.4), and so f is projective.

□

Remark (5.5.4). —

(i) Let $f : X \to Y$ be a morphism such that f is proper, such that there exists a *very ample* (relative to f) $\mathcal{O}_X$-module $\mathscr{L}$, and such that the quasi-coherent $\mathcal{O}_Y$-module $\mathscr{E} = f_*(\mathscr{L})$ is *of finite type*. Then f is a *projective* morphism: indeed (4.4.4), there is then a Y-immersion $r : X \to \mathbf{P}(\mathscr{E})$, and, since f is proper, r is a *closed* immersion (5.4.4). We will see in Chapter III, §3, that when Y is *locally Noetherian*, the third condition above ($\mathscr{E}$ being of finite type) is a consequence of the first two, and so the first two conditions *characterise*, in this case, the projective morphisms, and if Y is quasi-compact, then we can replace the second condition (the existence of a very ample (relative to f) $\mathcal{O}_X$-module $\mathscr{L}$) by the hypothesis that there exists an *ample* (relative to f) $\mathcal{O}_X$-module (4.6.11).

(ii) Let Y be a quasi-compact scheme such that there exists an ample $\mathcal{O}_Y$-module. For a Y-scheme X to be *projective*, it is necessary and sufficient for it to be Y-isomorphic to a *closed* Y-subscheme of a projective bundle of the form $\mathbf{P}^r_Y$. The condition is clearly sufficient. Conversely, if X is projective over Y, then it is quasi-projective, and so there exists a Y- **II** | 105
immersion j of X into some $\mathbf{P}^r_Y$ (5.3.3) that is *closed*, by (5.4.4) and (5.5.3).

(iii) The argument of (5.5.3) shows that, for every prescheme Y and every integer $r \geqslant 0$, the structure morphism $\mathbf{P}^r_Y \to Y$ is *surjective*, because if we set $\mathscr{S} = \mathbf{S}_{\mathcal{O}_Y}(\mathcal{O}_Y^{r+1})$, then we evidently have $\mathscr{S}_y = \mathbf{S}_{k(y)}(k(y)^{r+1})$ (1.7.3), and so $(\mathscr{S}_n)_y \neq 0$ for any $y \in Y$ or any $n \geqslant 0$.

(iv) It follows from the examples of Nagata [Nag58] that there exist proper morphisms that are not quasi-projective.

Proposition (5.5.5). —

(i) *A closed immersion is a projective morphism.*

(ii) *If $f : X \to Y$ and $g : Y \to Z$ are projective morphisms, and if Z is a quasi-compact scheme or a prescheme whose underlying space is Noetherian, then $g \circ f$ is projective.*

(iii) *If $f : X \to Y$ is a projective S-morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is projective for every extension $S' \to S$ of the base prescheme.*

(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are projective S-morphisms, then so is $f \times_S g$.*

(v) *If $g \circ f$ is a projective morphism, and if g is separated, then f is projective.*

(vi) *If f is projective, then so is f_{red}.*

Proof. (i) follows immediately from (3.1.7). We have to show (iii) and (iv) separately, because of the restriction introduced on Z in (ii) (cf. (**I**, 3.5.1)). To show (iii), we restrict to the case where $S = Y$ (**I**, 3.3.11), and the claim then immediately follows from (5.5.1, b) and (3.5.3). To show (iv), we are immediately led to the case where $X = \mathbf{P}(\mathscr{E})$ and $X = \mathbf{P}(\mathscr{E}')$, where $\mathscr{E}$ (resp. $\mathscr{E}'$) is a quasi-coherent $\mathcal{O}_Y$-module (resp. quasi-coherent $\mathcal{O}_{Y'}$-module) of finite type. Let p and p' be the canonical projections of $T = Y \times_S Y'$ to Y and Y' (respectively); by (4.1.3.1), we have $\mathbf{P}(p^*(\mathscr{E})) = \mathbf{P}(\mathscr{E}) \times_Y T$ and $\mathbf{P}(p'^*(\mathscr{E}')) = \mathbf{P}(\mathscr{E}') \times_{Y'} T$; whence

$$\begin{aligned}\mathbf{P}(p^*(\mathscr{E})) \times_T \mathbf{P}(p'^*(\mathscr{E}')) &= (\mathbf{P}(\mathscr{E}) \times_Y T) \times_T (T \times_{Y'} \mathbf{P}(\mathscr{E}')) \\ &= \mathbf{P}(\mathscr{E}) \times_Y (T \times_{Y'} \mathbf{P}(\mathscr{E}')) = \mathbf{P}(\mathscr{E}) \times_S \mathbf{P}(\mathscr{E}')\end{aligned}$$

by replacing T with $Y \times_S Y'$, and using (**I**, 3.3.9.1). But $p^*(\mathscr{E})$ and $p'^*(\mathscr{E}')$ are of finite type over T (**0**, 5.2.4), and thus so is $p^*(\mathscr{E}) \otimes_{\mathcal{O}_T} p'^*(\mathscr{E}')$; since $\mathbf{P}(p^*(\mathscr{E})) \times_T \mathbf{P}(p'^*(\mathscr{E}'))$ can be identified with a closed subprescheme of $p^*(\mathscr{E}) \otimes_{\mathcal{O}_T} p'^*(\mathscr{E}')$ (4.3.3), this proves (iv). To show (v) and (vi), we can apply (**I**, 5.5.13), because every closed subprescheme of a projective Y-scheme is a projective Y-scheme, by (5.5.1, a).

It remains to prove (ii); by the hypothesis on Z, this follows from (5.5.3), (5.3.4, ii), and (5.4.2, ii). □

Proposition (5.5.6). — *If X and X' are projective Y-schemes, then $X \sqcup X'$ is a projective Y-scheme.*

Proof. This is an evident consequence of (5.5.2) and (4.3.6). □

Proposition (5.5.7). — *Let X be a projective Y-scheme, and $\mathscr{L}$ a Y-ample $\mathcal{O}_X$-module; then, for every section f of $\mathscr{L}$ over X, X_f is affine over Y.*

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Proof. Since the question is local on Y, we can assume that $Y = \operatorname{Spec}(A)$; furthermore, $X_{f^{\otimes n}} = X_f$, so by replacing $\mathscr{L}$ with some suitable $\mathscr{L}^{\otimes n}$, we can assume that $\mathscr{L}$ is very ample relative to the structure morphism $q : X \to Y$ (4.6.11). The canonical homomorphism $\sigma : q^*(q_*(\mathscr{L})) \to \mathscr{L}$ is thus surjective, and the corresponding morphism

$$r = r_{\mathscr{L},\sigma} : X \longrightarrow P = \mathbf{P}(q_*(\mathscr{L}))$$

is an immersion such that $\mathscr{L} = r^*(\mathscr{O}_P(1))$ (4.4.4); furthermore, since X is proper over Y, the immersion r is closed (5.4.4). But by definition, $f \in \Gamma(Y, q_*(\mathscr{L}))$, and $\sigma^\flat$ is the identity of $q_*(\mathscr{L})$; it then follows from Equation (3.7.3.1) that we have $X_f = r^{-1}(D_+(f))$; so X_f is a closed subprescheme of the affine scheme $D_+(f)$, and is thus also an affine scheme. ☐

In the particular case where $Y = X$, we obtain (taking (4.6.13, i) into account) the following corollary, whose direct proof is immediate anyway:

Corollary (5.5.8). — *Let X be a prescheme, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module. For every section f of $\mathscr{L}$ over X, X_f is affine over X (and thus also an affine scheme whenever X is an affine scheme).*

5.6. Chow's lemma.

Theorem (5.6.1). — (Chow's lemma). *Let S be a prescheme, and X an S-scheme of finite type. Suppose that the following conditions are satisfied:*

(a) *S is Noetherian;*
(b) *S is a quasi-compact scheme, and X has a finite number of irreducible components.*

Under these hypotheses,

(i) *there exists a* quasi-projective *S-scheme X', and an S-morphism $f : X' \to X$ that is both*projective *and* surjective*;*
(ii) *we can take X' and f to be such that there exists an open subset $U \subset X$ for which $U' = f^{-1}(U)$ is dense in X', and for which the restriction of f to U' is an isomorphism $U' \simeq U$; and*
(iii) *if X is reduced (resp. irreducible, integral), then we can assume that X' is reduced (resp. irreducible, integral).*

Proof. The proof proceeds in multiple steps.

(A) We can first restrict to the case where X is *irreducible*. Indeed, in hypothesis (a), X is Noetherian, and so, in the two hypotheses, the irreducible components X_i of X are finite in number. If the theorem is shown to be true for each of the reduced closed preschemes of X having the X_i as their underlying spaces, and if X'_i and $f_i : X'_i \to X_i$ are the prescheme and the morphism corresponding to X_i (respectively), then the prescheme X' given by the *sum* of the X'_i, and the morphism $f : X' \to X$ whose restriction to each X'_i is $j_i \circ f_i$ (where j_i is the canonical injection $X_i \to X$) satisfy the conclusion of the theorem. It is immediate that X' is reduced if all of the X'_i are; furthermore, we can satisfy (ii) by taking U to be the union of the sets $U_i \cap \complement\left(\bigcup_{j \neq i} X_j\right)$. Finally, since the X'_i are quasi-projective over S, so is X' (5.3.6); similarly, the morphisms $X'_i \to X$ are projective by (5.5.5, i) and (5.5.5, ii), and so f is **II** | 107
projective (5.5.6), and is clearly surjective, by definition.

(B) Now suppose that X is *irreducible*. Since the structure morphism $r : X \to S$ is of finite type, there exists a finite cover (S_i) of S by affine open subsets, and for each i there is a finite cover (T_{ij}) of $r^{-1}(S_i)$ by affine open subsets, and the morphisms $T_{ij} \to S_i$ are of finite type, and so quasi-projective (5.3.4, i); since in both hypotheses (a) and (b) the immersion $S_i \to S$ is quasi-compact, it is also quasi-projective (5.3.4, i), and so the restriction of r to T_{ij} is a quasi-projective morphism (5.3.4, ii). Denote the T_{ij} by U_k ($1 \leqslant k \leqslant n$). There exists, for each index k, an open immersion $\phi_k : U_k \to P_k$, where P_k is projective over S ((5.3.2) and (5.5.2)). Let $U = \bigcap_k U_k$; since X is irreducible, and the U_k nonempty, U is nonempty, and thus dense in X; the restrictions of the ϕ_k to U define a morphism

$$\phi : U \longrightarrow P = P_1 \times_S P_2 \times_S \cdots \times_S P_n$$

such that the diagrams

(5.6.1.1)
$$\begin{array}{ccc} U & \xrightarrow{\phi} & P \\ {\scriptstyle j_k}\downarrow & & \downarrow{\scriptstyle p_k} \\ U_k & \xrightarrow{\phi_k} & P_k \end{array}$$

commute, where j_k is the canonical injection $U \to U_k$, and p_k the canonical projection $P \to P_k$. If j is the canonical injection $U \to X$, then the morphism $\psi = (j, \phi)_S : U \to X \times_S P$ is an *immersion* (**I**, 5.3.14). In hypothesis (a), $X \times_S P$ is locally Noetherian ((3.4.1), (**I**, 6.3.7), and (**I**, 6.3.8)); in hypothesis (b), $X \times_S P$ is a quasi-compact scheme ((**I**, 5.5.1) and (**I**, 6.6.4)); in both cases, the *closure* X' in $X \times_S P$ of the subprescheme Z associated to ψ (and so with underlying space $\psi(U)$) exists, and ψ factors as

(5.6.1.2)
$$\psi : U \xrightarrow{\psi'} X' \xrightarrow{h} X \times_S P$$

where ψ' is an *open immersion* and h a *closed immersion* (**I**, 9.5.10). Let $q_1 : X \times_S P \to X$ and $q_2 : X \times_S P \to P$ be the canonical projections; we set

(5.6.1.3)
$$f : X' \xrightarrow{h} X \times_S P \xrightarrow{q_1} X,$$

(5.6.1.4)
$$g : X' \xrightarrow{h} X \times_S P \xrightarrow{q_2} P.$$

We will see that X' and f satisfy the conclusion of the theorem.

(C) First we show that f is *projective* and *surjective*, and that the restriction of f to $U' = f^{-1}(U)$ is an *isomorphism* from U' to U. Since the P_k are projective over S, so is P (5.5.5, iv), and so $X \times_S P$ is projective over X (5.5.5, iii), and thus so is X', which is a closed subprescheme of $X \times_S P$. Furthermore, we have $f \circ \psi' = q_1 \circ (h \circ \psi') = q_1 \circ \psi = j$, so $f(X')$ contains the open everywhere-dense subset U of X; but f is a *closed* morphism (5.5.3), so $f(X') = X$. Now note that $q_1^{-1}(U) = U \times_S P$ is induced on an open subset of $X \times_S P$, and, by definition, the prescheme $U' = h^{-1}(U \times_S P)$ is induced by X' on the open subset U'; it is thus the closure *relative to* $U \times_S P$ of the prescheme Z (**I**, 9.5.8). But the immersion ψ factors as **II** | 108
$U \xrightarrow{\Gamma_\phi} U \times_S P \xrightarrow{j \times 1} X \times_S P$, and since P is separated over S, the graph morphism Γ_ϕ is a closed immersion (**I**, 5.4.3), and so Z is a *closed* subprescheme of $U \times_S P$, whence $U' = Z$. Since ψ is an immersion, the restriction of f to U' is an isomorphism onto U, and the inverse of ψ'; finally, by the definition of X', U' is dense in X'.

(D) We now show that g is an *immersion*, which will imply that X' is *quasi-projective* over S, because P is projective over S. Set

$$\begin{aligned} V_k &= \phi_k(U_k) && \text{(open subset of } P_k) \\ W_k &= p_k^{-1}(V_k) && \text{(open subset of } P) \\ U'_k &= f^{-1}(U_k) && \text{(open subset of } X') \\ U''_k &= g^{-1}(W_k) && \text{(open subset of } X'). \end{aligned}$$

It is clear that the U'_k form an open cover of X'; we will first see that the U''_k also form an open cover of X', by showing that $U'_k \subset U''_k$. For this, it will suffice to show that the diagram

(5.6.1.5)
$$\begin{array}{ccc} U'_k & \xrightarrow{g|U'_k} & P \\ {\scriptstyle f|U'_k}\downarrow & & \downarrow{\scriptstyle p_k} \\ U_k & \xrightarrow{\phi_k} & P_k \end{array}$$

commutes. But the prescheme $U'_k = h^{-1}(U_k \times_S P)$ is induced by X' on the open subset U'_k, and is thus the closure of $Z = U' \subset U'_k$ relative to U'_k (**I**, 9.5.8). To show the commutativity of (5.6.1.5), it thus suffices (since P_k is an S-scheme) to show that composing the diagram with the canonical injection $U' \to U'_k$ (or, equivalently, thanks to the isomorphism from U'

to U, with ψ) gives us a commutative diagram (**I**, 9.5.6). But, by definition, the diagram thus obtains is exactly (5.6.1.1), whence our claim.

The W_k thus form an open cover of $g(X')$; to show that g is an immersion, it suffices to show that each of the restrictions $g|U''_k$ is an immersion into W_k (**I**, 4.2.4). For this, consider the morphism $u_k : W_k \xrightarrow{p_k} V_k \xrightarrow{\phi_k^{-1}} U_k \to X$; since X is separated over S, the graph morphism $\Gamma_{u_k} : W_k \to X \times_S W_k$ is a closed immersion (**I**, 5.4.3), and so the graph $T_k = \Gamma_{u_k}(W_k)$ is a closed subprescheme of $X \times_S W$; if we show that $U' \to X \times_S W_k$ factors through this subprescheme, then the map from the subprescheme induced by X' on the open subset X''_k of X' to $X \times_S W_k$ will also factor through this graph, by (**I**, 9.5.8). Since the restriction of q_2 to T_k is an isomorphism onto W_k, the restriction of g to X''_k will be an immersion into W_k, and our claim will be proven. Let v_k be the canonical injection $U' \to X \times_S W_k$; we have to show that there exists a morphism $w_k : U' \to W_k$ such that $v_k = \Gamma_{u_k} \circ w_k$. By the definition of the product, it suffices to prove that $q_1 \circ v_k = u_k \circ q_2 \circ v_k$ (**I**, 3.2.1), or, by composing on the right with the isomorphism $\psi' : U \to U'$, that $q_1 \circ \psi = u_k \circ q_2 \circ \psi$. But since $q_1 \circ \psi = j$ **II** | 109
and $q_2 \circ \psi = \phi$, our claim follows from the commutativity of (5.6.1.1), taking into account the definition of u_k.

(E) It is clear that since U, and thus U', is irreducible, so is the X' from the preceding construction, and the morphism f is thus *birational* (**I**, 2.2.9). If in addition X is reduced, then so is U', and hence X' is also reduced (**I**, 9.5.9). This finishes the proof.

□

Corollary (5.6.2). — *Suppose that one of the hypotheses,* (a) *and* (b), *of* (5.6.1) *is satisfied. For X to be proper over S, it is necessary and sufficient for there to exist a projective scheme X' over S, and a surjective $S-$morphism $f : X' \to X$ (which is thus projective, by* (5.5.5, v)*). Whenever this is the case, we can further choose f to be such that there exists a dense open subset U of X for which the restriction of f to $f^{-1}(U)$ is an isomorphism $f^{-1}(U) \simeq U$, and for which $f^{-1}(U)$ is dense in X'. If in addition X is irreducible (resp. reduced), then we can assume that X' is also irreducible (resp. reduced); when X and X' are irreducible, f is a birational morphism.*

Proof. The condition is sufficient, by (5.5.3) and (5.4.3, ii). It is necessary because, with the notation of (5.6.1), if X is proper over S, then X' is proper over S, because it is projective over X, and thus proper over X (5.5.3), and our claim follows from (5.4.2, ii); furthermore, since X' is quasi-projective over S, it is projective over S, by (5.5.3). □

Corollary (5.6.3). — *Let S be a locally Noetherian prescheme, and X an S-scheme of finite type over S, with structure morphism $f_0 : X \to S$. For X to be proper over S, it is necessary and sufficient that, for every morphism* of finite type *$S' \to S$, $(f_0)_{(S')} : X_{(S')} \to S'$ be a closed morphism. It even suffices for this condition to be verified only for every S-prescheme of the form $S' = S \otimes_{\mathbf{Z}} \mathbf{Z}[T_1, \ldots, T_n]$ (where the T_i are indeterminates).*

Proof. The condition being clearly necessary, we now show that it is sufficient. Since the question is local on S and S' (5.4.1), we can suppose that S and S' are affine and Noetherian. By Chow's lemma, there exists a projective S-scheme P, an immersion $j : X' \to P$, and a surjective projective morphism $f : X' \to X$, such that the diagram

$$\begin{array}{ccc} X & \xleftarrow{f} & X' \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle j} \\ S & \xleftarrow{r} & P \end{array}$$

commutes. Since P is of finite type over S, the first hypothesis implies that the projection $q_2 : X \times_S P \to P$ is a *closed* morphism. But the immersion j is the composition of q_2 and the morphism $f \times 1$ from $X' \times_S P$ to $X \times_S P$; but f, being projective, is proper (5.5.3), and so $f \times 1$ is closed. We thus conclude that j is a closed immersion, and thus proper (5.4.2, i). Furthermore, the structure morphism $r : P \to S$ is projective, and thus proper (5.5.3), so $f_0 \circ f = r \circ j$ is proper (5.4.2, ii); finally, since f is surjective, f_0 is proper, by (5.4.3).

To prove the proposition using only the second, weaker hypothesis (where S' is of the form $S \otimes_{\mathbf{Z}} \mathbf{Z}[T_1, \ldots, T_n]$), it suffices to show that it implies the first. But, if S' is affine and of finite type over $S = \operatorname{Spec}(A)$, then we have $S' = \operatorname{Spec}(A[c_1, \ldots, c_n])$ (**I**, 6.3.3), and S' is thus isomorphic **II** | 110
to a closed subprescheme of $S'' = \operatorname{Spec}(A[T_1, \ldots, T_n])$ (where the T_i are indeterminates). In the commutative diagram

$$\begin{array}{ccc} X \times_S S' & \xrightarrow{1_X \times j} & X \times_S S'' \\ {\scriptstyle (f_0)_{(S')}} \downarrow & & \downarrow {\scriptstyle (f_0)_{(S'')}} \\ S' & \xrightarrow{\quad j \quad} & S'' \end{array}$$

both j and $1_X \times j$ are closed immersions (**I**, 4.3.1), and $(f_0)_{(S')}$ is closed by hypothesis; thus $(f_0)_{(S'')}$ is also closed. □

§6. INTEGRAL MORPHISMS AND FINITE MORPHISMS

6.1. Preschemes integral over another prescheme.

§7. VALUATIVE CRITERIA

7.1. Reminder on valuation rings.

§8. BLOWUP SCHEMES; BASED CONES; PROJECTIVE CLOSURE

8.1. Blowup preschemes.

(8.1.1). Let Y be a prescheme, and, for every integer $n \geqslant 0$, let $\mathscr{I}_n$ be a quasi-coherent sheaf of ideals of $\mathscr{O}_Y$; suppose that the following conditions are satisfied:

(8.1.1.1) $$\mathscr{I}_0 = \mathscr{O}_Y, \ \mathscr{I}_n \subset \mathscr{I}_m \text{ for } m \leqslant n,$$

(8.1.1.2) $$\mathscr{I}_m \mathscr{I}_n \subset \mathscr{I}_{m+n} \text{ for any } m, n.$$ **II** | 153

We note that these hypotheses imply

(8.1.1.3) $$\mathscr{I}_1^n \subset \mathscr{I}_n.$$

Set

(8.1.1.4) $$\mathscr{S} = \bigoplus_{n \geqslant 0} \mathscr{I}_n.$$

It follows from (8.1.1.1) and (8.1.1.2) that $\mathscr{S}$ is a quasi-coherent graded $\mathscr{O}_Y$-algebra, and thus defines a Y-scheme $X = \operatorname{Proj}(\mathscr{S})$. If $\mathscr{J}$ is an *invertible* sheaf of ideals of $\mathscr{O}_Y$, then $\mathscr{I}_n \otimes_{\mathscr{O}_Y} \mathscr{J}^{\otimes n}$ is canonically identified with $\mathscr{I}_n \mathscr{J}^n$. If we then replace the $\mathscr{I}_n$ by the $\mathscr{I}_n \mathscr{J}^n$, and, in doing so, replace $\mathscr{S}$ by a quasi-coherent $\mathscr{O}_Y$-algebra $\mathscr{S}_{(\mathscr{J})}$, then $X_{(\mathscr{J})} = \operatorname{Proj}(\mathscr{S}_{(\mathscr{J})})$ is canonically isomorphic to X (3.1.8).

(8.1.2). Suppose that Y is *locally integral*, so that the sheaf $\mathscr{R}(Y)$ of rational functions is a quasi-coherent $\mathscr{O}_Y$-algebra (**1**, 7.3.7). We say that a $\mathscr{O}_Y$-submodule $\mathscr{I}$ of $\mathscr{R}(Y)$ is a *fractional ideal* of $\mathscr{R}(Y)$ if it is of *finite type* (**0**, 5.2.1). Suppose we have, for all $n \geqslant 0$, a quasi-coherent fractional ideal $\mathscr{I}_n$ of $\mathscr{R}(Y)$, such that $\mathscr{I}_0 = \mathscr{O}_Y$, and such that condition (8.1.1.2) (but not necessarily the second condition (8.1.1.1)) is satisfied; we can then again define a quasi-coherent graded $\mathscr{O}_Y$-algebra by Equation (8.1.1.4), and the corresponding Y-scheme $X = \operatorname{Proj}(\mathscr{S})$; we will again have a canonical isomorphism from X to $X_{\mathscr{J}}$ for every *invertible* fractional ideal $\mathscr{J}$ of $\mathscr{R}(Y)$.

Definition (8.1.3). — Let Y be a prescheme (resp. a locally integral prescheme), and $\mathscr{I}$ a quasi-coherent ideal of $\mathscr{O}_Y$ (resp. a quasi-coherent fractional ideal of $\mathscr{R}(Y)$). We say that the Y-scheme $X = \operatorname{Proj}(\bigoplus_{n \geqslant 0} \mathscr{I}^n)$ is obtained by blowing up the ideal $\mathscr{I}$, or is the blow-up prescheme of Y relative to $\mathscr{I}$. When $\mathscr{I}$ is a quasi-coherent ideal of $\mathscr{O}_Y$, and Y' is the closed subprescheme of Y defined by $\mathscr{I}$, we also say that X is the Y-scheme obtained by blowing up Y'.

By definition, $\mathscr{S} = \bigoplus_{n \geqslant 0} \mathscr{I}^n$ is then generated by $\mathscr{S}_1 = \mathscr{I}$; if $\mathscr{I}$ is an $\mathscr{O}_Y$-module of *finite type*, then X is *projective* over Y (5.5.2). Without any hypotheses on $\mathscr{I}$, the $\mathscr{O}_X$-module $\mathscr{O}_X(1)$ is *invertible* (3.2.5) and *very ample*, by (4.4.3) applied to the structure morphism $X \to Y$.

We note that, if $j : X \to Y$ is the structure morphism, then the restriction of f to $f^{-1}(Y - Y')$ is an *isomorphism* to $Y - Y'$ whenever $\mathscr{I}$ is an *ideal of* $\mathscr{O}_Y$ and Y' is the closed subprescheme that it defines: indeed, the question being local on Y, it suffices to assume that $\mathscr{I} = \mathscr{O}_Y$, and our claim then follows from (3.1.7).

If we replace $\mathscr{I}$ by $\mathscr{I}^d$ ($d > 0$), then the blow-up Y-scheme X is replaced by a canonically isomorphic Y-scheme X' (8.1.1); similarly, for every *invertible* ideal (resp. *invertible* fractional ideal) $\mathscr{J}$, the blow-up prescheme $X_{(\mathscr{J})}$ relative to the ideal $\mathscr{I}\mathscr{J}$ is canonically isomorphic to X (8.1.1).

In particular, whenever $\mathscr{I}$ is an *invertible* ideal (resp. *invertible* fractional ideal), the Y-scheme obtained by blowing up $\mathscr{I}$ is *isomorphic to* Y (3.1.7).

Proposition (8.1.3). — *Let Y be an integral prescheme.*

(i) *For every sequence $(\mathscr{I}_n)$ of quasi-coherent fractional ideals of $\mathscr{R}(Y)$ that satisfies (8.1.1.2) and such that $\mathscr{I}_0 = \mathscr{O}_Y$, the Y-scheme $X = \operatorname{Proj}(\bigoplus_{n \geqslant 0} \mathscr{I}^n)$ is integral, and the structure morphism $f : X \to Y$ is dominant.* **II** | 154

(ii) *Let $\mathscr{I}$ be a quasi-coherent fractional ideal of $\mathscr{R}(Y)$, and let X be the Y-scheme given by the blow up of Y relative to $\mathscr{I}$. If $\mathscr{I} \neq 0$, then the structure morphism $f : X \to Y$ is then birational and surjective.*

Proof.

(i) This follows from the fact that $\mathscr{S} = \bigoplus_{n \geqslant 0} \mathscr{I}_n$ is an *integral* $\mathscr{O}_Y$-algebra ((3.1.12) and (3.1.14)), since, for all $y \in Y$, $\mathscr{O}_y$ is an integral ring (**I**, 5.1.4).

(ii) By (i), X is integral; if, furthermore, x and y are the generic points of X and Y (respectively), then we have $f(x) = y$, and it remains to show that $k(x)$ is of rank 1 over $k(y)$. But x is also the generic point of the fibre $f^{-1}(y)$; if ψ is the canonical morphism $Z \to Y$, where $Z = \operatorname{Spec}(k(y))$, then the prescheme $f^{-1}(y)$ can be identified with $\operatorname{Proj}(\mathscr{S}')$, where $\mathscr{S}' = \psi^*(\mathscr{S})$ (3.5.3). But it is clear that $\mathscr{S}' = \bigoplus_{n \geqslant 0} (\mathscr{I}_y)^n$, and, since $\mathscr{I}$ is a quasi-coherent fractional ideal of $\mathscr{R}(Y)$ that is not zero, $\mathscr{I}_y \neq 0$ (**I**, 7.3.6), whence $\mathscr{I}_y = k(y)$; then $\operatorname{Proj}(\mathscr{S}')$ can be identified with $\operatorname{Spec}(k(y))$ (3.1.7), whence the conclusion.

□

We show a *converse* of (8.1.4) in (**III**, 2.3.8).

(8.1.5). We return to the setting and notation of (8.1.1). By definition, the injection homomorphisms $\mathscr{I}_{n+1} \to \mathscr{I}_n$ (8.1.1.1) define, for every $k \in \mathbf{Z}$, an injective homomorphism of degree zero of graded $\mathscr{S}$-modules

(8.1.5.1) $$u_k : \mathscr{S}_+(k+1) \longrightarrow \mathscr{S}(k);$$

since $\mathscr{S}_+(k+1)$ and $\mathscr{S}(k+1)$ are canonically **(TN)**-isomorphic, they give a canonical correspondence between u_k and an injective homomorphism of $\mathscr{O}_X$-modules (3.4.2):

(8.1.5.2) $$\widetilde{u}_k : \mathscr{O}_X(k+1) \longrightarrow \mathscr{O}_X(k).$$

Recall as well (3.2.6) that we have defined canonical homomorphisms

(8.1.5.3) $$\lambda : \mathscr{O}_X(h) \otimes_{\mathscr{O}_X} \mathscr{O}_X(k) \longrightarrow \mathscr{O}_X(h+k)$$

and, since the diagram

$$\begin{array}{ccc} \mathscr{S}(h) \otimes_{\mathscr{S}} \mathscr{S}(k) \otimes_{\mathscr{S}} \mathscr{S}(l) & \longrightarrow & \mathscr{S}(h+k) \otimes_{\mathscr{S}} \mathscr{S}(l) \\ \downarrow & & \downarrow \\ \mathscr{S}(h) \otimes_{\mathscr{S}} \mathscr{S}(k+l) & \longrightarrow & \mathscr{S}(h+k+l) \end{array}$$

commutes, it follows from the functoriality of the λ (3.2.6) that the homomorphisms (8.1.5.3) define the structure of a *quasi-coherent graded $\mathscr{O}_X$-algebra* on

(8.1.5.4) $$\mathscr{S}_X = \bigoplus_{n \in \mathbf{Z}} \mathscr{O}_X(n).$$

Furthermore, the diagram

$$\begin{array}{ccc} \mathscr{S}(h)\otimes_{\mathscr{S}}\mathscr{S}(k+1) & \longrightarrow & \mathscr{S}(h+k+1) \\ {\scriptstyle 1\otimes u_k}\downarrow & & \downarrow{\scriptstyle u_{k+h}} \\ \mathscr{S}(h)\otimes_{\mathscr{S}}\mathscr{S}(k) & \longrightarrow & \mathscr{S}(h+k) \end{array}$$

commutes; the functoriality of the λ then implies that we have a commutative diagram

$$\begin{array}{ccc} \mathscr{O}_X(h)\otimes_{\mathscr{O}_X}\mathscr{O}_X(k+1) & \xrightarrow{\lambda} & \mathscr{O}_X(h+k+1) \\ {\scriptstyle 1\otimes \widetilde{u}_k}\downarrow & & \downarrow{\scriptstyle \widetilde{u}_{k+h}} \\ \mathscr{O}_X(h)\otimes_{\mathscr{O}_X}\mathscr{O}_X(k) & \xrightarrow{\lambda} & \mathscr{O}_X(h+k) \end{array}$$

where the horizontal arrows are the canonical homomorphisms. We can thus say that the $\widetilde{u}_k$ define an *injective homomorphism* (of degree zero) *of graded* $\mathscr{S}_X$*-modules*

(8.1.5.6) $$\widetilde{u}:\mathscr{S}_X(1)\longrightarrow\mathscr{S}_X.$$

(8.1.6). Keeping the notation from (8.1.5), we now note that, for $n\geqslant 0$, the composite homomorphism $\widetilde{v}_n=\widetilde{u}_{n-1}\circ\widetilde{u}_{n-2}\circ\ldots\circ\widetilde{u}_0$ is an *injective* homomorphism $\mathscr{O}_X(n)\to\mathscr{O}_X$; we denote by $\mathscr{I}_{n,X}$ its image, which is thus a quasi-coherent ideal of $\mathscr{O}_X$, *isomorphic* to $\mathscr{O}_X(n)$. Furthermore, the diagram

$$\begin{array}{ccc} \mathscr{O}_X(m)\otimes_{\mathscr{O}_X}\mathscr{O}_X(n) & \xrightarrow{\lambda} & \mathscr{O}_X(m+n) \\ {\scriptstyle \widetilde{v}_m\otimes\widetilde{v}_n}\downarrow & & \downarrow{\scriptstyle \widetilde{v}_{m+n}} \\ \mathscr{O}_X & \xrightarrow[\mathrm{id}]{} & \mathscr{O}_X \end{array}$$

commutes for $m\geqslant 0$, $n\geqslant 0$. We thus deduce the following inclusions:

(8.1.6.1) $$\mathscr{I}_{0,X}=\mathscr{O}_X,\quad \mathscr{I}_{n,X}\subset\mathscr{I}_{m,X}\qquad\text{for } 0\leqslant m\leqslant n;$$

(8.1.6.2) $$\mathscr{I}_{m,X}\mathscr{I}_{n,X}\subset\mathscr{I}_{m+n,X}\qquad\text{for } m\geqslant 0,\ n\geqslant 0.$$ **II | 156**

Proposition (8.1.7). — *Let Y be a prescheme, $\mathscr{I}$ a quasi-coherent ideal of $\mathscr{O}_Y$, and $X=\operatorname{Proj}(\bigoplus_{n\geqslant 0}\mathscr{I}^n)$ the Y-scheme given by blowing up $\mathscr{I}$. We then have, for all $n>0$, a canonical isomorphism*

(8.1.7.1) $$\mathscr{O}_X(n)\xrightarrow{\sim}\mathscr{I}^n\mathscr{O}_X=\mathscr{I}_{n,X}$$

(cf. (**0**, 4.3.5)*), and thus that $\mathscr{I}^n\mathscr{O}_X$ is a very-ample invertible $\mathscr{O}_X$-module if $n>0$.*

Proof. The last claim is immediate, since $\mathscr{O}_X(1)$ is invertible (3.2.5) and very ample for Y by definition ((4.4.3) and (4.4.9)). Also by definition, the image of v_n is exactly $\mathscr{I}^n\mathscr{S}$, and (8.1.7.1) then follows from the exactness of the functor $\widetilde{\mathscr{M}}$ (3.2.4) and from Equation (3.2.4.1). □

Corollary (8.1.8). — *Under the hypotheses of* (8.1.7)*, if $f:X\to Y$ is the structure morphism, and Y' the closed subprescheme of Y defined by $\mathscr{I}$, then the closed subprescheme $X'=f^{-1}(Y')$ of X is defined by $\mathscr{I}\mathscr{O}_X$ (which is canonically isomorphic to $\mathscr{O}_X(1)$), from which we obtain a canonical short exact sequence*

(8.1.8.1) $$0\longrightarrow\mathscr{O}_X(1)\longrightarrow\mathscr{O}_X\longrightarrow\mathscr{O}_{X'}\longrightarrow 0.$$

Proof. This follows from (8.1.7.1) and from (**I**, 4.4.5). □

(8.1.9). Under the hypotheses of (8.1.7), we can be more precise about the structure of the $\mathscr{I}_{n,X}$. Note that the homomorphism

$$\widetilde{u}_{-1}:\mathscr{O}_X\longrightarrow\mathscr{O}_X(-1)$$

canonically corresponds to a section s of $\mathscr{O}_X(-1)$ over X, which we call the *canonical section* (relative to $\mathscr{I}$) (**0**, 5.1.1). In the diagram in (8.1.5.5), the horizontal arrows are isomorphisms (3.2.7); by replacing h with k, and k with -1 in this diagram, we obtain that $\widetilde{u}_k=1_k\otimes\widetilde{u}_{-1}$ (where 1_k denotes the identity on $\mathscr{O}_X(h)$), or, equivalently, that the homomorphism $\widetilde{u}_k$ is given exactly by *tensoring with the canonical section s* (for all $k\in\mathbf{Z}$). The homomorphism $\widetilde{u}$ (8.1.5.6) can then be understood in the same way.

Thus, for all $n \geqslant 0$, the homomorphism $\widetilde{v}_n : \mathcal{O}_X(n) \to \mathcal{O}_X$ is given exactly by tensoring with $s^{\otimes n}$; we thus deduce:

Corollary (8.1.10). — *With the notation of (8.1.8), the underlying space of X' is the set of $x \in X$ such that $s(x) = 0$, where s denotes the canonical section of $\mathcal{O}_X(-1)$.*

Proof. Indeed, if c_x is a generator of the fibre $(\mathcal{O}_X(1))_x$ at a point x, then $s_x \otimes c_x$ is canonically identified with a generator of the fibre of $\mathcal{I}_{1,X}$ at the point x, and is thus invertible if and only if $s_x \notin \mathfrak{m}_x(\mathcal{O}_x(-1))_x$, or, equivalently, if and only if $s(x) \neq 0$. □

Proposition. — *Let Y be an integral prescheme, $\mathcal{I}$ a quasi-coherent fractional ideal of $\mathcal{R}(Y)$, and X the Y-scheme given by blowing up $\mathcal{I}$. Then $\mathcal{I}\mathcal{O}_X$ is an invertible $\mathcal{O}_X$-module that is very ample for Y.*

Proof. The question being local on Y (4.4.5), we can reduce to the case where $Y = \operatorname{Spec}(A)$, with A some integral ring of ring of fractions K, and $\mathcal{I} = \widetilde{\mathfrak{I}}$, with $\mathfrak{I}$ some fractional ideal of K; there then exists an element $a \neq 0$ of A such that $a\mathfrak{I} \subset A$. Let $S = \bigoplus_{n\geqslant 0} \mathfrak{I}^n$; the map $x \mapsto ax$ is an A-isomorphism from $\mathfrak{I}^{n+1} = (S(1))_n$ to $a\mathfrak{I}^{n+1} = a\mathfrak{I}S_n \subset \mathfrak{I}^n = S_n$, and thus defines a (TN)-isomorphism of degree zero of graded S-modules $S_+(1) \to a\mathfrak{I}S$. On the other hand, $x \mapsto a^{-1}x$ is an isomorphism of degree zero of graded S-modules $a\mathfrak{I}S \xrightarrow{\sim} \mathfrak{I}S$. We thus obtain, by composition (3.2.4), an isomorphism of $\mathcal{O}_X$-modules $\mathcal{O}_X(1) \xrightarrow{\sim} \mathcal{I}\mathcal{O}_X$, and, since S is generated by $S_1 = \mathfrak{I}$, $\mathcal{O}_X(1)$ is invertible (3.2.5) and very ample ((4.4.3) and (4.4.9)), whence our claim. □ II | 157

8.2. Preliminary results on the localisation of graded rings.

(8.2.1). Let S be a graded ring, but not assumed (for the moment) to be only in positive degree. We define

(8.2.1.1) $$S^{\geqslant} = \bigoplus_{n\geqslant 0} S_n, \qquad S^{\leqslant} = \bigoplus_{n\leqslant 0} S_n$$

which are both graded subrings of S, in only positive and negative degrees (respectively). If f is a homogeneous elements of degree d (positive or negative) of S, then the ring of fraction $S_f = S'$ is again endowed with the structure of a graded ring, by taking S'_n ($n \in \mathbf{Z}$) to be the set of the x/f^k for $x \in S_{n+kd}$ ($k \geqslant 0$); we define $S_{(f)} = S'_0$, and will write $S_f^{\geqslant}$ and $S_f^{\leqslant}$ for $S'^{\geqslant}$ and $S'^{\leqslant}$ (respectively). If $d > 0$, then

(8.2.1.2) $$(S^{\geqslant})_f = S_f$$

since, if $x \in S_{n+kd}$ with $n + kd < 0$, then we can write $x/f^k = xf^h/f^{h+k}$, and we also have that $n + (h+k)d > 0$ for h sufficiently large and > 0. We thus conclude, by definition, that

(8.2.1.3) $$(S^{\geqslant})_{(f)} = (S_f^{\geqslant})_0 = S_{(f)}.$$

If M is a graded S-module, then we similarly define

(8.2.1.4) $$M^{\geqslant} = \bigoplus_{n\geqslant 0} M_n, \qquad M^{\leqslant} = \bigoplus_{n\leqslant 0} M_n$$

which are (respectively) a graded $S^{\geqslant}$-module and a graded $S^{\leqslant}$-module, and their intersection is the S_0 module M_0. If $f \in S_d$, then we define M_f to be the graded S_f-module whose elements of degree n are the z/f^k for $z \in M_{n+kd}$ ($k \geqslant 0$); we denote by $M_{(f)}$ the set of elements of degree zero of M_f, and this is an $S_{(f)}$-module, and we will write $M_f^{\geqslant}$ and $M_f^{\leqslant}$ to mean $(M_f)^{\geqslant}$ and $(M_f)^{\leqslant}$ (respectively). If $d > 0$, then we see, as above, that

(8.2.1.5) $$(M^{\geqslant})_f = M_f$$

and

(8.2.1.6) $$(M^{\geqslant})_{(f)} = (M_f^{\geqslant})_0 = M_{(f)}.$$

(8.2.2). Let $\mathbf{z}$ be an indeterminate, we we will call the *homogenisation variable*. If S is a graded ring (in positive or negative degrees), then the polynomial algebra[1]

$$\widehat{S} = S[\mathbf{z}] \tag{8.2.2.1}$$

is a graded S-algebra, where we define the degree of $f\mathbf{z}^n$ ($n \geqslant 0$), with f homogeneous, as II | 158

$$\deg(f\mathbf{z}^n) = n + \deg f. \tag{8.2.2.2}$$

Lemma (8.2.3). — (i) *There are canonical isomorphisms of (non-graded) rings*

$$\widehat{S}_{(\mathbf{z})} \xrightarrow{\sim} \widehat{S}/(\mathbf{z}-1)\widehat{S} \xrightarrow{\sim} S. \tag{8.2.3.1}$$

(ii) *There is a canonical isomorphism of (non-graded) rings*

$$\widehat{S}_{(f)} \xrightarrow{\sim} S_f^{\leqslant} \tag{8.2.3.2}$$

for all $f \in S_d$ with $d > 0$.

Proof. The first of the isomorphisms in (8.2.3.1) was defined in (2.2.5), and the second is trivial; the isomorphism $\widehat{S}_{(\mathbf{z})} \xrightarrow{\sim} S$ thus defined thus gives a correspondence between $x\mathbf{z}^n/\mathbf{z}^{n+k}$ (where $\deg(x) = k$ for $k \geqslant -n$) and the element x. The homomorphism (8.2.3.2) gives a correspondence between $x\mathbf{z}^n/f^k$ (where $\deg(x) = kd - n$) and the element x/f^k of degree $-n$ in $S_f^{\leqslant}$, and it is again clear that this does indeed give an isomorphism. □

(8.2.4). Let M be a graded S-module. It is clear that the S-module

$$\widehat{M} = M \otimes_S \widehat{S} = M \otimes_S S[\mathbf{z}] \tag{8.2.4.1}$$

is the direct sum of the S-modules $M \otimes S\mathbf{z}^n$, and thus of the abelian groups $M_k \otimes S\mathbf{z}^n$ ($k \in \mathbf{Z}$, $n \geqslant 0$); we define on $\widehat{M}$ the structure of a graded $\widehat{S}$-module by setting

$$\deg(x \otimes \mathbf{z}^n) = n + \deg x \tag{8.2.4.2}$$

for all homogeneous x in M. We leave it to the reader to prove the analogue of (8.2.3):

Lemma (8.2.5). — (i) *There is a canonical di-isomorphism of (non-graded) modules*

$$\widehat{M}_{(\mathbf{z})} \xrightarrow{\sim} M. \tag{8.2.5.1}$$

(ii) *For all $f \in S_d$ ($d > 0$), there is a di-isomorphism of (non-graded) modules*

$$\widehat{M}_{(f)} \xrightarrow{\sim} M_f^{\leqslant}. \tag{8.2.5.2}$$

(8.2.6). Let S be a *positively*-graded ring, and consider the decreasing sequence of graded ideals of S

$$S_{[n]} = \bigoplus_{m \geqslant n} S_m \qquad (n \geqslant 0) \tag{8.2.6.1}$$

(so, in particular, we have $S_{[0]} = S$ and $S_{[1]} = S_+$). Since it is evident that $S_{[m]}S_{[n]} \subset S_{[m+n]}$, we can define a *graded ring* $S^\natural$ by setting

$$S^\natural = \bigoplus_{n \geqslant 0} S_n^\natural \quad \text{with} \quad S_n^\natural = S_{[n]}. \tag{8.2.6.2}$$

$S_0^\natural$ is then the ring S considered as a *non-graded* ring, and $S^\natural$ is thus an $S_0^\natural$-algebra. For every homogeneous element $f \in S_d$ ($d > 0$), we denote by $f^\natural$ the element f considered as belonging to $S_{[d]} = S_d^\natural$. With this notation:

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Lemma (8.2.7). — *Let S be a positively-graded ring, and f a homogeneous element of S_d ($d > 0$). There are canonical ring isomorphisms*

$$S_f \xrightarrow{\sim} \bigoplus_{n \in \mathbf{Z}} S(n)_{(f)} \tag{8.2.7.1}$$

$$(S_f^{\geqslant})_{f/1} \xrightarrow{\sim} S_f \tag{8.2.7.2}$$

[1] This should not be confused with the use of the notation $\widehat{S}$ to denote the completed separation of a ring.

(8.2.7.3)
$$S^\natural_{(f^\natural)} \xrightarrow{\sim} S^{\geqslant}_f$$

where the first two are isomorphisms of graded rings.

Proof. It is immediate, by definition, that we have $(S_f)_n = (S(n)_f)_0$, whence the isomorphism in (8.2.7.1), which is exactly the identity. Next, since $f/1$ is invertible in S_f, there is a canonical isomorphism $S_f \xrightarrow{\sim} (S^{\geqslant}_f)_{f/1} = (S_f)_{f/1}$, by (8.2.1.2) applied to S_f; the inverse isomorphism is, by definition, the isomorphism in (8.2.7.2). Finally, if $x = \sum_{m \geqslant n} y_m$ is an element of $S_{[n]}$ with $n = kd$, then the element $x/(f^\natural)^k$ corresponds to the element $\sum_m y_m/f^k$ of $S^{\geqslant}_f$, and we can quickly verify that this defines an isomorphism (8.2.7.3). □

(8.2.8). If M is a graded S-module, then we similarly define, for all $n \in \mathbf{Z}$,

(8.2.8.1)
$$M_{[n]} = \bigoplus_{m \geqslant n} M_m$$

and, since $S_{[m]}M_{[n]} \subset M_{[m+n]}$ $(m \geqslant 0)$, we can define a graded $S^\natural$-module $M^\natural$ by setting

(8.2.8.2)
$$M^\natural = \bigoplus_{n \in \mathbf{Z}} \quad \text{with} \quad M^\natural_n = M_{[n]}.$$

We leave to the reader the proof of:

Lemma (8.2.9). — *With the notation of (8.2.7) and (8.2.8), there are canonical di-isomorphisms of modules*

(8.2.9.1)
$$M_f \xrightarrow{\sim} \bigoplus_{n \in \mathbf{Z}} M(n)_{(f)}$$

(8.2.9.2)
$$(M^{\geqslant}_f)_{f/1} \xrightarrow{\sim} M_f$$

(8.2.9.3)
$$M^\natural_{(f^\natural)} \xrightarrow{\sim} M^{\geqslant}_f$$

where the first two are di-isomorphisms of graded modules.

Lemma (8.2.10). — *Let S be a positively-graded ring.*

(i) *For $S^\natural$ to be an $S^\natural_0$-algebra of finite type (resp. a Noetherian $S^\natural_0$-algebra), it is necessary and sufficient for S to be an $S^\natural_0$-algebra of finite type (resp. a Noetherian $S^\natural_0$-algebra).*
(ii) *For $S^\natural_{n+1} = S^\natural_1 S^\natural_n$ $(n \geqslant n_0)$, it is necessary and sufficient for $S_{n+1} = S_1 S_n$ $(n \geqslant n_0)$.*
(iii) *For $S^\natural_n = S^\natural_1$ $(n \geqslant n_0)$, it is necessary and sufficient for $S_n = S_1^n$ $(n \geqslant n_0)$.*
(iv) *If (f_α) is a set of homogeneous elements of S_+ such that S_+ is the radical in S_+ of the ideal of S_+ generated by the f_α, then $S^\natural_+$ is the radical in $S^\natural_+$ of the ideal of $S^\natural_+$ generated by the $f^\natural_\alpha$.*

Proof. (i) If $S^\natural$ is an $S^\natural_0$-algebra of finite type, then $S_+ = S^\natural_1$ is a module of finite type over $S = S^\natural_0$, by (2.1.6, i), and so S is an S_0-algebra of finite type (2.1.4); if $S^\natural$ is a Noetherian ring, then so too is $S^\natural_0 = S$ (2.1.5). Conversely, if S is an S_0-algebra of finite type, then we know (2.1.6, ii) **II** | 160 that there exist $h > 0$ and $m_0 > 0$ such that $S_{n+h} = S_h S_n$ for $n \geqslant m_0$; we can clearly assume that $m_0 \geqslant h$. Furthermore, the S_m are S_0-modules of finite type (2.1.6, i). So, if $n \geqslant m_0 + h$, then $S^\natural_n = S_h S^\natural_{n-h} = S^\natural_h S^\natural_{n-h}$; and if $m < m_0 + h$ then, letting $E = S_{m_0} + \ldots + S_{m_0+h-1}$, we have that
$$S^\natural_m = S_m + \ldots + S_{m_0+h-1} + S_h E + S_h^2 E + \ldots.$$
For $1 \leqslant m \leqslant m_0$, let G_m be the union of the finite systems of generators of the S_0-modules S_i for $m \leqslant i \leqslant m_0 + h - 1$, thought of as a subset of $S_{[m]}$. For $m_0 + 1 \leqslant m \leqslant m_0 + h - 1$, let G_m be the union of the finite system of generators of the S_0-modules S_i for $m \leqslant i \leqslant m_0 + h - 1$ and of $S_h E$, thought of as a subset of $S_{[m]}$. It is clear that $S^\natural_m = S^\natural_0 G_m$ for $1 \leqslant m \leqslant m_0 + h - 1$, and thus the union G of the G_m for $1 \leqslant m \leqslant m_0 + h - 1$ is a system of generators of the $S^\natural_0$-algebra $S^\natural$. We thus conclude that, if $S = S^\natural_0$ is a Noetherian ring, then so too is $S^\natural$.

(ii) It is clear that, if $S_{n+1} = S_1 S_n$ for $n \geqslant n_0$, then $S^\natural_{n+1} = S_1 S^\natural_n$, and *a fortiori* $S^\natural_{n+1} = S^\natural_1 S^\natural_n$ for $n \geqslant n_0$. Conversely, this last equality can be written as

$$S_{n+1} + S_{n+2} + \ldots = (S_1 + S_2 + \ldots)(S_n + S_{n+1} + \ldots)$$

and comparing terms of degree $n+1$ (in S) on both sides gives that $S_{n+1} = S_1 S_n$.

(iii) If $S_n = S_1^n$ for $n \geqslant n_0$, then $S^\natural_n = S_1^n + S_1^{n+1} + \ldots$; since $S^\natural_1$ contains $S_1 + S_1^2 + \ldots$, we have that $S^\natural_n \subset S_1^{\natural n}$, and thus $S^\natural_n = S_1^{\natural n}$ for $n \geqslant n_0$. Conversely, the only terms of $S_1^{\natural n} = (S_1 + S_2 + \ldots)^n$ that are of degree n in S are those of S_1^n; the equality $S^\natural_n = S_1^{\natural n}$ thus implies that $S_n = S_1^n$.

(iv) It suffices to show that, if an element $g \in S_{k+h}$ is considered as an element of $S^\natural_k$ ($k > 0$, $h \geqslant 0$), then there exists an integer $n > 0$ such that g^n is a linear combination (in $S^\natural_{kn}$) of the $f^\natural_\alpha$ with coefficients in $S^\natural$. By hypothesis, there exists an integer m_0 such that, for $m \geqslant m_0$, we have, *in S*, that $g^m = \sum_\alpha c_{\alpha m} f_\alpha$, where the indices α here are *independent of m*; furthermore, we can clearly assume that the $c_{\alpha m}$ are homogeneous, with

$$\deg(c_{\alpha m}) = m(k+h) - \deg f_\alpha$$

in S. So take m_0 sufficiently large enough to ensure that $km_0 > \deg f_\alpha$ for all the f_α that appear in g^{m_0}; for all α, let $c'_{\alpha m}$ be the element $c_{\alpha m}$ considered as having degree $km - \deg f_\alpha$ *in* $S^\natural$; we then have, in $S^\natural$, that $g^m = \sum_\alpha c'_{\alpha m} f^\natural_\alpha$, which finishes the proof.

□

(8.2.11). Consider the graded S_0-algebra

(8.2.11.1) $$S^\natural \otimes_S S_0 = S^\natural / S_+ S^\natural = \bigoplus_{n \geqslant 0} S_{[n]} / S_+ S_{[n]}.$$

Since S_n is a quotient S_0-module of $S_{[n]}/S_+S_{[n]}$, there is a canonical homomorphism of graded S_0-algebras

(8.2.11.2) $$S^\natural \otimes_S S_0 \longrightarrow S$$

which is clearly *surjective*, and thus corresponds (2.9.2) to a canonical *closed immersion*

(8.2.11.3) $$\operatorname{Proj}(S) \longrightarrow \operatorname{Proj}(S^\natural \otimes_S S_0).$$

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Proposition (8.2.12). — *The canonical morphism (8.2.11.3) is bijective. For the homomorphism (8.2.11.2) to be (TN)-bijective, it is necessary and sufficient for there to exist some n_0 such that $S_{n+1} = S_1 S_n$ for $n \geqslant n_0$. If this latter condition is satisfied, then (8.2.11.3) is an isomorphism; the converse is true whenever S is Noetherian.*

Proof. To prove the first claim, it suffices (2.8.3) to show that the kernel $\mathfrak{I}$ of the homomorphism (8.2.11.2) consists of *nilpotent* elements. But if $f \in S_{[n]}$ is an element whose class modulo $S_+S_{[n]}$ belongs to this kernel, then this implies that $f \in S_{[n+1]}$; then f^{n+1}, considered as an element of $S_{[n(n+1)]}$, is also an element of $S_+S_{[n(n+1)]}$, since it can be written as $f \cdot f^n$; so the class of f^{n+1} modulo $S_+S_{[n(n+1)]}$ is zero, which proves our claim. Since the hypothesis that $S_{n+1} = S_1 S_n$ for $n \geqslant n_0$ is equivalent to $S^\natural_{n+1} = S^\natural_1 S^\natural_n$ for $n \geqslant n_0$ (8.2.10, ii), this hypothesis is equivalent, by definition, to the fact that (8.2.11.2) is (TN)-injective, and thus (TN)-bijective, and so (8.2.11.3) is an isomorphism, by (2.9.1). Conversely, if (8.2.11.3) is an isomorphism, then the sheaf $\widetilde{\mathfrak{I}}$ on $\operatorname{Proj}(S^\natural \otimes_S S_0)$ is zero (2.9.2, i); since $S^\natural \otimes_S S_0$ is Noetherian, as a quotient of $S^\natural$ (8.2.10, i), we conclude from (2.7.3) that $\mathfrak{I}$ satisfies condition (TN), and so $S^\natural_{n+1} = S^\natural_1 S^\natural_n$ for $n \geqslant n_0$, and this finishes the proof, by (8.2.10, ii). □

(8.2.13). Consider now the canonical injections $(S_+)^n \to S_{[n]}$, which define an injective homomorphism of degree zero of graded rings

(8.2.13.1) $$\bigoplus_{n \geqslant 0} (S_+)^n \longrightarrow S^\natural.$$

Proposition (8.2.14). — *For the homomorphism (8.2.13.1) to be a (TN)-isomorphism, it is necessary and sufficient for there to exist some n_0 such that $S_n = S_1^n$ for all $n \geqslant n_0$. Whenever this is the case, the morphisms corresponding to (8.2.13.1) is everywhere defined and also an isomorphism*

$$\mathrm{Proj}(S^\natural) \xrightarrow{\sim} \mathrm{Proj}(\bigoplus_{n\geqslant 0}(S_+)^n);$$

the converse is true whenever S is Noetherian.

Proof. The first two claims are evident, given (8.2.10, iii) and (2.9.1). The third will follow from (8.2.10, i and iii) and the following lemma:

Lemma (8.2.14.1). — *Let T be a positively-graded ring that is also a T_0-algebra of finite type. If the morphism corresponding to the injective homomorphism $\bigoplus_{n\geqslant 0} T_1^n \to T$ is everywhere defined and also an isomorphism $\mathrm{Proj}(T) \to \mathrm{Proj}(\bigoplus_{n\geqslant 0} T_1^n)$, then there exists some n_0 such that $T_n = T_1^n$ for $n \geqslant n_0$.*

Let g_i ($1 \leqslant i \leqslant r$) be generators of the T_0-module T_1. The hypothesis implies first of all that the $D_+(g_i)$ cover $\mathrm{Proj}(T)$ (2.8.1). Let $(h_j)_{1\leqslant j\leqslant s}$ be a system of homogeneous elements of T_+, with $\deg(h_j) = n_j$, that form, with the g_i, a system of generators of the ideal T_+, or, equivalently (2.1.3), a system of generators of T as a T_0-algebra; if we set $T' = \bigoplus_{n\geqslant 0} T_1^n$, then the element $h_j/g_i^{n_j}$ of the ring $T_{(g_i)}$ must, by hypothesis, belong to the subring $T'_{(g_i)}$, and so there exists some integer k such that $T_1^k h_j \subset T_1^{k+n_j}$ for all j. We thus conclude, by induction on r, that $T_1^k h_j^r \subset T'$ for all $r \geqslant 1$, and, by definition of the h_j, we thus have that $T_1^k T \subset T'$. Also, there exists, for all j, an integer m_j such that $h_j^{m_j}$ belongs to the ideal of T generated by the g_i (2.3.14), so $h_j^{m_j} \in T_1 T$, and $h_j^{m_j k} \in T_1^k T \subset T'$. **II | 162**
There is thus an integer $m_0 \geqslant k$ such that $h_j^m \in T_1^{mn}$ for $m \geqslant m_0$. So, if q is the largest of the integers n_j, then $n_0 = qsm_0 + k$ is the required number. Indeed, an element of S_n, for $n \geqslant n_0$, is the sum of monomials belonging to $T_1^\alpha u$, where u is a product of powers of the h_j; if $\alpha \geqslant k$, then it follows from the above that $T_1^\alpha u \subset T_1^n$; in the other case, one of the exponents of the h_j is $\geqslant m_0$, so $u \in T_1^\beta v$, where $\beta \geqslant k$ and v is again a product of powers of the h_j; we can then reduce to the previous case, and so we conclude that $T_1^\alpha u \subset T_1^n$ in all cases. □

Remark (8.2.15). — The condition $S_n = S_1^n$ for $n \geqslant n_0$ clearly implies that $S_{n+1} = S_1 S_n$ for $n \geqslant n_0$, but the converse is not necessarily true, even if we assume that S is Noetherian. For example, let K be a field, $A = K[\mathbf{x}]$, and $B = K[\mathbf{y}]/\mathbf{y}^2 K[\mathbf{y}]$, where $\mathbf{x}$ and $\mathbf{y}$ are indeterminates, with $\mathbf{x}$ taken to have degree 1 and $\mathbf{y}$ to have degree 2, and let $S = A \otimes_K B$, so that S is a graded algebra over K that has a basis given by the elements 1, $\mathbf{x}^n$ ($n \geqslant 1$), and $\mathbf{x}^n\mathbf{y}$ ($n \geqslant 0$). It is immediate that $S_{n+1} = S_1 S_n$ for $n \geqslant 2$, but $S_1^n = K\mathbf{x}^n$ while $S_n = K\mathbf{x}^n + K\mathbf{x}^n\mathbf{y}$ for $n \geqslant 2$.

8.3. Based cones.

(8.3.1). Let Y be a prescheme; in all of this section, we will consider only *Y-preschemes* and *Y-morphisms*. Let $\mathscr{S}$ be a quasi-coherent *positively*-graded $\mathscr{O}_Y$-algebra; *we further assume that $\mathscr{S}_0 = \mathscr{O}_Y$*. Following the notation introduced in (8.2.2), we let

$$\widehat{\mathscr{S}} = \mathscr{S}[\mathbf{z}] = \mathscr{S} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y[\mathbf{z}] \tag{8.3.1.1}$$

which we consider as a positively-graded $\mathscr{O}_Y$-algebra by defining the degrees as in (8.2.2.2), so that, for every affine open subset U of Y, we have

$$\Gamma(U, \widehat{\mathscr{S}}) = (\Gamma(U, \mathscr{S}))[\mathbf{z}].$$

In what follows, we write

$$X = \mathrm{Proj}(\mathscr{S}), \quad C = \mathrm{Spec}(\mathscr{S}), \quad \widehat{C} = \mathrm{Proj}(\widehat{\mathscr{S}}) \tag{8.3.1.2}$$

(where, in the definition of C, we consider $\mathscr{S}$ as a non-graded $\mathscr{O}_Y$-algebra), and we say that C (resp. $\widehat{C}$) is the *affine cone* (resp. *projective cone*) defined by $\mathscr{S}$; we will sometimes say "cone" instead of "affine cone". By an abuse of language, we also say that C (resp. $\widehat{C}$) is the *affine cone based at X (?)* (resp. the *projective cone based at X (?)*)[2], with the implicit understanding that the prescheme X is

[2] *[Trans.] A more literal translation of the French (*cône projetant (affine/projectif)*) would be the* projecting (affine/projective) cone*, but it seems that this terminology already exists to mean something else.*

given in the form $\operatorname{Proj}(\mathscr{S})$; finally, we say that $\widehat{C}$ is the *projective closure* of C (with the data of $\mathscr{S}$ being implicit in the structure of C).

Proposition (8.3.2). — *There exist canonical Y-morphisms*

(8.3.2.1) $$Y \xrightarrow{\varepsilon} C \xrightarrow{i} \widehat{C}$$

(8.3.2.2) $$X \xrightarrow{j} \widehat{C}$$

such that ε and j are closed immersions, and i is an affine morphism, which is a dominant open immersion, for which

(8.3.2.3) $$i(C) = \widehat{C} \setminus j(X);$$

furthermore, $\widehat{C}$ is the smallest closed subprescheme of $\widehat{C}$ containing $i(C)$.

Proof. To define i, consider the open subset of $\widehat{C}$ given by

(8.3.2.4) $$\widehat{C}_{\mathbf{z}} = \operatorname{Spec}(\widehat{\mathscr{S}}/(\mathbf{z}-1)\widehat{\mathscr{S}})$$

(3.1.4), where $\mathbf{z}$ is canonically identified with a section of $\mathscr{S}$ over Y. The isomorphism $i : C \xrightarrow{\sim} \widehat{C}_{\mathbf{z}}$ then corresponds to the canonical isomorphism (8.2.3.1)

$$\widehat{\mathscr{S}}/(\mathbf{z}-1)\widehat{\mathscr{S}} \xrightarrow{\sim} \mathscr{S}.$$

The morphism ε corresponds to the augmentation homomorphism $\mathscr{S} \to \mathscr{S}_0 = \mathscr{O}_Y$, which has kernel $\mathscr{S}_+$ (1.2.7), and, since the latter is surjective, ε is a closed immersion (1.4.10). Finally, j corresponds (3.5.1) to the surjective homomorphism of degree zero $\widehat{\mathscr{S}} \to \mathscr{S}$, which restricts to the identity on $\mathscr{S}$ and is zero on $\mathbf{z}\widehat{\mathscr{S}}$, which is its kernel; j is everywhere defined, and is a closed immersion, by (3.6.2).

To prove the other claims of (8.3.2), we can clearly restrict to the case where $Y = \operatorname{Spec}(A)$ is affine, and $\mathscr{S} = \widetilde{S}$, with S a graded A-algebra, whence $\widehat{\mathscr{S}} = (\widehat{S})^{\sim}$; the homogeneous elements f of S_+ can then be identified with sections of $\widehat{\mathscr{S}}$ over Y, and the open subset of $\widehat{C}$, denoted $D_+(f)$ in (2.3.3), can then be written as $\widehat{C}_f$ (3.1.4); similarly, the open subset of C denoted $D(f)$ in (**I**, 1.1.1) can be written as C_f (**0**, 5.5.2). With this in mind, it follows from (2.3.14) and from the definition of $\widehat{S}$ that, in this case, the open subsets $\widehat{C}_{\mathbf{z}} = i(C)$ and $\widehat{C}_f$ (with f homogeneous in S_+) form a *cover* of $\widehat{C}$. Furthermore, with this notation,

(8.3.2.5) $$i^{-1}(\widehat{C}_f) = C_f;$$

indeed, $\widehat{C}_f \cap i(C) = \widehat{C}_f \cap \widehat{C}_{\mathbf{z}} = \widehat{C}_{f\mathbf{z}} = \operatorname{Spec}(\widehat{S}_{(f\mathbf{z})})$. But, if $d = \deg(f)$, then $\widehat{S}_{(f\mathbf{z})}$ is canonically isomorphic to $(\widehat{S}_{(\mathbf{z})})_{f/\mathbf{z}^d}$ (2.2.2), and it follows from the definition of the isomorphism in (8.2.3.1) that the image of $(\widehat{S}_{(\mathbf{z})})_{f/\mathbf{z}^d}$ under the corresponding isomorphism of rings of fractions is exactly S_f. Since $C_f = \operatorname{Spec}(S_f)$, this proves (8.3.2.5) and shows, at the same time, that the morphism i is affine; furthermore, the restriction of i to C_f, thought of as a morphism to $\widehat{C}_f$, corresponds (**I**, 1.7.3) to the canonical homomorphism $\widehat{S}_{(f)} \to \widehat{S}_{(f\mathbf{z})}$, and, by the above and (8.2.3.2), we can claim the following result:

(8.3.2.6). If $Y = \operatorname{Spec}(A)$ is affine, and $\mathscr{S} = \widetilde{S}$, then, for every homogeneous f in S_+, $\widehat{C}_f$ is canonically identified with $\operatorname{Spec}(S_f^{\leqslant})$, and the morphism $C_f \to \widehat{C}_f$ given by restricting i then corresponds to the canonical injection $S_f^{\leqslant} \to S_f$.

Now note that (for Y affine) the complement of $\widehat{C}_{\mathbf{z}}$ in $\widehat{C} = \operatorname{Proj}(\widehat{S})$ is, by definition, the set of graded prime ideals of $\widehat{S}$ containing $\mathbf{z}$, which is exactly $j(X)$, by definition of j, which proves (8.3.2.3). **II** | 164

Finally, to prove the last claim of (8.3.2), we can assume that Y is affine. With the above notation, note that, in the ring $\widehat{S}$, $\mathbf{z}$ is not a zero divisor; since $i(C) = \widehat{C}$, it suffices to prove the following lemma:

Lemma (8.3.2.7). — *Let T be a positively-graded ring, $Z = \mathrm{Proj}(T)$, and g a homogeneous element of T of degree $d > 0$. If g is not a zero divisor in T, then Z is the smallest closed subprescheme of Z that contains $Z_g = D_+(g)$.*

By (**I**, 4.1.9), the question is local on Z; for every homogeneous element $h \in T_e$ ($e > 0$), it thus suffices to prove that Z_h is the smallest closed subprescheme of Z_h that contains Z_{gh}; it follows from the definitions and from (**I**, 4.3.2) that this condition is equivalent to asking for the canonical homomorphism $T_{(h)} \to T_{(gh)}$ to be *injective*. But this homomorphism can be identified with the canonical homomorphism $T_{(h)} \to (T_{(h)})_{g^e/h^d}$ (2.2.3). But since g^e is not a zero divisor in T, g^e/h^d is not a zero divisor in T_h (nor *a fortiori* in $T_{(h)}$), since the fact that $(g^e/h^d)(t/h^m) = 0$ (for $t \in T$ and $m > 0$) implies the existence of some $n > 0$ such that $h^n g^e t = 0$, whence $h^n t = 0$, and thus $t/h^m = 0$ in T_h. This thus finishes the proof (**0**, 1.2.2). □

(8.3.3). We will often identify the affine cone C with the subprescheme induced by the projective cone $\widehat{C}$ on the open subset $i(C)$ by means of the open immersion i. The closed subprescheme of C associated to the closed immersion ε is called the *vertex prescheme* (?) of C; we also say that ε, which is a Y-section of C, is the *vertex section* (?), or the *null section*, or C; we can identify Y with the vertex prescheme (?) of C by means of ε. Also, $i \circ \varepsilon$ is a Y-section of $\widehat{C}$, and thus also a closed immersion (**I**, 5.4.6), corresponding to the canonical surjective homomorphism of degree zero $\widehat{\mathscr{S}} = \mathscr{S}[\mathbf{z}] \to \mathscr{O}_Y[\mathbf{z}]$ (3.1.7), whose kernel is $\mathscr{S}_+[\mathbf{z}] = \mathscr{S}_+\widehat{\mathscr{S}}$; the subprescheme of $\widehat{C}$ associated to this closed immersion is also called the *vertex prescheme* (?) of $\widehat{C}$, and $i \circ \varepsilon$ the *vertex section* (?) of $\widehat{C}$; it can be identified with Y by means of $i \circ \varepsilon$. Finally, the closed subprescheme of $\widehat{C}$ associated to j is called the *part at infinity* of $\widehat{C}$, and can be identified with X by means of j.

(8.3.4). The subpreschemes of C (resp. $\widehat{C}$) induced on the *open* subsets

(8.3.4.1) $$E = C - \varepsilon(Y), \qquad \widehat{E} = \widehat{C} - i(\varepsilon(Y))$$

are called (by an abuse of language) the *pointed affine cone* and the *pointed projective cone* (respectively) defined by $\mathscr{S}$; we note that, despite this nomenclature, *E is not necessarily affine* over Y, nor $\widehat{E}$ projective over Y (8.4.3). When we identify C with $i(C)$, we thus have the underlying spaces

(8.3.4.2) $$C \cup \widehat{E} = \widehat{C}, \qquad C \cap \widehat{E} = E$$

so that $\widehat{C}$ can be considered as being obtained by *gluing* the open subpreschemes C and $\widehat{E}$; furthermore, by (8.3.2.3),

(8.3.4.3) $$E = \widehat{E} - j(X).$$

If $Y = \mathrm{Spec}(A)$ is affine, then, with the notation of (8.3.2),

(8.3.4.4) $$E = \bigcup C_f, \qquad \widehat{E} = \bigcup \widehat{C}_f, \qquad C_f = C \cap \widehat{C}_f$$

where f runs over the set of homogeneous elements of S_+ (or only a subset M of this set, with M generating an ideal of S_+ whose radical in S_+ is S_+ itself, or, equivalently, such that the X_f for $f \in M$ cover X (2.3.14)). The gluing of C and $\widehat{C}_f$ along C_f is thus determined by the injection morphisms $C_f \to C$ and $C_f \to \widehat{C}_f$, which, as we have seen (8.3.2.6), correspond (respectively) to the canonical homomorphisms $S \to S_f$ and $S_f^{\leqslant} \to S_f$.

Proposition (8.3.5). — *With the notation of (8.3.1) and (8.3.4), the morphism associated* (3.5.1) *to the canonical injection $\varphi : \mathscr{S} \to \widehat{\mathscr{S}} = \mathscr{S}[\mathbf{z}]$ is a surjective affine morphism (called the canonical retraction)*

(8.3.5.1) $$p : \widehat{E} \longrightarrow X$$

such that

(8.3.5.2) $$p \circ j = 1_X.$$

Proof. To prove the proposition, we can restrict to the case where Y is affine. Taking into account the expression in (8.3.4.4) for $\widehat{E}$, the fact that the domain of definition $G(\varphi)$ of p is equal to $\widehat{E}$ will follow from the first of the following claims:

(8.3.5.3). If $Y = \operatorname{Spec}(A)$ is affine, and $\mathscr{S} = \widetilde{S}$, then, for all homogeneous $f \in S_+$,

$$p^{-1}(X_f) = \widehat{C}_f \tag{8.3.5.4}$$

and the restriction of p to $\widehat{C}_f = \operatorname{Spec}(S_f^{\leqslant})$, thought of as a morphism from $\widehat{C}_f$ to X_f, corresponds to the canonical injection $S_{(f)} \to S_f^{\leqslant}$. If, further, $f \in S_1$, then $\widehat{C}_f$ is isomorphic to $X_f \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ (where T is an indeterminate).

Indeed, Equation (8.3.5.4) is exactly a particular case of (2.8.1.1), and the second claim is exactly the definition of $\operatorname{Proj}(\varphi)$ whenever Y is affine (2.8.1). Then Equation (8.3.5.2) and the fact that p is surjective show that the composition $\mathscr{S} \to \widehat{\mathscr{S}} \to \mathscr{S}$ of the canonical homomorphisms is the identity on $\mathscr{S}$. Finally, the last claim of (8.3.5.3) follows from the fact that $S_f^{\leqslant}$ is isomorphic to $S_{(f)}[T]$ whenever $f \in S_1$ (2.2.1). □

Corollary (8.3.6). — *The restriction*

$$\pi : E \longrightarrow X \tag{8.3.6.1}$$

of p to E is a surjective affine morphism. If Y is affine and f homogeneous in S_+, then

$$\pi^{-1}(X_f) = C_f \tag{8.3.6.2}$$

and the restriction of π to C_f corresponds to the canonical injection $S_{(f)} \to S_f$. If, further, $f \in S_1$, then C_f is isomorphic to $X_f \otimes_{\mathbf{Z}} \mathbf{Z}[T, T^{-1}]$ (where T is an indeterminate).

Proof. Equation (8.3.6.2) follows immediately from (8.3.5.3) and (8.3.2.5), and shows the surjectivity of π; we have already seen that the immersion i, restricted to C_f, corresponds to the injection **II** | 166
$S_f^{\leqslant} \to S_f$ (8.3.2). Finally, the last claim is a consequence of the fact that, for $f \in S_1$, S_f is isomorphic to $S_{(f)}[T, T^{-1}]$ (2.2.1). □

Remark (8.3.7). — Whenever Y is affine, the elements of the underlying space of E are the (not-necessarily-graded) prime ideals $\mathfrak{p}$ of S not containing S_+, by definition of the immersion ε (8.3.2). For such an ideal $\mathfrak{p}$, the $\mathfrak{p} \cap S_n$ clearly satisfy the conditions of (2.1.9), and so there exists exactly one *graded* prime ideal $\mathfrak{q}$ of S such that $\mathfrak{q} \cap S_n = \mathfrak{p} \cap S_n$ for all n; the map $\pi : E \to X$ of underlying spaces can then be understood via the equation

$$\pi(\mathfrak{p}) = \mathfrak{q}. \tag{8.3.7.1}$$

Indeed, to prove this equation, it suffices to consider some homogeneous f in S_+ such that $\mathfrak{p} \in D(f)$, and to note that $\mathfrak{q}_{(f)}$ is the inverse image of $\mathfrak{p}_f$ under the injection $S_{(f)} \to S_f$.

Corollary (8.3.8). — *If $\mathscr{S}$ is generated by $\mathscr{S}_1$, then the morphisms p and π are of finite type; for all $x \in X$, the fibre $p^{-1}(x)$ is isomorphic to* $\operatorname{Spec}(k(x)[T])$*, and the fibre π^{-1} isomorphic to* $\operatorname{Spec}(k(x)[T, T^{-1}])$

Proof. This follows immediately from (8.3.5) and (8.3.6) by noting that, whenever Y is affine and S is generated by S_1, the X_f, for $f \in S_1$, form a cover of X (2.3.14). □

Remark (8.3.9). — The pointed affine cone corresponding to the graded $\mathscr{O}_Y$-algebra $\mathscr{O}_Y[T]$ (where T is an indeterminate) can be identified with $G_m = \operatorname{Spec}(\mathscr{O}_Y[T, T^{-1}])$, since it is exactly C_T, as we have seen in (8.3.2) (see (8.4.4) for a more general result). This prescheme is canonical endowed with the structure of a "*Y-scheme in commutative groups*". This idea will be explained in detail later on, but, for now, can be quickly summarised as follows. A Y-scheme in groups is a Y-scheme G endowed with two Y-morphisms, $p : G \times_Y G \to G$ and $s : G \to G$, that satisfy conditions formally analogous to the axioms of the composition law and the symmetry law of a group: the diagram

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{p \times 1} & G \times G \\ {\scriptstyle 1 \times p}\downarrow & & \downarrow{\scriptstyle p} \\ G \times G & \xrightarrow[p]{} & G \end{array}$$

should commute ("associativity"), and there should be a condition which corresponds to the fact that, for groups, the maps

$$(x,y) \longmapsto (x,x^{-1},y) \longmapsto (x,x^{-1}y) \longmapsto x(x^{-1}y)$$

and

$$(x,y) \longmapsto (x,x^{-1},y) \longmapsto (x,yx^{-1}) \longmapsto (yx^{-1})x$$

should both reduce to $(x,y) \mapsto y$; the sequence of morphisms corresponding, for example, to the first composite map is

$$G \times G \xrightarrow{(1,s)\times 1} G \times G \times G \xrightarrow{1\times p} G \times G \xrightarrow{p} G$$

and the reader should write down the second sequence. **II** | 167

It is immediate (**I**, 3.4.3) that the data of a structure of a Y-scheme in groups on a Y-scheme G is equivalent to the data, for *every* Y-prescheme Z, of a *group* structure on the set $\operatorname{Hom}_Y(Z,G)$, where these structures should be such that, for every Y-morphism $Z \to Z'$, the corresponding map $\operatorname{Hom}_Y(Z',G) \to \operatorname{Hom}_Y(Z,G)$ is a group homomorphism. In the particular case of G_m that we consider here, $\operatorname{Hom}_Y(Z,G)$ can be identified with the set of Z-sections of $Z \times_Y G_m$ (**I**, 3.3.14), and thus with the set of Z-sections of $\operatorname{Spec}(\mathscr{O}_Z[T,T^{-1}])$; finally, the same reasoning as in (**I**, 3.3.15) shows that this set is canonically identified with the set of *invertible* elements of the ring $\Gamma(Z,\mathscr{O}_Z)$, and the group structure on this set is the structure coming from the multiplication in the ring $\Gamma(Z,\mathscr{O}_Z)$. The reader can verify that the morphisms p and s from above are obtained in the following way: they correspond, by (1.2.7) and (1.4.6), to the homomorphisms of $\mathscr{O}_Y$-algebras

$$\pi : \mathscr{O}_Y[T,T^{-1}] \longrightarrow \mathscr{O}_Y[T,T^{-1},T',T'^{-1}]$$
$$\sigma : \mathscr{O}_Y[T,T^{-1}] \longrightarrow \mathscr{O}_Y[T,T^{-1}]$$

and are entirely defined by the data of $\pi(T) = TT'$ and $\sigma(T) = T^{-1}$.

With this in mind, G_m can be considered as a "*universal domain of operators*" for every *affine cone* $C = \operatorname{Spec}(\mathscr{S})$, where $\mathscr{S}$ is a quasi-coherent positively-graded $\mathscr{O}_Y$-algebra. This means that we can canonically define a Y-morphism $G_m \times_Y C \to C$ which has the formal properties of an external law of a set endowed with a group of operators; or, again, as above for schemes in groups, we can give, for every Y-prescheme Z, an external law on $\operatorname{Hom}_Y(Z,C)$, having the group $\operatorname{Hom}_Y(Z,G_m)$ as its set of operators, with the usual axioms of sets endowed with a group of operators, and a compatibility condition with respect to the Y-morphisms $Z \to Z'$. In the current case, the morphism $G_m \times_Y C \to C$ is defined by the data of a homomorphism of $\mathscr{O}_Y$-algebras $\mathscr{S} \to \mathscr{S} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y[T,T^{-1}] = \mathscr{S}[T,T^{-1}]$, which associates, to each section $s_n \in \Gamma(U,\mathscr{S}_n)$ (where U is an open subset of Y), the section $s_nT^n \in \Gamma(U,\mathscr{S} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y[T,T^{-1}])$.

Conversely, suppose that we are given a quasi-coherent, *a priori non-graded*, $\mathscr{O}_Y$-algebra, and, on $C = \operatorname{Spec}(\mathscr{S})$, a structure of a "*$Y$-scheme in sets endowed with a group of operators*" that has the Y-scheme in groups G_m as its domain of operators; then we canonically obtain a *grading* of $\mathscr{O}_Y$-algebras on $\mathscr{S}$. Indeed, the data of a Y-morphism $G_m \times_Y C \to C$ is equivalent to that of a homomorphism of $\mathscr{O}_Y$-algebras $\psi : \mathscr{S} \to \mathscr{S}[T,T^{-1}]$, which can be written as $\psi = \sum_{n\in\mathbf{Z}} \psi_n T^n$, where the $\psi_n : \mathscr{S} \to \mathscr{S}$ are homomorphisms of $\mathscr{O}_Y$-modules (with $\psi_n(s) = 0$ except for finitely many n for every section $s \in \Gamma(U,\mathscr{S})$, for any open subset U of Y). We can then prove that the axioms of sets endowed with a group of operators imply that the $\psi_n(\mathscr{S}) = \mathscr{S}_n$ define a grading (in positive or negative degree) of $\mathscr{O}_Y$-algebras on $\mathscr{S}$, with the ψ_n being the corresponding projectors. We also have the notation of a structure of an "*affine cone*" on every affine Y-scheme, defined in a "geometric" way without any reference to any prior grading. We will not further develop this point of view here, and we leave the **II** | 168
work of precisely formulating the definitions and results corresponding to the information given above to the reader.

8.4. Projective closure of a vector bundle.

(8.4.1). Let Y be a prescheme, and $\mathscr{E}$ a quasi-coherent $\mathscr{O}_Y$-module. If we take $\mathscr{S}$ to be the graded $\mathscr{O}_Y$-algebra $\mathbf{S}_{\mathscr{O}_Y}(\mathscr{E})$, then Definition (8.3.1.1) shows that $\widehat{\mathscr{S}}$ can be identified with $\mathbf{S}_{\mathscr{O}_Y}(\mathscr{E} \oplus \mathscr{O}_Y)$. With the affine cone $\operatorname{Spec}(\mathscr{S})$ defined by $\mathscr{S}$ being, by definition, $\mathbf{V}(\mathscr{E})$, and $\operatorname{Proj}(\mathscr{S})$ being, by definition, $\mathbf{P}(\mathscr{E})$, we see that:

Proposition (8.4.2). — *The projective closure of a vector bundle $\mathbf{V}(\mathscr{E})$ on Y is canonically isomorphic to $\mathbf{P}(\mathscr{E} \oplus \mathscr{O}_Y)$, and the part at infinity of the latter is canonically isomorphic to $\mathbf{P}(\mathscr{E})$.*

Remark (8.4.3). — Take, for example, $\mathscr{E} = \mathscr{O}_Y^r$ with $r \geqslant 2$; then the pointed cones E and $\widehat{E}$ defined by $\mathscr{S}$ are nether affine nor projective on Y if $Y \neq \varnothing$. The second claim is immediate, because $\widehat{C} = \mathbf{P}(\mathscr{O}_Y^{r+1})$ is projective on Y, and the underlying spaces of E and $\widehat{E}$ are non-closed open subsets of $\widehat{C}$, and so the canonical immersions $E \to \widehat{C}$ and $\widehat{E} \to \widehat{C}$ are not projective (5.5.3), and we conclude by appealing to (5.5.5, v). Now, supposing, for example, that $Y = \operatorname{Spec}(A)$ is affine, and $r = 2$, then $C = \operatorname{Spec}(A[T_1, T_2])$, and E is then the prescheme induced by C on the open subset $D(T_1) \cup D(T_2)$; but we have already seen that the latter is not affine (**I**, 5.5.11); *a fortiori* $\widehat{E}$ cannot be affine, since E is the open subset where the section $\mathbf{z}$ over $\widehat{E}$ does not vanish (8.3.2).

However:

Proposition (8.4.4). — *If $\mathscr{L}$ is an invertible $\mathscr{O}_Y$-module, then there are canonical isomorphisms for both the pointed cones E and $\widehat{E}$ corresponding to $C = \mathbf{V}(\mathscr{L})$:*

(8.4.4.1)
$$\operatorname{Spec}\left(\bigoplus_{n\in\mathbf{Z}} \mathscr{L}^{\otimes n}\right) \xrightarrow{\sim} E$$

(8.4.4.2)
$$\mathbf{V}(\mathscr{L}^{-1}) \xrightarrow{\sim} \widehat{E}.$$

Furthermore, there exists a canonical isomorphism from the projective closure of $\mathbf{V}(\mathscr{L})$ to the projective closure of $\mathbf{V}(\mathscr{L}^{-1})$ that sends the null section (resp. the part at infinity) of the former to the part at infinity (resp. the null section) of the second.

Proof. We have here that $\mathscr{S} = \bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes n}$; the canonical injection

$$\mathscr{S} \longrightarrow \bigoplus_{n\in\mathbf{Z}} \mathscr{L}^{\otimes n}$$

defines a canonical dominant morphism

(8.4.4.3)
$$\operatorname{Spec}\left(\bigoplus_{n\in\mathbf{Z}} \mathscr{L}^{\otimes n}\right) \longrightarrow \mathbf{V}(\mathscr{L}) = \operatorname{Spec}\left(\bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes n}\right)$$

and it suffices to prove that this morphism is an isomorphism from the scheme $\operatorname{Spec}(\bigoplus_{n\in\mathbf{Z}} \mathscr{L}^{\otimes n})$ to E. The question being local on Y, we can assume that $Y = \operatorname{Spec}(A)$ is affine and that $\mathscr{L} = \mathscr{O}_Y$, **II** | 169
and so $\mathscr{S} = (A[T])^{\sim}$ and $\bigoplus_{n\in\mathbf{Z}} \mathscr{L}^{\times n} = (A[T, T^{-1}])^{\sim}$. But $A[T, T^{-1}]$ is the ring of fractions $A[T]_T$ of $A[T]$, and thus (8.4.4.3) identifies $\bigoplus_{n\in\mathbf{Z}} \mathscr{L}^{\otimes n}$ (?) with the prescheme induced by $C = \mathbf{V}(\mathscr{L})$ on the open subset $D(T)$; the complement $V(T)$ of this open subset in C is the underlying space of the closed subprescheme of C defined by the ideal $TA[T]$, which is exactly the null section of C, and so $E = D(T)$.

The isomorphism in (8.4.4.2) will be a consequence of the last claim, since $\mathbf{V}(\mathscr{L}^{-1})$ is the complement of the part at infinity of its projective closure, and $\widehat{E}$ is the complement of the null section of the projective closure $C = \mathbf{V}(\mathscr{L})$. But these projective closures are $\mathbf{P}(\mathscr{L}^{-1} \oplus \mathscr{O}_Y)$ and $\mathbf{P}(\mathscr{L} \oplus \mathscr{O}_Y)$ (respectively); but we can write $\mathscr{L} \oplus \mathscr{O}_Y = \mathscr{L} \otimes (\mathscr{L}^{-1} \oplus \mathscr{O}_Y)$. The existence of the desired canonical isomorphism then follows from (4.1.4), and everything reduces to showing that this isomorphism swaps the null sections and the parts at infinity. For this, we can reduce to the case where $Y = \operatorname{Spec}(A)$ is affine, $L = Ac$, and $L^{-1} = Ac'$, with the canonical isomorphism $L \otimes L^{-1} \to A$ sending $c \otimes c'$ to the element 1 of A. Then $\mathbf{S}(L \oplus A)$ is the tensor product of $A[\mathbf{z}]$ with $\bigoplus_{n\geqslant 0} Ac^{\otimes n}$, and $\mathbf{S}(L^{-1} \oplus A)$ is the tensor product of $A[\mathbf{z}]$ with $\bigoplus_{n\geqslant 0} Ac'^{\otimes n}$, and the isomorphism defined in (4.1.4) sends $\mathbf{z}^h \otimes c'^{\otimes(n-h)}$ to the element $\mathbf{z}^{n-h} \otimes c^{\otimes h}$. But, in $\mathbf{P}(\mathscr{L}^{-1} \oplus \mathscr{O}_Y)$, the part at infinity is the set of points where the section $\mathbf{z}$ vanishes, and the null section is the section of points where the section c' vanishes; since we have analogous definitions for $\mathbf{P}(\mathscr{L} \oplus \mathscr{O}_Y)$, the conclusion follows immediately from the above explanation. □

8.5. Functorial behaviour.

(8.5.1). Let Y and Y' be prescheme, $q : Y' \to Y$ a morphism, and $\mathscr{S}$ (resp. $\mathscr{S}'$) a *positively*-graded quasi-coherent $\mathscr{O}_Y$-algebra (resp. *positively*-graded quasi-coherent $\mathscr{O}_{Y'}$-algebra). Consider a q-morphism of graded algebras

(8.5.1.1) $$\varphi : \mathscr{S} \longrightarrow \mathscr{S}'.$$

We know (1.5.6) that this corresponds, canonically, to a morphism

$$\Phi = \operatorname{Spec}(\varphi) : \operatorname{Spec}(\mathscr{S}') \longrightarrow \operatorname{Spec}(\mathscr{S})$$

such that the diagram

(8.5.1.2) $$\begin{array}{ccc} C' & \xrightarrow{\Phi} & C \\ \downarrow & & \downarrow \\ Y' & \xrightarrow[q]{} & Y \end{array}$$

commutes, where we write $C = \operatorname{Spec}(\mathscr{S})$ and $C' = \operatorname{Spec}(\mathscr{S}')$. *Suppose, further, that* $\mathscr{S}_0 = \mathscr{O}_Y$ *and* $\mathscr{S}'_0 = \mathscr{O}_{Y'}$; let $\varepsilon : Y \to C$ and $\varepsilon : Y' \to C'$ be the canonical immersions (8.3.2); we then have a commutative diagram

(8.5.1.3) $$\begin{array}{ccc} Y' & \xrightarrow{q} & Y \\ {\scriptstyle\varepsilon'}\downarrow & & \downarrow{\scriptstyle\varepsilon} \\ C' & \xrightarrow[\Phi]{} & C \end{array}$$

which corresponds to the diagram **II** | 170

$$\begin{array}{ccc} \mathscr{S} & \xrightarrow{\varphi} & \mathscr{S}' \\ \downarrow & & \downarrow \\ \mathscr{O}_Y & \longrightarrow & \mathscr{O}_{Y'} \end{array}$$

where the vertical arrows are the augmentation homomorphisms, and so the commutativity follows from the hypothesis that φ is assumed to be a homomorphism of *graded* algebras.

Proposition (8.5.2). — *If E (resp. E') is the pointed affine cone defined by $\mathscr{S}$ (resp. $\mathscr{S}'$), then $\Phi^{-1}(E) \subset E'$; if, further,* $\operatorname{Proj}(\varphi) : G(\varphi) \to \operatorname{Proj}(\mathscr{S})$ *is everywhere defined (or, equivalently, if $G(\varphi) = \operatorname{Proj}(\mathscr{S}')$), then $\Phi^{-1}(E) = E'$, and conversely.*

Proof. The first claim follows from the commutativity of (8.5.1.3). To prove the second, we can restrict to the case where $Y = \operatorname{Spec}(A)$ and $Y' = \operatorname{Spec}(A')$ are affine, and $\mathscr{S} = \widetilde{S}$ and $\mathscr{S}' = \widetilde{S}'$. For every homogeneous f in S_+, writing $f' = \varphi(f)$, we have that $\Phi^{-1}(C_f) = C'_{f'}$ (**I**, 2.2.4.1); saying that $G(\varphi) = \operatorname{Proj}(S')$ implies that the radical (in S'_+) of the ideal generated by the $f' = \varphi(f)$ is S'_+ itself ((2.8.1) and (2.3.14)), and this is equivalent to saying that the $C'_{f'}$ cover E' (8.3.4.4). □

(8.5.3). The q-morphism φ canonically extends to a q-morphism of graded algebras

(8.5.3.1) $$\widehat{\varphi} : \widehat{\mathscr{S}} \longrightarrow \widehat{\mathscr{S}'}$$

by letting $\widehat{\varphi}(\mathbf{z}) = \mathbf{z}$. This induces a morphism

$$\widehat{\Phi} = \operatorname{Proj}(\widehat{\varphi}) : G(\widehat{\varphi}) \longrightarrow \widehat{C} = \operatorname{Proj}(\widehat{\mathscr{S}})$$

such that the diagram

$$\begin{array}{ccc} G(\widehat{\varphi}) & \xrightarrow{\widehat{\Phi}} & \widehat{C} \\ \downarrow & & \downarrow \\ Y' & \xrightarrow[q]{} & Y \end{array}$$

commutes (3.5.6). It follows immediately from the definitions that, if we write $i : C \to \widehat{C}$ and $i' : C' \to \widehat{C'}$ to mean the canonical open immersions (8.3.2), then $i'(C') \subset G(\widehat{\varphi})$, and the diagram

(8.5.3.2)
$$\begin{array}{ccc} C' & \xrightarrow{\Phi} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ G(\widehat{\varphi}) & \xrightarrow[\widehat{\Phi}]{} & \widehat{C} \end{array}$$

commutes. Finally, if we let $X = \operatorname{Proj}(\mathscr{S})$ and $X' = \operatorname{Proj}(\mathscr{S}')$, and if $j : X \to \widehat{C}$ and $j' : X' \to \widehat{C'}$ are the canonical closed immersions (8.3.2), then it follows from the definition of these immersions that $j'(G(\varphi)) \subset G(\widehat{\varphi})$, and that the diagram II | 171

(8.5.3.3)
$$\begin{array}{ccc} G(\varphi) & \xrightarrow{\operatorname{Proj}(\varphi)} & X \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\ G(\widehat{\varphi}) & \xrightarrow[\widehat{\Phi}]{} & \widehat{C} \end{array}$$

commutes.

Proposition (8.5.4). — *If $\widehat{E}$ (resp. $\widehat{E'}$) is the pointed projective cone defined by $\mathscr{S}$ (resp. by $\mathscr{S}'$), then $\widehat{\Phi}^{-1}(\widehat{E}) \subset \widehat{E'}$; furthermore, if $p : \widehat{E} \to X$ and $p' : \widehat{E'} \to X'$ are the canonical retractions, then $p'(\widehat{\Phi}^{-1}(\widehat{E})) \subset G(\widehat{\varphi})$, and the diagram*

(8.5.4.1)
$$\begin{array}{ccc} \widehat{\Phi}^{-1}(\widehat{E}) & \xrightarrow{\widehat{\Phi}} & \widehat{E} \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ G(\varphi) & \xrightarrow[\operatorname{Proj}(\varphi)]{} & X \end{array}$$

commutes. If $\operatorname{Proj}(\varphi)$ *is everywhere defined, then so too is $\widehat{\Phi}$, and we have that $\widehat{\Phi}^{-1}(\widehat{E}) = \widehat{E'}$*

Proof. The first claim follows from the commutativity of Diagrams (8.5.1.3) and (8.5.3.2), and the two following claims from the definition of the canonical retractions (8.3.5) and the definition of $\widehat{\varphi}$. To see that $\widehat{\Phi}$ is everywhere defined whenever $\operatorname{Proj}(\varphi)$ is, we can restrict to the case where $Y = \operatorname{Spec}(A)$ and $Y' = \operatorname{Spec}(A')$ are affine, and where $\mathscr{S} = \widetilde{S}$ and $\mathscr{S}' = \widetilde{S'}$; the hypothesis is that, when f runs over the set of homogeneous elements of S_+, the radical in S'_+ of the ideal generated in S'_+ by the $\varphi(f)$ is S'_+ itself; we thus immediately conclude that the radical in $(S'[\mathbf{z}])_+$ of the ideal generated by $\mathbf{z}$ and the $\varphi(f)$ is $(S'[\mathbf{z}])_+$ itself, whence our claim; this also shows that $\widehat{E'}$ is the union of the $\widehat{C'}_{\varphi(f)}$, and hence equal to $\widehat{\Phi}^{-1}(\widehat{E})$. ☐

Corollary (8.5.5). — *Whenever* $\operatorname{Proj}(\varphi)$ *is everywhere defined, the inverse image under $\widehat{\Phi}$ of the underlying space of the part at infinity (resp. of the vertex prescheme) of $\widehat{C'}$ is the underlying space of the part at infinity (resp. of the vertex prescheme) of $\widehat{C}$.*

Proof. This follows immediately from (8.5.4) and (8.5.2), taking into account the equalities (8.3.4.1) and (8.3.4.2). ☐

8.6. A canonical isomorphism for pointed cones.

(8.6.1). Let Y be a prescheme, $\mathscr{S}$ a quasi-coherent positively-graded $\mathscr{O}_Y$-algebra *such that* $\mathscr{S}_0 = \mathscr{O}_Y$, and let X be the Y-scheme $\operatorname{Proj}(\mathscr{S})$. We are going to apply the results of (8.5) to the case where $Y' = X$, and $q : X \to Y$ is the structure morphism; let

(8.6.1.1)
$$\mathscr{S}_X = \bigoplus_{n \in \mathbf{Z}} \mathscr{O}_X(n)$$

which is a quasi-coherent graded $\mathscr{O}_X$-algebra, with multiplication defined by means of the canonical homomorphisms (3.2.6.1) II | 172

$$\mathscr{O}_X(m) \otimes_{\mathscr{O}_X} \mathscr{O}_X(n) \longrightarrow \mathscr{O}_X(m+n)$$

whose associativity is ensured by the commutative diagram in (2.5.11.4). Let $\mathscr{S}'$ be the quasi-coherent positively-graded $\mathscr{O}_X$-subalgebra $\mathscr{S}_X^{\geqslant} = \bigoplus_{n\geqslant 0} \mathscr{O}_X(n)$ of $\mathscr{S}_X$.

Finally, consider the canonical q-morphism

(8.6.1.2) $$\alpha : \mathscr{S} \longrightarrow \mathscr{S}_X^{\geqslant}$$

defined in (3.3.2.3) as a homomorphism $\mathscr{S} \to q_*(\mathscr{S}_X)$, but which clearly sends $\mathscr{S}$ to $q_*(\mathscr{S}_X^{\geqslant})$. Write

(8.6.1.3) $$C_X = \operatorname{Spec}(\mathscr{S}_X^{\geqslant}), \quad \widehat{C}_X = \operatorname{Proj}(\mathscr{S}_X^{\geqslant}[\mathbf{z}]), \quad X' = \operatorname{Proj}(\mathscr{S}_X^{\geqslant})$$

and denote by E_X and $\widehat{E}_X$ the corresponding pointed affine and pointed projective cones (respectively); denote the canonical morphisms defined in (8.3) by $\varepsilon_X : X \to C_X$, $i_X : C \to \widehat{C}_X$, $j_X : X' \to \widehat{C}_X$, $p_X : \widehat{E}_X \to X'$, and $\pi_X : E_X \to X'$.

Proposition (8.6.2). — *The structure morphism $u : X' \to X$ is an* isomorphism, *and the morphism* $\operatorname{Proj}(\alpha)$ *is everywhere defined and identical to u. The morphism* $\operatorname{Proj}(\widehat{\alpha}) : \widehat{C}_X \to \widehat{C}$ *is everywhere defined, and its restrictions to $\widehat{E}_X$ and E_X are* isomorphisms *to $\widehat{E}$ and E (respectively). Finally, if we identify X' with X via u, then the morphisms p_X and π_X are identified with the structure morphisms of the X-preschemes $\widehat{E}_X$ and E_X.*

Proof. We can clearly restrict to the case where $Y = \operatorname{Spec}(A)$ is affine, and $\mathscr{S} = \widetilde{S}$; then X is the union of affine open subsets X_f, where f runs over the set of homogeneous elements of S_+, with the ring of each X_f being $S_{(f)}$. It follows from (8.2.7.1) that

(8.6.2.1) $$\Gamma(X_f, \mathscr{S}_X^{\geqslant}) = S_f^{\geqslant}.$$

So $u^{-1}(X_f) = \operatorname{Proj}(S_f^{\geqslant})$. But if $f \in S_d$ ($d > 0$), then $\operatorname{Proj}(S_f^{\geqslant})$ is canonically isomorphic to $\operatorname{Proj}((S_f^{\geqslant})^{(d)})$ (2.4.7), and we also know that $(S_f^{\geqslant})^{(d)} = (S^{(d)})_f^{\geqslant}$ can be identified with $S_{(f)}[T]$ (2.2.1) by the map $T \mapsto f/1$; we thus conclude (3.1.7) that the structure morphism $u^{-1}(X_f) \to X_f$ is an isomorphism, whence the first claim. To prove the second, note that the restriction $u^{-1}(X_f) \cap G(\alpha) \to X = \operatorname{Proj}(S)$ of $\operatorname{Proj}(\alpha)$ corresponds to the canonical map $x \mapsto x/1$ from S to $S_f^{\geqslant}$ (2.6.2); we thus deduce, first of all, that $G(\alpha) = X'$, and then, taking into account the fact that $u^{-1}(X_f) = (u^{-1}(X_f))_{f/1}$, that it follows from (2.8.1.1) that the image of $u^{-1}(X_f)$ under $\operatorname{Proj}(\alpha)$ is contained in X_f, and the restriction of $\operatorname{Proj}(\alpha)$ to $u^{-1}(X_f)$, thought of as a morphism to $X_f = \operatorname{Spec}(S_{(f)})$, is indeed identical to that of u. Finally, applying (8.3.5.4) to p_X instead of p, we see that $p_X^{-1}(u^{-1}(X_f)) = \operatorname{Spec}((S_f^{\geqslant})_{f/1}^{\leqslant})$, and this open subset is, by (8.5.4.1), the inverse image under $\operatorname{Proj}(\widehat{\alpha})$ of $p^{-1}(X_f) = \operatorname{Spec}(S_f^{\leqslant})$ (8.3.5.3). Taking (8.2.3.2) into account, the restriction of $\operatorname{Proj}(\widehat{\alpha})$ to $p_X^{-1}(u^{-1}(X_f))$ corresponds to the isomorphism inverse to (8.2.7.2), restricted to $S_f^{\leqslant}$, whence the third claim; the last claim is evident by definition. **II** | 173

We note also that it follows from the commutative diagram in (8.5.3.2) that *the restriction to C_X of* $\operatorname{Proj}(\widehat{\alpha})$ *is exactly the morphism* $\operatorname{Spec}(\alpha)$. □

Corollary (8.6.3). — *Considered as X-schemes, $\widehat{E}_X$ is canonically isomorphic to* $\operatorname{Spec}(\mathscr{S}_X^{\leqslant})$, *and E_X to* $\operatorname{Spec}(\mathscr{S}_X)$.

Proof. Since we know that the morphisms p_X and π_X are affine ((8.3.5) and (8.3.6)), it suffices (given (1.3.1)) to prove the corollary in the case where $Y = \operatorname{Spec}(A)$ is affine and $\mathscr{S} = \widetilde{S}$. The first claim follows from the existence of the canonical isomorphisms (8.2.7.2) $(S_f^{\geqslant})_{f/1}^{\leqslant} \xrightarrow{\sim} S_f^{\leqslant}$ and from the fact that these isomorphisms are compatible with the map sending f to fg (where f and g are homogeneous in S_+). Similarly, applying (8.3.6.2) to π_X instead of π, we see that $\pi_X^{-1}(u^{-1}(X_f)) = \operatorname{Spec}((S_f^{\geqslant})_{f/1})$ for f homogeneous in S_+, and the second claim then follows from the existence of the canonical isomorphisms (8.2.7.2) $(S_f^{\geqslant})_{f/1} \xrightarrow{\sim} S_f$.

We can then say that $\widehat{C}_X$, thought of as an X-scheme, is given by *gluing* the affine X-schemes $C_X = \operatorname{Spec}(\mathscr{S}_X^{\geqslant})$ and $\widehat{E}_X = \operatorname{Spec}(\mathscr{S}_X^{\leqslant})$ over X, where the intersection of the two affine X-schemes is the open subset $E_X = \operatorname{Spec}(\mathscr{S}_X)$. □

Corollary (8.6.4). — *Assume that $\mathscr{O}_X(1)$ is an invertible $\mathscr{O}_X$-module, and that $\mathscr{S}_X$ is isomorphic to $\bigoplus_{n\in\mathbf{Z}}(\mathscr{O}_X(1))^{\otimes n}$ (which will be the case, in particular, whenever $\mathscr{S}$ is generated by $\mathscr{S}_1$ ((3.2.5) and (3.2.7))). Then the pointed projective cone $\widehat{E}$ can be identified with the rank-1 vector bundle $\mathbf{V}(\mathscr{O}_X(-1))$ on X, and the pointed affine cone E with the subprescheme of this vector bundle induced on the complement of the null section. With this identification, the canonical retraction $\widehat{E}\to X$ is identified with the structure morphism of the X-scheme $\mathbf{V}(\mathscr{O}_X(-1))$. Finally, there exists a canonical Y-morphism $\mathbf{V}(\mathscr{O}_X(1))\to C$, whose restriction to the complement of the null section of $\mathbf{V}(\mathscr{O}_X(1))$ is an isomorphism from this complement to the pointed affine cone E.*

Proof. If we write $\mathscr{L}=\mathscr{O}_X(1)$, then $\mathscr{S}_X^{\geqslant}$ is identical to $\mathbf{S}_{\mathscr{O}_X}(\mathscr{L})$, and so $\widehat{E}_X$ is canonically identified with $\mathbf{V}(\mathscr{L}^{-1})$, by (8.6.3), and C_X with $\mathbf{V}(\mathscr{L})$. The morphism $\mathbf{V}(\mathscr{L})\to C$ is the restriction of $\operatorname{Proj}(\widehat{\alpha})$, and the claims of the corollary are then particular cases of (8.6.2). ☐

We note that the inverse image under the morphism $\mathbf{V}(\mathscr{O}_X(1))\to C$ of the underlying space of the vertex prescheme of C is the underlying space of the null section of $\mathbf{V}(\mathscr{O}_X(1))$ (8.5.5); but, in general, the corresponding subpreschemes of C and of $\mathbf{V}(\mathscr{O}_X(1))$ are not isomorphic. This problem will be studied below.

8.7. Blowing up based cones.

(8.7.1). Under the conditions of (8.6.1), we have, writing $r=\operatorname{Proj}(\widehat{\alpha})$, a commutative diagram

(8.7.1.1)
$$\begin{array}{ccc} X & \xrightarrow{i_X\circ\varepsilon_X} & \widehat{C}_X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle r} \\ Y & \xrightarrow{i\circ\varepsilon} & \widehat{C} \end{array}$$

by (8.5.1.3) and (8.5.3.2); furthermore, the restriction of r to the complement $\widehat{C}_X - i_X(\varepsilon_X(X))$ of the null section is an *isomorphism* to the complement $\widehat{C}-i(\varepsilon(Y))$ of the null section, by (8.6.2). If we suppose, to simplify things, that Y is affine, that $\mathscr{S}$ is of finite type and generated by $\mathscr{S}_1$, and that X is projective over Y and $\widehat{C}_X$ projective over X (5.5.1), then $\widehat{C}_X$ is projective over Y (5.5.5, ii), and *a fortiori* over $\widehat{C}$ (5.5.5, v). We then have a projective Y-morphism $r:\widehat{C}_X\to\widehat{C}$ (whose restriction to C_X is a projective Y-morphism $C_X\to C$) that *contracts X to Y* (?) and that induces an *isomorphism* when we restrict to the *complements of X and Y*. We thus have a connection between C_X and C, analogous to that which exists between a blow-up prescheme and the original prescheme (8.1.3). We will effectively show that C_X can be identified with the homogeneous spectrum of a graded $\mathscr{O}_C$-algebra. **II** | 174

(8.7.2). Keeping the notation of (8.6.1), consider, for all $n\geqslant 0$, the quasi-coherent ideal

(8.7.2.1)
$$\mathscr{S}_{[n]}=\bigoplus_{m\geqslant n}\mathscr{S}_m$$

of the graded $\mathscr{O}_Y$-algebra $\mathscr{S}$. It is clear that

(8.7.2.2)
$$\mathscr{S}_{[0]}=\mathscr{S},\qquad \mathscr{S}_{[n]}\subset\mathscr{S}_{[m]}\qquad\text{for } m\leqslant n$$

(8.7.2.3)
$$\mathscr{S}_n\mathscr{S}_{[m]}\subset\mathscr{S}_{[m+n]}.$$

Consider the $\mathscr{O}_C$-module associated to $\mathscr{S}_{[n]}$, which is a quasi-coherent ideal of $\mathscr{O}_C=\widetilde{\mathscr{S}}$ (1.4.4)

(8.7.2.4)
$$\mathscr{I}_n=(\mathscr{S}_{[n]})^{\sim}.$$

We thus deduce, from (8.7.2.2) and (8.7.2.3), using (1.4.4) and (1.4.8.1), the analogous formulas

(8.7.2.5)
$$\mathscr{I}_{[0]}=\mathscr{O}_C,\qquad \mathscr{I}_{[n]}\subset\mathscr{I}_{[m]}\qquad\text{for } m\leqslant n$$

(8.7.2.6)
$$\mathscr{I}_n\mathscr{I}_{[m]}\subset\mathscr{I}_{[m+n]}.$$

We are thus in the setting of (8.1.1), which leads us to introduce the quasi-coherent graded $\mathscr{O}_C$-algebra

(8.7.2.7)
$$\mathscr{S}^{\natural}=\bigoplus_{n\geqslant 0}\mathscr{I}_n=\left(\bigoplus_{n\geqslant 0}\mathscr{S}_{[n]}\right)^{\sim}.$$

Proposition (8.7.3). — *There is a canonical C-isomorphism*

$$h : C_X \xrightarrow{\sim} \operatorname{Proj}(\mathscr{S}^\natural). \tag{8.7.3.1}$$

Proof. Suppose first of all that $Y = \operatorname{Spec}(A)$ is affine, so that $\mathscr{S} = \widetilde{S}$, with S a positively-graded A-algebra, and $C = \operatorname{Spec}(S)$. Definition (8.2.7.4) then shows, with the notation of (8.2.6), that $\mathscr{S}^\natural = (S^\natural)^\sim$. To define (8.7.3.1), consider a homogeneous element $f \in S_d$ ($d > 0$) and the corresponding element $f^\natural \in S^\natural$ (8.2.6); the S-isomorphism in (8.2.7.3) then defines a C-isomorphism

$$\operatorname{Spec}(S_f^{\geqslant}) \xrightarrow{\sim} \operatorname{Spec}(S^\natural_{(f^\natural)}). \tag{8.7.3.2}$$

But with the notation of (8.6.2), if $v : C_X \to X$ is the structure morphism, then it follows from (8.6.2.1) that $v^{-1}(X_f) = \operatorname{Spec}(S_f^{\geqslant})$. We also have that $\operatorname{Spec}(S^\natural_{(f^\natural)}) = D_+(f^\natural)$, which means that (8.7.3.2) defines an isomorphism $v^{-1}(X_f) \to D_+(f^\natural)$. Furthermore, if $g \in S_e$ ($e > 0$), then the diagram II | 175

$$\begin{array}{ccc} v^{-1}(X_{fg}) & \xrightarrow{\sim} & D_+(f^\natural g^\natural) \\ \downarrow & & \downarrow \\ v^{-1}(X_f) & \xrightarrow{\sim} & D_+(f^\natural) \end{array}$$

commutes, by definition of the isomorphism in (8.2.7.3). Finally, by definition, S_+ is generated by the homogeneous f, and so it follows from (8.2.10, iv) and from (2.3.14) that the $D_+(f^\natural)$ form a cover of $\operatorname{Proj}(S^\natural)$, and that the $v^{-1}(X_f)$ form a cover of C_X, since the X_f form a cover of X; in this case, we have thus defined the isomorphism (8.7.3.1).

To prove (8.7.3) in the general case, it suffices to show that, if U and U' are affine open subsets of Y, given by rings A and A' (respectively), and such that $U' \subset U$, then, setting $\mathscr{S}|U = \widetilde{S}$ and $\mathscr{S}|U' = \widetilde{S'}$, the diagram

$$\begin{array}{ccc} C_{U'} & \longrightarrow & \operatorname{Proj}(S'^\natural) \\ \downarrow & & \downarrow \\ C_U & \longrightarrow & \operatorname{Proj}(S^\natural) \end{array} \tag{8.7.3.3}$$

commutes. But S is canonically identified with $S \otimes_A A'$, and so $S'^\natural$ is canonically identified with

$$S^\natural \otimes_S S' = S^\natural \otimes_A A';$$

thus $\operatorname{Proj}(S'^\natural) = \operatorname{Proj}(S^\natural) \times_U U'$ (2.8.10); similarly, if $X = \operatorname{Proj}(S)$ and $X' = \operatorname{Proj}(S')$, then $X' = X \times_U U'$ and $\mathscr{S}_{X'} = \mathscr{S}_X \otimes_{\mathscr{O}_U} U'$ (3.5.4), or, equivalently, $\mathscr{S}_{X'} = j^*(\mathscr{S}_X)$, where j is the projection $X' \to X$. We then (1.5.2) have that $C_{U'} = C_U \times_X X' = C_U \times_U U'$, and the commutativity of (8.7.3.3) is then immediate. □

Remark (8.7.4). — (i) The end of the proof of (8.7.3) can be immediately generalised in the following way. Let $g : Y' \to Y$ be a morphism, $\mathscr{S}' = g^*(\mathscr{S})$, and $X' = \operatorname{Proj}(\mathscr{S}')$; then we have a commutative diagram

$$\begin{array}{ccc} C_{X'} & \longrightarrow & \operatorname{Proj}(\mathscr{S}'^\natural) \\ \downarrow & & \downarrow \\ C_X & \longrightarrow & \operatorname{Proj}(\mathscr{S}^\natural) \end{array} \tag{8.7.4.1}$$

Now let $\varphi : \mathscr{S}'' \to \mathscr{S}$ be a homomorphism of graded $\mathscr{O}_Y$-algebras such that, if we write $X'' = \operatorname{Proj}(\mathscr{S}'')$, then $u = \operatorname{Proj}(\varphi) : X \to X''$ is everywhere defined; we also have a Y-morphism $v : C \to C''$ (with $C'' = \operatorname{Spec}(\mathscr{S}''))$ such that $\mathscr{A}(v) = \varphi$, and, since φ is a II | 176
homomorphism of graded algebras, φ induces a v-morphism of graded algebras $\psi : \mathscr{S}''^\natural \to \mathscr{S}^\natural$ (1.4.1). Furthermore, it follows from (8.2.10, iv) and from the hypothesis on φ that $\operatorname{Proj}(\psi)$

is everywhere defined. Finally, taking (3.5.6.1) into account, there is a canonical u-morphism $\mathscr{S}_{X''} \to \mathscr{S}_X$, whence (1.5.6) a morphism $w : C_{X''} \to C_X$. With this in mind, the diagram

(8.7.4.2)
$$\begin{array}{ccc} C_{X''} & \xrightarrow{\sim} & \operatorname{Proj}(\mathscr{S}''^{\natural}) \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle \operatorname{Proj}(\psi)} \\ C_X & \xrightarrow{\sim} & \operatorname{Proj}(\mathscr{S}^{\natural}) \end{array}$$

is commutative, as we can immediately verify by restricting to the case where Y is affine.

(ii) Note that, by (8.7.2.5) and (8.7.2.6), we have $\mathscr{I}_1^m \subset \mathscr{I}_m \subset \mathscr{I}_1$ for all $m > 0$. But, by definition, $\mathscr{I}_1 = (\mathscr{S}_+)^\sim$, and so $\mathscr{I}_1$ defines the closed subprescheme $\varepsilon(Y)$ in C ((1.4.10) and (8.3.2)); we thus conclude that, for all $m > 0$, *the support of $\mathscr{O}_C/\mathscr{I}_m$ is contained in the underlying space of the vertex prescheme $\varepsilon(Y)$*; on the inverse image of the pointed affine cone E, the structure morphism $\operatorname{Proj}(\mathscr{S}^{\natural}) \to C$ thus restricts to an *isomorphism* (by (8.7.3) and (8.7.1)). Furthermore, by canonically identifying C with an open subset of $\widehat{C}$ (8.3.3), we can clearly extend the ideals $\mathscr{I}_m$ of $\mathscr{O}_C$ to ideals $\mathscr{J}_m$ of $\mathscr{O}_{\widehat{C}}$, by asking for it to agree with $\mathscr{O}_{\widehat{C}}$ on the open subset $\widehat{E}$ of $\widehat{C}$. If we define $\mathscr{T} = \bigoplus_{n \geqslant 0} \mathscr{J}_m$, which is a quasi-coherent graded $\mathscr{O}_{\widehat{C}}$-algebra, we can extend the isomorphism (8.7.3.1) to a $\widehat{C}$-isomorphism

(8.7.4.3)
$$\widehat{C}_X \xrightarrow{\sim} \operatorname{Proj}(\mathscr{T}).$$

Indeed, over $\widehat{E}$, it follows from the above that $\operatorname{Proj}(\mathscr{T})$ is canonically identified with $\widehat{E}$, and we thus define the isomorphism (8.7.4.3) over $\widehat{E}$ by asking for it to agree with the canonical isomorphism $\widehat{E}_X \to \widehat{E}$ (8.6.2); it is clear that this isomorphism and (8.7.3.1) then agree over $\widehat{E}$.

Corollary (8.7.5). — *Suppose that there exists some $n_0 > 0$ such that*

(8.7.5.1)
$$\mathscr{S}_{n+1} = \mathscr{S}_1 \mathscr{S}_n \qquad \textit{for } n \geqslant n_0.$$

Then the vertex subprescheme (?) of C_X (isomorphic to X) is the inverse image under the canonical morphism $r : C_X \to C$ of the vertex subprescheme of C (isomorphic to Y). Conversely, if this property is true, and if we further assume that Y is Noetherian and that $\mathscr{S}$ is of finite type, then there exists some $n_0 > 0$ such that (8.7.5.1) holds true.

Proof. The first claim being local on Y, we can assume that $Y = \operatorname{Spec}(A)$ is affine, so that $\mathscr{S} = \widetilde{S}$, with S a positively-graded A-algebra. The claim then follows from (8.2.12), since $\operatorname{Proj}(S^{\natural} \otimes_S S_0) = C_X \times_C \varepsilon(Y)$ (by the identification in (8.7.3.1)), or, in other words, since this prescheme is the inverse image of $\varepsilon(Y)$ in C_X (**I**, 4.4.1). The converse also follows from (8.2.12) whenever Y is Noetherian affine and S is of finite type. If Y is Noetherian (but not necessarily affine) and $\mathscr{S}$ is of finite type, **II** | 177
then there exists a finite cover of Y by Noetherian affine open subsets U_i, and we then deduce from the above that, for all i, there exists an integer n_i such that $\mathscr{S}_{n+1}|U_i = (\mathscr{S}_1|U_i)(\mathscr{S}_n|U_i)$ for $n \geqslant n_i$; the largest of the n_i then ensures that (8.7.5.1) holds true. □

(8.7.6). Now consider the C-prescheme Z given by *blowing up* the *vertex subprescheme* $\varepsilon(Y)$ in the affine cone C; by Definition (8.1.3), it is exactly the prescheme $\operatorname{Proj}(\bigoplus_{n \geqslant 0} \mathscr{S}_+^n)$; the canonical injection

(8.7.6.1)
$$\iota : \bigoplus_{n \geqslant 0} \mathscr{S}_+^n \longrightarrow \mathscr{S}^{\natural}$$

defines (by the identification in (8.7.3)) a canonical dominant C-morphism

(8.7.6.2)
$$G(\iota) \longrightarrow Z$$

where $G(\iota)$ is an open subset of C_X (3.5.1); note that it could be the case that $G(\iota) \neq C_X$, as shown by the example where $Y = \operatorname{Spec}(K)$, with K a field, and $\mathscr{S} = \widetilde{S}$, with $S = K[\mathbf{y}]$, where $\mathbf{y}$ is an indeterminate *of degree* 2; if R_n denotes the set $(S_+)^n$, thought of as a subset of $S_{[n]} = S_n^{\natural}$, then $S_+^{\natural}$ is not the radical in $S_+^{\natural}$ of the ideal generated by the union of the R_n (cf. (2.3.14)).

Corollary (8.7.7). — *Assume that there exists some $n_0 > 0$ such that*

(8.7.7.1) $$\mathscr{S}_n = \mathscr{S}_1^n \quad \text{for } n \geqslant n_0.$$

Then the canonical morphism (8.7.6.2) is everywhere defined, and is an isomorphism $C_X \xrightarrow{\sim} Z$. Conversely, if this property is true, and if we further assume that Y is Noetherian and that $\mathscr{S}$ is of finite type, then there exists some n_0 such that (8.7.7.1) holds true.

Proof. The first claim is local on Y, and thus follows from (8.2.14); the converse follows similarly, arguing as in (8.7.5). □

Remark (8.7.8). — Since condition (8.7.7.1) implies (8.7.5.1), we see that, whenever it holds true, not only can C_X be identified with the prescheme given by blowing up the vertex (identified with Y) of the affine cone C, but also the vertex (identified with X) of C_X can be identified with the closed subprescheme given by the inverse image of the vertex Y of C. Furthermore, hypothesis (8.7.7.1) implies that, on $X = \operatorname{Proj}(\mathscr{S})$, the $\mathscr{O}_X$-modules $\mathscr{O}_X(n)$ are invertible ((3.2.5) and (3.2.9)), and that $\mathscr{O}_X(n) = \mathscr{L}^{\otimes n}$ with $\mathscr{L} = \mathscr{O}_X(1)$ ((3.2.7) and (3.2.9)); by Definition (8.6.1.1), C_X is thus the *vector bundle* $\mathbf{V}(\mathscr{L})$ on X, and its vertex is the *null section* of this vector bundle.

8.8. Ample sheaves and contractions.

(8.8.1). Let Y be a prescheme, $f : X \to Y$ a *separated* and *quasi-compact* morphism, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module that is *ample relative to f*. Consider the positively-graded $\mathscr{O}_Y$-algebra

(8.8.1.1) $$\mathscr{S} = \mathscr{O}_Y \oplus \bigoplus_{n\geqslant 1} f_*(\mathscr{L}^{\otimes n})$$

which is quasi-coherent (**I**, 9.2.2, a). There is a canonical homomorphisms of graded $\mathscr{O}_X$-algebras **II** | 178

(8.8.1.2) $$\tau : f^*(\mathscr{S}) \longrightarrow \bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes n}$$

which, in degrees $\geqslant 1$, agrees with the canonical homomorphism $\sigma : f^*(f_*(\mathscr{L}^{\otimes n})) \to \mathscr{L}^{\otimes n}$ (**0**, 4.4.3), and is the identity in degree 0. The hypothesis that $\mathscr{L}$ is f-ample then implies ((4.6.3) and (3.6.1)) that the corresponding Y-morphism

(8.8.1.3) $$r = r_{\mathscr{L},\tau} : X \longrightarrow P = \operatorname{Proj}(\mathscr{S})$$

is everywhere defined and is a *dominant open immersion*, and that

(8.8.1.4) $$r^*(\mathscr{O}_P(n)) = \mathscr{L}^{\otimes n} \qquad \text{for all } n \in \mathbf{Z}.$$

Proposition (8.8.2). — *Let $C = \operatorname{Spec}(\mathscr{S})$ be the affine cone defined by $\mathscr{S}$; if $\mathscr{L}$ is f-ample, then there exists a canonical Y-morphism*

(8.8.2.1) $$g : V = \mathbf{V}(\mathscr{L}) \longrightarrow C$$

such that the diagram

(8.8.2.2) $$\begin{array}{ccccc} X & \xrightarrow{j} & \mathbf{V}(\mathscr{L}) & \xrightarrow{\pi} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\varepsilon} & C & \xrightarrow{\psi} & Y \end{array}$$

commutes, where ψ and π are the structure morphisms, and j and ε the canonical immersions sending X and Y (respectively) to the null section of $\mathbf{V}(\mathscr{L})$ and the vertex prescheme of C (respectively). Furthermore, the restriction of g to $\mathbf{V}(\mathscr{L}) - j(X)$ is an open immersion

(8.8.2.3) $$\mathbf{V}(\mathscr{L}) - j(X) \longrightarrow E = C - \varepsilon(Y)$$

into the pointed affine cone E corresponding to $\mathscr{S}$.

Proof. With the notation of (8.8.1), let $\mathscr{S}_P^{\geqslant} = \bigoplus_{n\geqslant 0} \mathscr{O}_P(n)$ and $C_p = \operatorname{Spec}(\mathscr{S}_P^{\geqslant})$. We know (8.6.2) that there is a canonical morphism $h = \operatorname{Spec}(\alpha) : C_p \to C$ such that the diagram

(8.8.2.4)
$$\begin{array}{ccc} C_P & \longrightarrow & P \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{\psi} & Y \end{array}$$

commutes; furthermore, if $\varepsilon_P : P \to C_P$ is the canonical immersion, then the diagram

(8.8.2.5)
$$\begin{array}{ccc} P & \xrightarrow{p} & C_P \\ {\scriptstyle \varepsilon_P}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{\varepsilon} & C \end{array}$$

commutes (8.7.1.1), and, finally, the restriction of H to the pointed affine cone E_P is an *isomorphism* $E_P \xrightarrow{\sim} E$ (8.6.2). It follows from (8.8.1.4) that

$$r^*(\mathscr{S}_P^{\geqslant}) = \mathbf{S}_{\mathscr{O}_X}(\mathscr{L})$$

and so we have a canonical P-morphism $q : \mathbf{V}(\mathscr{L}) \to C_P$, with the commutative diagram **II** | 179

(8.8.2.6)
$$\begin{array}{ccc} \mathbf{V}(\mathscr{L}) & \xrightarrow{\pi} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle r} \\ C_P & \longrightarrow & P \end{array}$$

identifying $\mathbf{V}(\mathscr{L})$ with the product $C_P \times_P X$ (1.5.2); since r is an open immersion, so too is q (**I**, 4.3.2). Furthermore, the restriction of q to $\mathbf{V}(\mathscr{L}) - j(X)$ sends this prescheme to E_p, by (8.5.2), and the diagram

(8.8.2.7)
$$\begin{array}{ccc} X & \xrightarrow{j} & \mathbf{V}(\mathscr{L}) \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle q} \\ P & \xrightarrow{\varepsilon_P} & C_P \end{array}$$

is commutative (since it is a particular case of (8.5.1.3)). The claims of (8.8.2) immediately follow from these facts, by taking g to be the composite morphism $h \circ q$. □

Remark (8.8.3). — Assume further that Y is a *Noetherian* prescheme, and that f is a *proper* morphism. Since r is then *proper*(5.4.4), and thus closed, and since it is also a dominant open immersion, r is necessarily an *isomorphism* $X \xrightarrow{\sim} P$. Furthermore, we will see, in Chapter III (**III**, 2.3.5.1), that $\mathscr{S}$ is then necessarily an $\mathscr{O}_Y$-algebra *of finite type*. It then follows that $\mathscr{S}^\natural$ is an $\mathscr{S}_0^\natural$-algebra *of finite type* ((8.2.10, i) and (8.7.2.7)); since C_P is C-isomorphic to $\operatorname{Proj}(\mathscr{S}^\natural)$ (8.7.3), we see that the morphism $h : C_P \to C$ is *projective*; since the morphism r is an isomorphism, so too is $q : \mathbf{V}(\mathscr{L}) \to C_P$, and we thus conclude that the morphism $g : \mathbf{V}(\mathscr{L}) \to C$ is *projective*. Furthermore, since the restriction of h to E_P is an isomorphism to E, and since q is an isomorphism, the restriction (8.8.2.3) of g is an isomorphism $\mathbf{V}(\mathscr{L}) - j(X) \xrightarrow{\sim} E$.

If we further assume that L is *very ample* for f, then, as we will also see in Chapter III (**III**, 2.3.5.1), there exists some integer $n_0 > 0$ such that $\mathscr{S}_n = \mathscr{S}_1^n$ for $n \geqslant n_0$. We then conclude, by (8.7.7), that $\mathbf{V}(\mathscr{L})$ can be identified with the prescheme Z given by *blowing up the vertex prescheme* (identified with Y) *in the affine cone* C, and that the *null section* of $\mathbf{V}(\mathscr{L})$ (identified with Y) is the *inverse image* of the vertex subprescheme Y of C.

Some of the above results can in fact be proven even without the Noetherian hypothesis:

Corollary (8.8.4). — *Let Y be a prescheme (resp. a quasi-compact scheme), $f : X \to Y$ a proper morphism, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module that is ample relative to f. Then the morphism in* (8.8.2.1) *is proper (resp. projective), and its restriction* (8.8.2.3) *is an isomorphism.*

Proof. To prove that g is proper, we can restrict to the case where Y is affine, and it then suffices to consider the case where Y is a quasi-compact scheme. The same arguments as in (8.8.3) first of all show that r is an *isomorphism* $X \xrightarrow{\sim} P$; then q is also an isomorphism, and, since the restriction of h to E_P is an isomorphism $E_P \xrightarrow{\sim} E$, we have already seen that (8.8.2.3) is an isomorphism. It remains only to prove that g is *projective*.

Since f is of finite type, by hypothesis, we can apply (3.8.5) to the homomorphism τ from (8.8.1.2): II | 180
there is an integer $d > 0$ and a quasi-coherent $\mathscr{O}_Y$-submodule $\mathscr{E}$ of finite type of $\mathscr{S}_d$ such that, if $\mathscr{S}'$ is the $\mathscr{O}_Y$-subalgebra of $\mathscr{S}$ generated by $\mathscr{E}$, and $\tau' = \tau \circ q^*(\varphi)$ (where φ is the canonical injection $\mathscr{S}' \to \mathscr{S}$), then $r' = r_{\mathscr{L},\tau'}$ is an immersion

$$X \longrightarrow P' = \operatorname{Proj}(\mathscr{S}').$$

Furthermore, since φ is injective, r' is also a *dominant immersion* (3.7.6); the same argument as for r then shows that r' is a *surjective closed immersion*; since r' factors as $X \xrightarrow{r} \operatorname{Proj}(\mathscr{S}) \xrightarrow{\Phi} \operatorname{Proj}(\mathscr{S}')$, where $\Phi = \operatorname{Proj}(\varphi)$, we thus conclude that Φ is also a *surjective closed immersion*. But this implies that Φ is an *isomorphism*; we can restrict to the case where $Y = \operatorname{Spec}(A)$ is affine, and $\mathscr{S} = \widetilde{S}$ and $\mathscr{S}' = \widetilde{S}'$, with S a graded A-algebra and S' a graded subalgebra of S. For every homogeneous element $t \in S'$, we have that $S'_{(t)}$ is a subring of $S_{(t)}$; if we return to the definition of $\operatorname{Proj}(\varphi)$ (2.8.1), we see that it suffices to prove that, if B' is a subring of a ring B, and if the morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(B')$ corresponding to the canonical injection $B' \to B$ is a closed immersion, then this morphism is necessarily an *isomorphism*; but this follows from (**I**, 4.2.3). Furthermore, $\Phi^*(\mathscr{O}_{P'}(n)) = \mathscr{O}_P(n)$ ((3.5.2, ii) and (3.5.4)), and so $r'^*(\mathscr{O}_{P'}(n))$ is isomorphic to $\mathscr{L}^{\otimes n}$ (4.6.3). Let $\mathscr{S}'' = \mathscr{S}'^{(d)}$, so that (3.1.8, i) X is canonically identified with $P'' = \operatorname{Proj}(\mathscr{S}'')$, and $\mathscr{L}'' = \mathscr{L}^{\otimes d}$ with $\mathscr{O}_{P''}(1)$ (3.2.9, ii).

Now, if $C'' = \operatorname{Spec}(\mathscr{S}'')$, then $\mathscr{S}^{\geqslant}_{P''} = \bigoplus_{n\geqslant 0} \mathscr{O}_{P''}(n)$ can be identified with $\bigoplus_{n\geqslant 0} \mathscr{L}''^{\otimes n}$, and thus $C_{P''} = \operatorname{Spec}(\mathscr{S}^{\geqslant}_{P''})$ with $\mathbf{V}(\mathscr{L}'')$; we also know (8.7.3) that $C_{P''}$ is C''-isomorphic to $\operatorname{Proj}(\mathscr{S}''^{\natural})$; by the definition of $\mathscr{S}''$, we know that $\mathscr{S}''^{\natural}$ is generated by $\mathscr{S}''^{\natural}_1$, and that $\mathscr{S}''^{\natural}_1$ is of finite type over $\mathscr{S}''^{\natural}_0 = \mathscr{S}''$ ((8.2.10, i and iii)), and so $\operatorname{Proj}(\mathscr{S}''^{\natural})$ is *projective* over C'' (5.5.1). Consider the diagram

$$\begin{array}{ccc} \mathbf{V}(\mathscr{L}) & \xrightarrow{\ g\ } & \operatorname{Spec}(\mathscr{S}) = C \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ \mathbf{V}(\mathscr{L}'') & \xrightarrow{\ g''\ } & \operatorname{Spec}(\mathscr{S}'') = C'' \end{array} \tag{8.8.4.1}$$

where g and g'' correspond, by (1.5.6), to the canonical j-morphisms

$$\mathscr{S} \longrightarrow \bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes n} \quad \text{and} \quad \mathscr{S}'' \longrightarrow \bigoplus_{n\geqslant 0} \mathscr{L}''^{\otimes n}$$

(3.3.2.3) (see (8.8.5) below), and v and u to the inclusion morphisms $\mathscr{S}'' \to \mathscr{S}$ and $\bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes nd} \to \bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes n}$ (respectively); it is immediate (3.3.2) that this diagram is commutative. We have just seen that g'' is a projective morphism; we also know that u is a *finite* morphism. Since the question is local on X, we can assume that X is affine of ring A, and that $\mathscr{L} = \mathscr{O}_X$; everything then reduces to noting that the ring $A[T]$ is a module of finite type over its subring $A[T^d]$ (with T an indeterminate). Since Y is a quasi-compact scheme, and since C'' is affine over Y, we know that C'' is also a quasi-compact scheme, and so $g'' \circ u$ is a projective morphism (5.5.5, ii); by commutativity of (8.8.4.1), $v \circ g$ is also II | 181
projective, and, since v is affine, thus separated, we finally conclude that g is projective (5.5.5, v). ☐

(8.8.5). Consider again the situation in (8.8.1). We will see that the morphism $g : \mathbf{V}(\mathscr{L}) \to C$ can be also be defined in a way that works for any invertible (but not necessarily ample) $\mathscr{O}_X$-module $\mathscr{L}$. For this, consider the f-morphism

$$\tau^\flat : \mathscr{S} \longrightarrow \bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes n} \tag{8.8.5.1}$$

corresponding to the morphism τ from (8.8.1.2). This induces (1.5.6) a morphism $g' : V \to C$ such that, if $\pi : V \to X$ and $\psi : C \to Y$ are the structure morphisms, the diagrams

(8.8.5.2)
$$\begin{array}{ccc} X & \xleftarrow{\pi} & V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g'} \\ Y & \xleftarrow{\psi} & C \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{j} & V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g'} \\ Y & \xrightarrow{\varepsilon} & C \end{array}$$

commute ((8.5.1.2) and (8.5.1.3)). We will show that (if we assume that $\mathscr{L}$ is f-ample) *the morphisms g and g' are identical*.

The question being local on Y, we can assume that $Y = \operatorname{Spec}(A)$ is affine, and (by (8.8.1.3)) identify X with an open subset of $P = \operatorname{Proj}(S)$, where $S = A \oplus \bigoplus_{n\geqslant 0} \Gamma(X, \mathscr{L}^{\otimes n})$; we then deduce, by (8.8.1.4), that $\Gamma(X, \mathscr{O}_P(n)) = \Gamma(X, \mathscr{L}^{\otimes n})$ for all $n \in \mathbf{Z}$. Taking into account the definition of $h = \operatorname{Spec}(\alpha)$, where α is the canonical p-morphism $\widetilde{S} \to \mathscr{S}_P^{\geqslant}$ (8.6.1.2), we have to show that the restriction to X of $\alpha^\sharp : p^*(\widetilde{S}) \to \mathscr{S}_P^{\geqslant}$ is identical to τ. Taking (**0**, 4.4.3) into account, it suffices to show that, if we compose the canonical homomorphism $\alpha_n : S_n \to \Gamma(P, \mathscr{O}_P(n))$ with the restriction homomorphism $\Gamma(P, \mathscr{O}_P(n)) \to \Gamma(X, \mathscr{O}_P(n)) = \Gamma(X, \mathscr{L}^{\otimes n})$, then we obtain the identity, for all $n > 0$; but this follows immediately from the definition of the algebra S and of α_n (2.6.2).

Proposition (8.8.6). — *Assume (with the notation of* (8.8.5)*) that, if we write $f = (f_0, \lambda)$, then the homomorphism $\lambda : \mathscr{O}_Y \to j_*(\mathscr{O}_X)$ is bijective; then:*

- (i) *if we write $g = (g_0, \mu)$, then $\mu : \mathscr{O}_C \to g_*(\mathscr{O}_V)$ is an isomorphism; and*
- (ii) *if X is integral (resp. locally integral and normal), then C is integral (resp. normal).*

Proof. Indeed, the f-morphism $\tau^\flat$ is then an *isomorphism*

$$\tau^\flat : \mathscr{S} = \psi_*(\mathscr{O}_C) \longrightarrow f_*(\pi_*(\mathscr{O}_V)) = \psi_*(g_*(\mathscr{O}_V))$$

and the Y-morphism g can be considered as that for which the homomorphism $\mathscr{A}(g)$ (1.1.2) is equal to $\tau^\flat$. To see that μ is an isomorphism of $\mathscr{O}_C$-modules, it suffices (1.4.2) to see that $\mathscr{A}(\mu) : \psi_*(\mathscr{O}_C) \to \psi_*(g_*(\mathscr{O}_V))$ is an isomorphism. But, by Definition (1.1.2), we have that $\mathscr{A}(\mu) = \mathscr{A}(g)$, whence the conclusion of (i).

To prove (ii), we can restrict to the case where Y is affine, and so $\mathscr{S} = \widetilde{S}$, with $S = \bigoplus_{n\geqslant 0} \Gamma(X, \mathscr{L}^{\otimes n})$; **II** | 182
the hypothesis that X is integral implies that the ring S is integral (**I**, 7.4.4), and thus so too is C (**I**, 5.1.4). To show that C is normal, we will use the following lemma:

Lemma (8.8.6.1). — *Let Z be a normal integral prescheme. Then the ring $\Gamma(Z, \mathscr{O}_Z)$ is integral and integrally closed.*

Proof. It follows from (**I**, 8.2.1.1) that $\Gamma(Z, \mathscr{O}_Z)$ is the intersection, in the field of rational functions $R(Z)$, of the integrally closed rings $\mathscr{O}_z$ over all $z \in Z$. □

With this in mind, we first show that V is *locally integral* and *normal*; for this, we can restrict to the case where $X = \operatorname{Spec}(A)$ is affine, with ring A integral and integrally closed (6.3.8), and where $\mathscr{L} = \mathscr{O}_X$. Since then $V = \operatorname{Spec}(A[T])$, and $A[T]$ is integral and integrally closed [Jaf60, p. 99], this proves our claim. For every affine open subset U of C, $g^{-1}(U)$ is quasi-compact, since the morphism g is quasi-compact; since V is locally integral, the connected components of $g^{-1}(U)$ are open integral preschemes in $g^{-1}(U)$, and thus finite in number, and, since V is normal, these preschemes are also normal (6.3.8). Then $\Gamma(U, \mathscr{O}_C)$, which is equal to $\Gamma(g^{-1}(U), \mathscr{O}_V)$, by (i), is the direct sum (?) of finitely-many integral and integrally closed rings (8.8.6.1), which proves that C is normal (6.3.4). □

8.9. Grauert's ampleness criterion: statement. We intend to show that the properties proven in (8.8.2) *characterise* f-ample $\mathscr{O}_X$-modules, and, more precisely, to prove the following criterion:

Theorem (8.9.1). — (Grauert's criterion). *Let Y be a prescheme, $p : X \to Y$ a separated and quasi-compact morphism, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module. For $\mathscr{L}$ to be ample relative to p, it is necessary and sufficient for there to exist a Y-prescheme C, a Y-section $\varepsilon : Y \to C$ of C, and a Y-morphism $q : \mathbf{V}(\mathscr{L}) \to C$, satisfying the following properties:*

(i) *the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\ j\ } & \mathbf{V}(\mathscr{L}) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ Y & \xrightarrow{\ \varepsilon\ } & C \end{array} \tag{8.9.1.1}$$

commutes, where j is the null section of the vector bundle $\mathbf{V}(\mathscr{L})$; and

(ii) *the restriction of q to $\mathbf{V}(\mathscr{L}) - j(X)$ is a quasi-compact open immersion*

$$\mathbf{V}(\mathscr{L}) - j(X) \longrightarrow X$$

whose image does not intersect $\varepsilon(Y)$.

Note that, if C is *separated* over Y, we can, in condition (ii), remove the hypothesis that the open immersion is quasi-compact; to see that this property (of quasi-compactness) is in fact a consequence of the other conditions, we can restrict to the case where Y is affine, and the claim then follows from (**I**, 5.5.1)i and (**I**, 5.5.10). We can also remove the same hypothesis if we assume that X is Noetherian, **II** | 183
since then V is also Noetherian, and the claim follows from (**I**, 6.3.5).

Corollary (8.9.2). — *If the morphism $p : X \to Y$ is proper, then we can, in the statement of Theorem (8.9.1), assume that q is proper, and replace "open immersion" by "isomorphism".*

In a more suggestive manner, we can say (whenever $p : X \to Y$ is proper) that *$\mathscr{L}$ is ample relative to p if and only if we can* "contract" *the null section of the vector bundle $\mathbf{V}(\mathscr{L})$ to the base prescheme Y.* An important particular case is that where Y is the spectrum of a field, and where the operation of "contraction" consists of contract the null section $\mathbf{V}(\mathscr{L})$ *to a single point*.

(8.9.3). The necessity of the conditions in Theorem (8.9.1) and Corollary (8.9.2) follow immediately from (8.8.2) and (8.8.4).

To show that the conditions of (8.9.1) suffices, consider a slightly more general situation. For this, let (with the notation of (8.8.2))

$$\mathscr{S}' = \bigoplus_{n \geqslant 0} \mathscr{L}^{\otimes n}$$

and

$$V = \mathbf{V}(\mathscr{L}) = \operatorname{Spec}(\mathscr{S}').$$

The closed subprescheme $j(X)$, null section of $\mathbf{V}(\mathscr{L})$, is defined by the quasi-coherent sheaf of ideals $\mathscr{J} = (\mathscr{S}'_+)^\sim$ of $\mathscr{O}_V$ (1.4.10). This $\mathscr{O}_V$-module is *invertible*, since this property is local on X, and this reduces to remarking that the ideal $TA[T]$ in a ring of polynomials $A[T]$ is a free cyclic $A[T]$-module. Furthermore, it is immediate (again, because the question is local on X) that

$$\mathscr{L} = j^*(\mathscr{J})$$

and

$$j_*(\mathscr{L}) = \mathscr{J}/\mathscr{J}^2.$$

Now, if

$$\pi : \mathbf{V}(\mathscr{L}) \longrightarrow X$$

is the structure morphism, then $\pi_*(\mathscr{J}) = \mathscr{S}'_+$ and $\pi_*(\mathscr{J}/\mathscr{J}^2) = \mathscr{L}$; there are thus canonical homomorphisms $\mathscr{L} \to \pi_*(\mathscr{J}) \to \mathscr{L}$, the first being the canonical injection $\mathscr{L} \to \mathscr{S}'_+$, and the second the canonical projection from $\mathscr{S}'_+$ to $\mathscr{S}'_1 = \mathscr{L}$, and their composition being the identity. We can also canonically embed $\pi_*(\mathscr{J}) = \mathscr{S}'_+ = \bigoplus_{n \geqslant 1} \mathscr{L}^{\otimes n}$ into the *product* $\prod_{n \geqslant 1} \mathscr{L}^{\otimes n} = \varprojlim_n \pi_*(\mathscr{J}/\mathscr{J}^{n+1})$ (since $\pi_*(\mathscr{J}/\mathscr{J}^{n+1}) = \mathscr{L} \oplus \mathscr{L}^{\otimes 2} \oplus \ldots \oplus \mathscr{L}^{\otimes n}$), and we thus have canonical homomorphisms

$$\mathscr{L} \longrightarrow \varprojlim \pi_*(\mathscr{J}/\mathscr{J}^{n+1}) \longrightarrow \mathscr{L} \tag{8.9.3.1}$$

whose composition is the identity.

With this in mind, the generalisation of (8.9.1) that we are going to prove is the following:

Proposition (8.9.4). — *Let Y be a prescheme, V a Y-prescheme, and X a closed subprescheme of V defined by an ideal $\mathscr{J}$ of $\mathscr{O}_V$, which is an* invertible *$\mathscr{O}_V$-module; if $j : X \to V$ is the canonical injection, then let* II | 184
$\mathscr{L} = j^*(\mathscr{J}) = \mathscr{J} \otimes_{\mathscr{O}_V} \mathscr{O}_X$, so that $j_*(\mathscr{L}) = \mathscr{J} / \mathscr{J}^2$. Assume that the structure morphism $p : X \to Y$ is separated and quasi-compact, and that the following conditions are satisfied:

(i) *there exists a Y-morphism $\pi : V \to X$ of finite type such that $\pi \circ j = 1_X$, and so $\pi_*(\mathscr{J} / \mathscr{J}^2) = \mathscr{L}$;*
(ii) *there exists a homomorphism of $\mathscr{O}_X$-modules $\varphi : \mathscr{L} \to \varprojlim \pi_*(\mathscr{J} / \mathscr{J}^{n+1})$ such that the composition*
$$\mathscr{L} \xrightarrow{\varphi} \varprojlim \pi_*(\mathscr{J} / \mathscr{J}^{n+1}) \xrightarrow{\alpha} \pi_*(\mathscr{J} / \mathscr{J}^2) = \mathscr{L}$$
(where α is the canonical homomorphism) is the identity;
(iii) *there exists a Y-prescheme C, a Y-section ε of C, and a Y-morphism $q : V \to C$ such that the diagram*

(8.9.4.1)
$$\begin{array}{ccc} X & \xrightarrow{j} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Y & \xrightarrow[\varepsilon]{} & C \end{array}$$

commutes; and
(iv) *the restriction of q to $W = V - j(X)$ is a quasi-compact open immersion into C, whose image does not intersect $\varepsilon(Y)$.*

Then $\mathscr{L}$ is ample relative to p.

8.10. Grauert's ampleness criterion: proof.

Lemma (8.10.1). — *Let $\pi : V \to X$ be a morphism, $j : X \to V$ an X-section of V that is also a closed immersion, and $\mathscr{J}$ a quasi-coherent ideal of $\mathscr{O}_V$ that defines the closed subprescheme of V associated to j. Then the following all hold true.*

(i) *For all $n \geqslant 0$, $\pi_*(\mathscr{O}_V / \mathscr{J}^{n+1})$ and $\pi_*(\mathscr{J} / \mathscr{J}^{n+1})$ are quasi-coherent $\mathscr{O}_X$-modules, and $\pi_*(\mathscr{O}_V / \mathscr{J}) = \mathscr{O}_X$ and $\pi_*(\mathscr{J} / \mathscr{J}^2) = j^*(\mathscr{J})$.*
(ii) *If $X = \{\xi\} = \operatorname{Spec}(k)$, where k is a field, then $\varprojlim \pi_*(\mathscr{O}_V / \mathscr{J}^{n+1})$ is isomorphic to the separated completion of the local ring $\mathscr{O}_{j(\xi)}$ for the $\mathfrak{m}_{j(\xi)}$-preadic topology.*
(iii) *Assume that $\mathscr{J}$ is an invertible $\mathscr{O}_V$-module (which implies that*
$$\mathscr{L} = j^*(\mathscr{J}) = \pi_*(\mathscr{J} / \mathscr{J}^2)$$
is an invertible $\mathscr{O}_X$-module), and that there exists a homomorphism $\varphi : \mathscr{L} \to \varprojlim \pi_*(\mathscr{J} / \mathscr{J}^{n+1})$ such that the composition $\mathscr{L} \xrightarrow{\varphi} \varprojlim \pi_*(\mathscr{J} / \mathscr{J}^{n+1}) \xrightarrow{\alpha} \pi_*(\mathscr{J} / \mathscr{J}^2)$ (where α is the canonical homomorphism) is the identity. If we write $\mathscr{S} = \bigoplus_{n \geqslant 0} \mathscr{L}^{\otimes n}$, then φ canonically induces an isomorphism of $\mathscr{O}_X$-algebras from the completion $\widehat{\mathscr{S}}$ of $\mathscr{S}$ relative to its canonical filtration (the completion being isomorphic to the product $\prod_{n \geqslant 0} \mathscr{L}^{\otimes n}$) to $\varprojlim \pi_*(\mathscr{O}_V / \mathscr{J}^{n+1})$.

Proof. Note first of all that the support of the $\mathscr{O}_V$-module $\mathscr{O}_V / \mathscr{J}^{n+1}$ is $j(X)$, and the support of $\mathscr{J} / \mathscr{J}^{n+1}$ is contained in $j(X)$. In the case of (ii), $j(X)$ is a closed point $j(\xi)$ of V, and, by definition, II | 185
$\pi_*(\mathscr{O}_V / \mathscr{J}^{n+1})$ is the fibre of $\mathscr{O}_V / \mathscr{J}^{n+1}$ at the point $j(\xi)$, or, equivalently, setting $C = \mathscr{O}_{j(\xi)}$, and denoting by $\mathfrak{m}$ the maximal ideal of C, the C-module $C/\mathfrak{m}^{n+1}$; claim (ii) is then evident.

To prove (i), note that the question is local on X; we can thus restrict to the case where X is affine. Let U be an affine open subset of V; then $j(X) \cap U$ is an affine open subset of $j(X)$, so $U_0 = \pi(j(X) \cap U)$, which is isomorphic to it, is an affine open subset of X; for every affine open subset $W_0 \subset U_0$ in X, $W = \pi^{-1}(W_0) \cap U$ is an affine open subset of V, since X is a scheme (**I**, 5.5.10); in particular, $U' = U \cap \pi^{-1}(U_0)$ is an affine open subset of V, and clearly $\pi(U') = U_0$ and $j(U_0) = j(X) \cap U$. Then, by definition, $\Gamma(W_0, \pi_*(\mathscr{O}_V / \mathscr{J}^{n+1})) = \Gamma(\pi^{-1}(W_0), \mathscr{O}_V / \mathscr{J}^{n+1})$; but since every point of $\pi^{-1}(W_0)$ not belonging to $j(W_0)$ has an open neighbourhood in $\pi^{-1}(W_0)$ not intersecting $j(X)$, and in which $\mathscr{O}_V / \mathscr{J}^{n+1}$ is thus zero, it is clear that the sections of $\mathscr{O}_V / \mathscr{J}^{n+1}$ over $\pi^{-1}(W_0)$ and over W are in bijective correspondence. In other words, if π' is the restriction of π to U', then the $(\mathscr{O}_X|U_0)$-modules $\pi_*(\mathscr{O}_V / \mathscr{J}^{n+1})|U_0$ and $\pi'_*((\mathscr{O}_V / \mathscr{J}^{n+1})|U')$ are identical. Since U' and U_0 are affine, and since the U_0 cover X, we thus conclude (**I**, 1.6.3) that $\pi_*(\mathscr{O}_V / \mathscr{J}^{n+1})$ is quasi-coherent, and the proof is identical for $\pi_*(\mathscr{J} / \mathscr{J}^{n+1})$.

Finally, to prove (iii), note that $\mathscr{S}$ is exactly $\mathbf{S}_{\mathscr{O}_X}(\mathscr{L})$; so φ canonically induces a homomorphism of $\mathscr{O}_X$-algebras $\psi : \mathscr{S} \to \varprojlim \pi_*(\mathscr{O}_V / \mathscr{J}^{n+1})$ (1.7.4); furthermore, this homomorphism sends $\mathscr{L}^{\otimes n}$ to $\varprojlim_m \pi_*(\mathscr{J}^n / \mathscr{J}^{n+1})$, and is thus continuous for the topologies considered, and indeed then extends to a homomorphism $\widehat{\psi} : \widehat{\mathscr{S}} \to \varprojlim \pi_*(\mathscr{O}_V / \mathscr{J}^{n+1})$. To see that this is indeed an isomorphism, we can, as in the proof of (i), restrict to the case where $X = \operatorname{Spec}(A)$ and $V = \operatorname{Spec}(B)$ are affine, with $\mathscr{J} = \widetilde{\mathfrak{J}}$, where $\mathfrak{J}$ is an ideal of B; there is an injection $A \to B$ corresponding to π that identifies A with a subring of B that is *complementary* to B, and $\mathscr{L}$ (resp. $\pi_*(\mathscr{O}_V / \mathscr{J}^{n+1})$) is the quasi-coherent $\mathscr{O}_X$-module associated to the A-module $L = \mathfrak{J}/\mathfrak{J}^2$ (resp. $B/\mathfrak{J}^{n+1}$). Since $\mathscr{J}$ is an *invertible* $\mathscr{O}_V$-module, we can further assume that $\mathfrak{J} = Bt$, where t is not a zero divisor in B. From the fact that $B = A \oplus Bt$, we deduce that, for all $n > 0$,

$$B = A \oplus At \oplus At^2 \oplus \ldots \oplus At^n \oplus Bt^{n+1}$$

and so there exists a canonical A-isomorphism from the ring of formal series $A[[T]]$ to $C = \varprojlim B/\mathfrak{J}^{n+1}$ that sends T to t. We also have that $L = A\bar{t}$, where $\bar{t}$ is the class of t modulo Bt^2, and the homomorphism φ sends, by hypothesis, $\bar{t}$ to an element $t' \in C$ that is congruent to t modulo Ct^2. We thus deduce, by induction on n, that

$$A \oplus At' \oplus \ldots \oplus At'^n \oplus Ct^{n+1} = A \oplus At \oplus \ldots \oplus At^n \oplus Ct^{n+1}$$

which proves that the homomorphism $\widehat{\psi}$ does indeed correspond to an isomorphism from $\prod_{n \geqslant 0} L^{\otimes n}$ to C. □

Lemma (8.10.2). — *Under the hypotheses of Lemma (8.10.1), let $g : X' \to X$ be a morphism, write $V' = V \times_X X'$, and let $\pi' : V' \to X'$ and $g : V' \to V$ be the canonical projections, so that we have the commutative diagram* II | 186

$$\begin{array}{ccc} V & \xleftarrow{g'} & V' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ X & \xleftarrow[g]{} & X' \end{array}$$

Then $j' = j \times 1_{X'}$ is an X'-section of V' that is also a closed immersion, and $\mathscr{J}' = g'^*(\mathscr{J})\mathscr{O}_{V'}$ is the quasi-coherent ideal of $\mathscr{O}_{V'}$ that defines the closed subprescheme of V' associated to j'. Furthermore, $\pi'_*(\mathscr{O}_{V'} / \mathscr{J}'n+1) = g^*(\pi_*(\mathscr{O}_V / \mathscr{J}^{n+1}))$. Finally, $\mathscr{J}'$ is an $\mathscr{O}_{V'}$-module that is canonically isomorphic to $g'^*(\mathscr{J})$, and is, in particular, invertible if $\mathscr{J}$ is an invertible $\mathscr{O}_V$-module.

Proof. The fact that j' is a closed immersion follows from (**I**, 4.3.1), and it is an X'-section of V' by functoriality of extension of the base prescheme. Furthermore, if Z (resp. Z') is the closed subprescheme of V (resp. V') associated to j (resp. j'), then $Z' = g'^{-1}(Z)$ (**I**, 4.3.1), and the second claim then follows from (**I**, 4.4.5). To prove the other claims, we see, as in (8.10.1), that we can restrict to the case where X, V, and X' (and thus also V') are affine; we keep the notation from the proof of (8.10.1), and let $X' = \operatorname{Spec}(A')$. Then $V' = \operatorname{Spec}(B')$, where $B' = B \otimes_A A'$, and $\mathscr{J}' = \widetilde{\mathfrak{J}''}$, where $\mathfrak{J}' = \operatorname{Im}(\mathfrak{J} \otimes_A A')$. Then $B'/\mathfrak{J}'^{n+1} = (B/\mathfrak{J}^{n+1}) \otimes_A A'$; furthermore, since $\mathfrak{J}$ is a direct factor (as an A-module) of B, $\mathfrak{J} \otimes_A A'$ is a direct factor (as an A'-module) of B', and is thus canonically identified with $\mathfrak{J}'$. □

Corollary (8.10.3). — *Assume that the hypotheses of Lemma (8.10.1) are satisfied, and assume further that π is of finite type, and that $\mathscr{J}$ is an invertible $\mathscr{O}_V$-module. Then, for all $x \in X$, the local ring at the point $j(x)$ of the fibre $\pi^{-1}(x)$ is a regular (thus integral) ring of dimension 1, whose completion is isomorphic to the formal series ring $k(x)[[T]]$ (where T is an indeterminate); furthermore, there exists exactly one irreducible component of $\pi^{-1}(x)$ that contains $j(x)$.*

Proof. Since $\pi^{-1}(x) = V \times_X \operatorname{Spec}(k(x))$, we are led, by (8.10.2), to the case where X is the spectrum of a field K. Since π is of finite type (**I**, 6.4.3, iv), $\mathscr{O}_{j(x)}$ is a Noetherian local ring, and thus separated for the $\mathfrak{m}_{j(x)}$-preadic topology (**0**, 7.3.5); it follows from (8.10.1, ii and iii) that the completion of this ring is isomorphic to $K[[T]]$, and so $\mathscr{O}_{j(x)}$ is regular and of dimension 1 ([CC, p. 17-01, th. 1]); finally, since $\mathscr{O}_{j(x)}$ is integral, $j(x)$ belongs to exactly one of the (finitely many) irreducible components of V (**I**, 5.1.4). □

Corollary (8.10.4). — *Suppose that the hypotheses of Lemma (8.10.1) are satisfied, and assume further that $\mathscr{J}$ is an invertible $\mathscr{O}_V$-module. Let $W = V - j(X)$; for every quasi-coherent ideal $\mathscr{K}$ of $\mathscr{O}_X$, let $\mathscr{K}_V = \pi^*(\mathscr{K})\mathscr{O}_V$ and $\mathscr{K}_W = \mathscr{K}_V|W$. Then $\mathscr{K}_V$ is the largest quasi-coherent ideal of $\mathscr{O}_V$ whose restriction to W is $\mathscr{K}_W$.*

Proof. Indeed, we see as in (8.10.1) that the question is local on X and V; we can thus reuse the notation from the proof of (8.10.1), with $\mathfrak{J} = Bt$, where t is not a zero divisor in B. Furthermore, we have $W = \operatorname{Spec}(B_t)$ and $\mathscr{K} = \widetilde{\mathfrak{K}}$, where $\mathfrak{K}$ is an ideal of A; whence $\pi^*(\mathscr{K})\mathscr{O}_V = (\mathfrak{K}.B)^\sim$ (**I**, 1.6.9), $\mathscr{K}_W = (\mathfrak{K}.B_t)^\sim$, and the largest ideal of B whose canonical image in B_t is $\mathfrak{K}.B_t$ is the inverse image of **II** | 187 $\mathfrak{K}.B_t$, that is, the set of $s \in B$ such that, for some integer $n > 0$, we have $t^n s \in \mathfrak{K}.B$. We have to show that this last relation implies that $s \in \mathfrak{K}.B$, or again that the canonical image of t is not a zero divisor in $B/\mathfrak{K}B = (A/\mathfrak{K}) \otimes_A B$, which follows from (8.10.2) applied to $X' = \operatorname{Spec}(A/\mathfrak{K})$. □

Corollary (8.10.5). — *Suppose that the hypotheses of (8.10.3) are satisfied; let $W = V - j(X)$, x be a point of X, $\mathscr{K}$ a quasi-coherent ideal of $\mathscr{O}_X$, and z the generic point of the irreducible component of $\pi^{-1}(x)$ that contains $j(x)$ (8.10.3).*

(i) *Let g be a section of $\mathscr{O}_V$ over V such that $g|W$ is a section of $\mathscr{K}_W$ over W (using the notation from (8.10.4)). Then g is a section of $\mathscr{K}_V$; if further $g(z) \neq 0$, and if, for every integer $m > 0$, we denote by g_m^x the germ at the point x of the canonical image g_m of g in $\Gamma(X, \pi_*(\mathscr{O}_V/\mathscr{J}^{m+1}))$, then there exists an integer $m > 0$ such that the image of g_m^x in*

$$(\pi_*(\mathscr{O}_V/\mathscr{J}^{m+1}))_x \otimes_{\mathscr{O}_x} k(x)$$

is $\neq 0$.

(ii) *Suppose further that the conditions of (8.10.1, iii) are fulfilled. Then, if there exists a section g of $\mathscr{K}_V$ over V such that $g(z) \neq 0$, then there exists an integer $n \geqslant 0$ and a section f of $\mathscr{K}.\mathscr{L}^{\otimes n} = \mathscr{K} \otimes \mathscr{L}^{\otimes n} \subset \mathscr{L}^{\otimes n}$ such that $f(x) \neq 0$. If g is a section of $\mathscr{J}$, we can take $n > 0$.*

Proof. (i) Since the ideal of $\mathscr{O}_W$ generated by $g|W$ is contained in $\mathscr{K}_W$ by hypothesis, the ideal of $\mathscr{O}_V$ generated by g is contained in $\mathscr{K}_V$ by (8.10.4), or, in other words, g is a section of $\mathscr{K}_V$. To prove the second claim of (i), we can again assume that X and V are affine, and reuse the notation from (8.10.1); the fibre $\pi^{-1}(x)$ is then affine of ring $B' = B \otimes_A k(x)$, and there exists in B' an element t' which is not a zero divisor and is such that $B' = k(x) \oplus B't'$. Since $j(x)$ is a specialisation of z and since $g(z) \neq 0$, we necessarily have that $g_{(j)x} \neq 0$. But $\mathscr{O}_{j(x)}$ is a separated local ring (8.10.3), and thus embeds into its completion, and the image of g in this completion is thus not null. But this completion is isomorphic to $\varprojlim_n (B'/B't'^{n+1})$ (8.10.3); if $g' = g \otimes 1 \in B'$, there then exists an integer m such that $g' \notin B't'^{m+1}$, or, again, the image g'_m of g' in $B'/B't'^{m+1}$ is not null. But since g'_m is exactly the image of g_m^x, our claim is proved.

(ii) By (8.10.1, iii), $\pi_*(\mathscr{O}_V/\mathscr{J}^{m+1})$ is isomorphic to the direct sum of the $\mathscr{L}^{\otimes k}$ for $0 \leqslant k \leqslant m$; we denote by f_k the section of $\mathscr{L}^{\otimes k}$ over X that is the component of the element of $\bigoplus_{k=0}^m \Gamma(X, \mathscr{L}^{\otimes k})$ which corresponds to g_m by this isomorphism. Choosing m as in (i), there is thus an index k such that $f_k(x) \neq 0$, by (i). To see that f_k is a section of $\mathscr{K}\mathscr{L}^{\otimes k}$, it suffices to consider, as above, the case where X and V are affine, and this follows immediately from the fact that $g \in \mathfrak{K}.B$ (with the notation from (8.10.4)). The final claim follows from the fact that the hypothesis $g \in \Gamma(V, \mathscr{J})$ implies that $f_0 = 0$.

□

(8.10.6). *Proof of (8.9.4).* The question is local on Y (4.6.4); since ε is a Y-section, we can thus replace C by an affine open neighbourhood U of a point of $\varepsilon(Y)$ such that $\varepsilon(Y) \cap U$ is closed in U. In other words, we can assume that C is affine, and that Y is a closed subprescheme of C (and thus also affine) defined by a quasi-coherent sheaf $\mathscr{I}$ of ideals of $\mathscr{O}_C$. Since p is separated and quasi-compact, **II** | 188 X is thus a quasi-compact scheme, and we are reduced to proving that $\mathscr{L}$ is *ample* (4.6.4). By criterion (4.5.2, a)), we must thus prove the following: for every quasi-coherent ideal $\mathscr{K}$ of $\mathscr{O}_X$ and every point $x \in X$ not belonging to the support of $\mathscr{O}_X/\mathscr{K}$, there exists an integer $n > 0$ and a section f of $\mathscr{K} \otimes \mathscr{L}^{\otimes n}$ over X such that $f(x) \neq 0$.

For this, set

$$\mathscr{K}_V = \pi^*(\mathscr{K})\mathscr{O}_V$$
$$\mathscr{K}_W = \mathscr{K}_V|W$$

where $W = V - j(X)$; since the restriction of q to W is a quasi-compact immersion to C, it follows from (**I**, 9.4.2) that $\mathscr{K}_W$ is the restriction to W of a quasi-coherent ideal $\mathscr{K}'_V$ of $\mathscr{O}_V$ of the form

$$\mathscr{K}'_V = q^*(\mathscr{K}_C)\mathscr{O}_V$$

where $\mathscr{K}_C$ is a quasi-coherent ideal of $\mathscr{O}_C$. Furthermore, since, by hypotheses, $q^{-1}(Y) \subset j(X)$, and since Y is defined by the ideal $\mathscr{I}$, the restriction to W of $q^*(\mathscr{I})\mathscr{O}_V$ is identical to that of $\mathscr{O}_V$, and so $\mathscr{K}_W$ is also the restriction to W of $q^*(\mathscr{I}\mathscr{K}_C)\mathscr{O}_V$, and we can thus suppose that $\mathscr{K}_C \subset \mathscr{I}$, whence

(8.10.6.1) $$\mathscr{K}'_V \subset q^*(\mathscr{I})\mathscr{O}_V \subset \mathscr{J}$$

taking into account (**I**, 4.4.6) and the commutativity of (8.9.4.1). Furthermore, we deduce from (8.10.4) that

(8.10.6.2) $$\mathscr{K}'_V \subset \mathscr{K}_V.$$

With this in mind, it follows from (8.10.3) that $j(x)$ belongs to exactly one irreducible component of $\pi^{-1}(x)$; let z be the generic point of this component, and let $z' = q(z)$. By (8.10.5), the proof will be finished (taking (8.10.6.1) and (8.10.6.2) into account) if we show the existence of a section g of $\mathscr{K}'_V$ over V such that $g(z) \neq 0$. But, by hypothesis, $\mathscr{K}$ has a restriction equal to that of $\mathscr{O}_X$ in an open neighbourhood of x; also, it follows from (8.10.3) that $z \neq j(x)$, and so $z \in W$, and thus $(\mathscr{K}_W)_z = \mathscr{O}_{V,z}$, whence, by definition, $(\mathscr{K}_C)_{z'} = \mathscr{O}_{C,z}$. Since C is affine, there is thus a section g' of $\mathscr{K}_C$ over C such that $g'(z') \neq 0$, and by taking g to be the section of $\mathscr{K}'_V$ corresponding canonically to g', we indeed have $g(z) \neq 0$, which finishes the proof.

Remark (8.10.7). — We ignore the question of whether or not condition (ii) in (8.9.4) is superfluous or not. In any case, the conclusion does not hold if we do not assume the existence of a Y-morphism $\pi : V \to X$ such that $\pi \circ j = 1_X$; we briefly point out how we can indeed construct a counterexample, whose details will not be developed until later on. We take $Y = \mathrm{Spec}(k)$, where k is a field, and $C = \mathrm{Spec}(A)$, where $A = k[T_1, T_2]$, and the Y-section ε corresponding to the augmentation homomorphism $A \to k$. We denote by C' the scheme induced by C by blowing up the closed point $a = \varepsilon(Y)$ of C; if D is the inverse image of a in C', we consider in D a closed point b, and we denote **II** | 189
by V the scheme induced by C' by blowing up b; X is the closed subprescheme of V given by the inverse image of a by the structure morphism $q : V \to C$. We now show that X is the union of two irreducible components, X_1 and X_2, where X_1 is the inverse image of b in V. It is immediate that the ideal $\mathscr{J}$ of $\mathscr{O}_V$ that defines X is again invertible, we we can show that $j^*(\mathscr{J}) = \mathscr{L}$ (where j is the canonical injection $X \to V$) is not ample, by considering the "degree" of the inverse image of $\mathscr{L}$ in X_1, which would be > 0 if $\mathscr{L}$ were ample, but we can show (by an elementary intersection calculation) that it is in fact equal to 0.

8.11. Uniqueness of contractions.

Lemma (8.11.1). — *Let U and V be preschemes, and $h = (h_0, \lambda) : U \to V$ a surjective morphism. Suppose that*

1. *$\lambda : \mathscr{O}_V \to h_*(\mathscr{O}_U) = (h_0)_*(\mathscr{O}_U)$ is an isomorphism;*
2. *the underlying space of V can be identified with the quotient of the underlying space of U by the relation $h_0(x) = h_0(y)$ (*a condition which always holds whenever the morphism h is *open* or *closed*, or, *a fortiori* when h is *proper*.)

Then, for every prescheme W, the map

(8.11.1.1) $$\mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(U, W)$$

that, to each morphism $v = (v_0, \nu)$ from V to W, associates the morphism $u = v \circ h = (u_0, \mu)$, is a bijection from $\mathrm{Hom}(V, W)$ *to the set of u such that u_0 is constant on every fibre $h_0^{-1}(x)$.*

Proof. It is clear that, if $u = v \circ h$, so that $u_0 = v_0 \circ h_0$, then u_0 is constant on every set $h_0^{-1}(x)$. Conversely, if u has this property, we will show that there exists exactly one $v \in \operatorname{Hom}(V, W)$ such that $u = v \circ h$. The existence and uniqueness of the continuous map $v_0 : V \to W$ such that $u_0 = v_0 \circ h_0$ follows from the hypotheses, since h_0 can be identified with the canonical map from U to U/R. We can also, replacing V by some isomorphic prescheme if necessary, suppose that λ is the identity; by hypothesis, μ is then a homomorphism $\mu : \mathscr{O}_W \to (u_0)_*(\mathscr{O}_U) = (v_0)_*((h_0)_*(\mathscr{O}_U))$ such that the corresponding homomorphism $\mu^\sharp : u_0^*(\mathscr{O}_W) \to \mathscr{O}_U$ is *local* on every fibre. Since $(v_0)_*((h_0)_*(\mathscr{O}_U)) = (v_0)_*(\mathscr{O}_V)$, we necessarily have that $\nu = u$, and everything then reduces to showing that the corresponding homomorphism $\nu^\sharp : v_0^*(\mathscr{O}_W) \to \mathscr{O}_V$ is local on every fibre. But every $y \in V$ is of the form $h_0(x)$ for some $x \in U$; let $z = v_0(y) = u_0(x)$. Then (**0**, 3.5.5) the homomorphism $\mu_x^\sharp$ factors as

$$\mu_x^\sharp : \mathscr{O}_z \xrightarrow{\nu_y^\sharp} \mathscr{O}_y \xrightarrow{\lambda_x^\sharp} \mathscr{O}_x.$$

By hypothesis, $\lambda_x^\sharp$ and $\mu_x^\sharp$ are local homomorphisms; thus $\lambda_x^\sharp$ sends every invertible element of $\mathscr{O}_y$ to an invertible element of $\mathscr{O}_x$; if $\nu_y^\sharp$ sent a non-invertible element of $\mathscr{O}_z$ to an invertible element of $\mathscr{O}_y$, then $\mu_x^\sharp$ would send this element of $\mathscr{O}_z$ to an invertible element of $\mathscr{O}_x$, contradicting the hypothesis, whence the lemma. □

Corollary (8.11.2). — *Let U be an integral prescheme, and V a normal prescheme; then every morphism $h : U \to V$ that is universally closed, birational, and radicial, is also an isomorphism.*

Proof. If $h = (h_0, \lambda)$, then it follows from the hypotheses that h_0 is injective and closed, and that $h_0(U)$ is dense in V, and so h_0 is a *homeomorphism* from U to V. To prove the corollary, it will suffice to show that $\lambda : \mathscr{O}_V \to (h_0)_*(\mathscr{O}_U)$ is an isomorphism: we can then apply (8.11.1), which proves that the map (8.11.1.1) is bijective (the fibres $h_0^{-1}(x)$ each consisting of a single point); thus h will be an isomorphism. The question clearly being local on V, we can suppose that $V = \operatorname{Spec}(A)$ is affine, of an integral and integrally closed ring (8.8.6.1); h then corresponds (**I**, 2.2.4) to a homomorphism $\varphi : A \to \Gamma(U, \mathscr{O}_U)$, and everything reduces to showing that φ is an isomorphism. But, if K is the field of fractions of A, then $\Gamma(U, \mathscr{O}_U)$ has, by hypothesis, K as its field of fractions, and A is a subring of $\Gamma(U, \mathscr{O}_U)$, with φ being the canonical injection (**I**, 8.2.7). Since the morphism h satisfies the hypotheses of (7.3.11), $\Gamma(U, \mathscr{O}_U)$ is a subring of the integral closure of A in K, and is thus identical to A by hypothesis. □ **II** | 190

Remark (8.11.3). — We will see in chapter III (**III**, 4.4.11) that, whenever V is a *locally Noetherian* prescheme, every morphism $h : U \to V$ that is proper and quasi-finite (in particular, every morphism satisfying the hypotheses of (8.11.2)) is necessarily *finite*. The conclusion of (8.11.2) then follows in this case from (6.1.15).

(8.11.4). We will now see that, in Grauert's criterion, we can often prove that the prescheme C and the "contraction" q are determined in an *essentially unique* manner.

Lemma (8.11.5). — *Let Y be a prescheme, $p : X \to Y$ a proper morphism, $\mathscr{L}$ a p-ample invertible $\mathscr{O}_X$-module, C a Y-prescheme, $\varepsilon : Y \to C$ a Y-section, and $q : V = \mathbf{V}(\mathscr{L}) \to C$ a Y-morphism, all such that the diagram in (8.9.1.1) commutes. Suppose further that, if $p = (p_0, \theta)$, then $\theta : \mathscr{O}_Y \to p_*(\mathscr{O}_X)$ is an isomorphism. Let $\mathscr{S}' = \bigoplus_{n \geqslant 0} p_*(\mathscr{L}^{\otimes n})$ and $C' = \operatorname{Spec}(\mathscr{S}')$, and let $q' : \mathbf{V}(\mathscr{L}) \to C'$ be the canonical Y-morphism* (8.8.5)*. Then there exists exactly one Y-morphism $u : C' \to C$ such that $q = u \circ q'$.*

Proof. The hypothesis on θ implies, in particular, that p is surjective; since, by (8.8.4), the restriction of q' to $\mathbf{V}(\mathscr{L}) - j(X)$ is an *isomorphism* to $C' - \varepsilon'(Y)$ (where ε is the vertex section of C'), it follows from (8.8.4) that q' is *proper* and *surjective*; furthermore, by (8.8.6), if we let $q' = (q'_0, \tau)$, then $\tau : \mathscr{O}_{C'} \to q'_*(\mathscr{O}_V)$ is an isomorphism. We are thus in a situation where we can apply (8.11.1), and we will have proven the lemma if we show that q is constant on every fibre $q'^{-1}(z')$, where $z' \in C'$. But this condition is trivially satisfied for $z' \notin \varepsilon'(Y)$. If $z' \in \varepsilon'(Y)$, then there exists exactly one $y \in Y$ such that $z' = \varepsilon'(y)$, and, by commutativity of (8.8.5.2) and the fact that q' sends $\mathbf{V}(\mathscr{L}) - j(X)$ to $C' - \varepsilon'(Y)$, $q'^{-1}(z') = j(p^{-1}(y))$; the commutativity of the diagram in (8.9.1.1) then proves our claim. □

Corollary (8.11.6). — *Under the hypotheses of (8.11.5), suppose further that q is proper, and that the restriction of q to $\mathbf{V}(\mathscr{L}) - j(X)$ is an isomorphism to $C - \varepsilon(Y)$. Then the morphism u is universally closed, surjective, and radicial, and its restriction to $C' - \varepsilon'(Y)$ is an isomorphism to $C - \varepsilon(Y)$.*

Proof. Since q' is an isomorphism from $\mathbf{V}(\mathscr{L}) - j(X)$ to $C' - \varepsilon'(Y)$ (8.8.4), the last claim follows immediately from the fact that $q = u \circ q'$. Furthermore, the commutativity of the diagrams in (8.8.5.2) and (8.9.1.1) shows that the restriction of u to the closed subprescheme $\varepsilon'(Y)$ of C' is an isomorphism to the closed subprescheme $\varepsilon(Y)$ of C, from which we immediately deduce that, for all $z' \in \varepsilon'(Y)$, if $z = u(z')$, then u defines an isomorphism from $k(z)$ to $k(z')$. These remarks prove that u is bijective and radicial; furthermore, if $\psi : C \to Y$ and $\psi' : C' \to Y$ are the structure morphisms, then $\psi' = \psi \circ u$, and, since ψ' is separated (1.2.4), so too is u (**I**, 5.5.1, v). We have already seen, in the proof of (8.11.5), that q' is surjective; since $q = u \circ q'$ is proper, we finally conclude, from (5.4.3) and (5.4.9), that u is universally closed. □ **II** | 191

Proposition (8.11.7). — *Let Y be a prescheme, X an* integral *prescheme, $p : X \to Y$ a proper morphism, $\mathscr{L}$ a p-ample invertible $\mathscr{O}_X$-module, C a* normal *Y-prescheme, $\varepsilon : Y \to C$ a Y-section, and $q : V = \mathbf{V}(\mathscr{L}) \to C$ a Y morphism, all such that the diagram in (8.9.1.1) commutes. Suppose further that, if $p = (p_0, \theta)$, then $\theta : \mathscr{O}_Y \to p_*(\mathscr{O}_X)$ is an isomorphism. Let $\mathscr{S}' = \bigoplus_{n \geqslant 0} p_*(\mathscr{L}^{\otimes n})$ and $C' = \operatorname{Spec}(\mathscr{S}')$, and let $q' : \mathbf{V}(\mathscr{L}) \to C'$ be the canonical Y-morphism* (8.8.5). *Then the unique Y-morphism $u : C' \to C$ such that $q = u \circ q'$ is an* isomorphism.

Proof. It follows from (8.8.6) that C' is integral; since u is a homeomorphism of the underlying subspaces $C' \to C$ (u being bijective and closed, by (8.11.6)), C is irreducible, thus integral, and, since the restriction of u to a non-empty open subset of C' is an isomorphism to an open subset of C, u is birational. Since C is assumed to be normal, it suffices to apply (8.11.2) to obtain the conclusion. □

Remark (8.11.8). — (i) The hypothesis that C is normal implies that X is also normal. Indeed, $C' = \operatorname{Spec}(\mathscr{S}')$ is then normal, being isomorphic to C, and integral, by (8.8.6); we thus conclude that $\operatorname{Proj}(\mathscr{S}')$ is *normal*. Indeed, the question is local on Y; if Y is affine, with $\mathscr{S}' = \widetilde{S}'$, then the ring $S' = \Gamma(C', \mathscr{S}')$ is integral and integrally closed (8.8.6.1), and so, for every homogeneous element $f \in S'_+$, the graded ring S'_f is integral and integrally closed [SZ60, t. I, p. 257 and 261], and thus so too is the ring $S'_{(f)}$ of its degree-zero terms, because the intersection of S'_f with the field of fractions of $S'_{(f)}$ is equal to $S'_{(f)}$; this proves our claim (6.3.4). Finally, since X is isomorphic to an open subprescheme of $\operatorname{Proj}(\mathscr{S}')$ (8.8.1), X is indeed normal. We can thus express (8.11.7) in the following form: *If X is integral and normal, and $p = (p_0, \theta) : X \to Y$ is a proper morphism such that $\theta : \mathscr{O}_Y \to p_*(\mathscr{O}_X)$ is an isomorphism, then, for every p-ample $\mathscr{O}_X$-module $\mathscr{L}$, there exists exactly one way of contracting the null section of $V = \mathbf{V}(\mathscr{L})$ to obtain a normal Y-scheme C and a proper Y-morphism $q : V \to C$.*

(ii) When p is proper, the hypothesis $p_*(\mathscr{O}_X) = \mathscr{O}_Y$ can be considered as an auxiliary hypothesis, not really restricting the generality of the result. Indeed, if it is not satisfied, then it suffices to replace Y with the Y-scheme $Y' = \operatorname{Spec}(p_*(\mathscr{O}_X))$, and to consider X as a Y'-scheme. We will return to this general method in chapter III, § 4.

8.12. Quasi-coherent sheaves on based cones.

(8.12.1). Let us use the hypotheses and notation of (8.3.1). Let $\mathscr{M}$ be a *quasi-coherent graded $\mathscr{S}$-module*; to avoid any confusion, we denote by $\widetilde{\mathscr{M}}$ the quasi-coherent $\mathscr{O}_C$-module associated to $\mathscr{M}$ (1.4.3) when $\mathscr{M}$ is considered as a *non-graded* $\mathscr{S}$-module, and by $\mathscr{P}roj_0(\mathscr{M})$ the quasi-coherent $\mathscr{O}_X$-module associated to $\mathscr{M}$, $\mathscr{M}$ being considered this time as a graded $\mathscr{S}$-module (in other words, the $\mathscr{O}_X$-module denoted by $\widetilde{\mathscr{M}}$ in (3.2.2)). In addition, we set **II** | 192

$$\mathscr{M}_X = \mathscr{P}roj_0(\mathscr{M}) = \bigoplus_{n \in \mathbf{Z}} \mathscr{P}roj_0(\mathscr{M}(n)); \tag{8.12.1.1}$$

the quasi-coherent graded $\mathscr{O}_X$-algebra $\mathscr{S}_X$ being defined by (8.6.1.1), $\mathscr{P}roj(\mathscr{M})$ is equipped with a structure of a *(quasi-coherent) graded $\mathscr{S}_X$-module*, by means of the canonical homomorphisms (3.2.6.1)

$$\mathscr{O}_X(m) \otimes_{\mathscr{O}_X} \mathscr{P}roj_0(\mathscr{M}(n)) \longrightarrow \mathscr{P}roj_0(\mathscr{S}(m) \otimes_{\mathscr{S}} \mathscr{M}(n)) \longrightarrow \mathscr{P}roj_0(\mathscr{M}(m+n)), \tag{8.12.1.2}$$

the verification of the axioms of sheaves of modules being done using the commutative diagram in (2.5.11.4).

If $Y = \mathrm{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$, and $\mathscr{M} = \widetilde{M}$, where S is a graded A-algebra and M is a graded S-module, then, for every homogeneous element $f \in S_+$, we have

(8.12.1.3) $$\Gamma(X_f, \mathscr{P}roj(\widetilde{M})) = M_f$$

by the definitions and (8.2.9.1).

Now consider the quasi-coherent graded $\widehat{\mathscr{S}}$-module

(8.12.1.4) $$\widehat{\mathscr{M}} = \mathscr{M} \otimes_{\mathscr{S}} \widehat{\mathscr{S}}$$

($\widehat{\mathscr{S}}$ being defined by (8.3.1.1)); this induces a quasi-coherent graded $\mathscr{O}_{\widehat{C}}$-module $\mathscr{P}roj_0(\widehat{\mathscr{M}})$, which we will also denote by

(8.12.1.5) $$\mathscr{M}^{\square} = \mathscr{P}roj_0(\widehat{\mathscr{M}}).$$

It is clear (3.2.4) that $\mathscr{M}^{\square}$ is an additive functor which is *exact* in $\mathscr{M}$, commuting with direct sums and with inductive limits.

Proposition (8.12.2). — *With the notation of (8.3.2), we have canonical functorial isomorphisms*

(8.12.2.1) $$i^*(\mathscr{M}^{\square}) \xrightarrow{\sim} \widetilde{\mathscr{M}}, \quad j^*(\mathscr{M}^{\square}) \xrightarrow{\sim} \mathscr{P}roj_0(\mathscr{M}).$$

Indeed, $i^*(\mathscr{M}^{\square})$ is canonically identified with $(\widehat{\mathscr{M}}/(\mathbf{z}-1)\widehat{\mathscr{M}})^{\sim}$ on $\mathrm{Spec}(\widehat{\mathscr{S}}/(\mathbf{z}-1)\widehat{\mathscr{S}})$ by (3.2.3); the first of the canonical isomorphisms (8.12.2.1) is then immediately induced (1.4.1) by the canonical isomorphism $\widehat{\mathscr{M}}/(\mathbf{z}-1)\widehat{\mathscr{M}} \xrightarrow{\sim} \mathscr{M}$. The canonical immersion $j : X \to C$ corresponds to the canonical homomorphism $\widehat{\mathscr{S}} \to \mathscr{S}$ with kernel $\mathbf{z}\widehat{\mathscr{S}}$ (8.3.2); the second homomorphism (8.12.2.1) is the particular case of the canonical homomorphism (3.5.2, ii), since here we have $\widehat{\mathscr{M}} \otimes_{\widehat{\mathscr{S}}} \mathscr{S} = \mathscr{M}$; to verify that this is an isomorphism, we can restrict to the case where $Y = \mathrm{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$, and $\mathscr{M} = \widetilde{M}$; by appealing to (2.8.8), the proof that, for all homogeneous f in S_+, the preceding homomorphism, restricted to X_f, restricts to an isomorphism, is then immediate.

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By an abuse of language, we again say, thanks to the existence of the first isomorphism (8.12.2.1), that $\mathscr{M}^{\square}$ is the *projective closure* of the $\mathscr{O}_X$-module $\widetilde{\mathscr{M}}$ (it being implicit that the data of the $\mathscr{O}_C$-module $\widetilde{\mathscr{M}}$ includes the grading of the $\mathscr{S}$-module $\mathscr{M}$).

(8.12.3). With the notation of (8.3.5), we have a canonical functorial homomorphism

(8.12.3.1) $$p^*(\mathscr{P}roj(\mathscr{M})) \longrightarrow \mathscr{M}^{\square}|\widehat{E}.$$

Indeed, this is a particular case of the homomorphism $\nu^{\sharp}$ defined more generally in (3.5.6). If $Y = \mathrm{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$, and $\mathscr{M} = \widetilde{M}$, then, by appealing to (2.8.8), the restriction of (8.12.3.1) to $p^{-1}(X_f) = \widehat{C}_f$ (for some homogeneous f in S_+) corresponds to the canonical homomorphism

(8.12.3.2) $$M_{(f)} \otimes_{S_{(f)}} S_f^{\leqslant} \longrightarrow M_f^{\leqslant}$$

taking into account (8.2.3.2) and (8.2.5.2).

(8.12.4). Let us place ourselves in the settings of (8.5.1), and assume its hypotheses and keep its notation. It follows from (1.5.6) that, for every quasi-coherent graded $\mathscr{S}$-module $\mathscr{S}$, we have, on one hand, a canonical isomorphism

(8.12.4.1) $$\Phi^*(\widetilde{\mathscr{M}}) \xrightarrow{\sim} (q^*(\mathscr{M}) \otimes_{q^*(\mathscr{S})} \mathscr{S}')^{\sim}$$

of $\mathscr{O}_{C'}$-modules; on the other hand, (3.5.6) implies the existence of a canonical $\mathrm{Proj}(\varphi)$-morphism

(8.12.4.2) $$\mathscr{P}roj_0\mathscr{M} \longrightarrow (\mathscr{P}roj_0(q^*(\mathscr{M})) \otimes_{q^*(\mathscr{S})} \mathscr{S}')|G(\varphi)$$

and also of a canonical $\widehat{\Phi}$-morphism

(8.12.4.3) $$\mathscr{P}roj_0\widehat{\mathscr{M}} \longrightarrow (\mathscr{P}roj_0(q^*(\widehat{\mathscr{M}})) \otimes_{q^*(\widehat{\mathscr{S}})} \widehat{\mathscr{S}'})|G(\widehat{\varphi}).$$

(8.12.5). Consider now the setting of (8.6.1), with the same notation; we thus take $Y' = X$, the morphism $q : X \to Y$ being the structure morphism, and φ the canonical q-morphism (8.6.1.2). We then have a canonical isomorphism

$$q^*(\mathscr{M}) \otimes_{q^*(\mathscr{S})} \mathscr{S}_X^{\geqslant} \xrightarrow{\sim} \mathscr{M}_X^{\geqslant} \tag{8.12.5.1}$$

by setting $\mathscr{M}_X^{\geqslant} = \bigoplus_{n \geqslant 0} \mathscr{P}roj_0(\mathscr{M}(n))$. We can indeed restrict to the case where $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$, and $\mathscr{M} = \widetilde{M}$, and define the isomorphism (8.12.5.1) on each of the affine open subsets X_f (where f is homogeneous in S_+), by verifying the compatibility with taking a homogeneous multiple of f. But the restriction to X_f of the left-hand side of (8.12.5.1) is $\widetilde{M}' = ((M \otimes_A S_{(f)}) \otimes_{S \otimes_A S_{(f)}} S_f^{\geqslant})^\sim$ by (8.6.2.1); since we have a canonical isomorphism from $M \otimes_A S_{(f)}$ to $M \otimes_S (S \otimes_A S_{(f)})$, we have an induced isomorphism from $\widetilde{M}'$ to $(M \otimes_S S_f^{\geqslant})^\sim$, and the latter is canonically isomorphic, by (8.2.9.1), to the restriction to X_f of the right-hand side of (8.12.5.1), and satisfies the required compatibility conditions. **II** | 194

Replacing $\mathscr{M}$ by $\widehat{\mathscr{M}}$, $\mathscr{S}$ by $\widehat{\mathscr{S}}$, and $\mathscr{S}_X$ by $(\mathscr{S}_X^{\geqslant})^\wedge$ in the previous argument, we similarly have a canonical isomorphism

$$q^*(\widehat{\mathscr{M}}) \otimes_{q^*(\widehat{\mathscr{S}})} (\mathscr{S}_X^{\geqslant})^\wedge \xrightarrow{\sim} (\mathscr{M}_X^{\geqslant})^\wedge. \tag{8.12.5.2}$$

If we recall (8.6.2) that the structure morphism $u : \operatorname{Proj}(\mathscr{S}_X^{\geqslant}) \to X$ is an isomorphism, then we deduce, first of all, from the above, that we have a canonical u-isomorphism

$$\mathscr{P}roj_0 \mathscr{M} \xrightarrow{\sim} \mathscr{P}roj_0(\mathscr{M}_X^{\geqslant}) \tag{8.12.5.3}$$

as a particular case of (8.12.4.2). We note that, with the notation from the proof of (8.6.2), this reduces to seeing that the canonical homomorphism $M_{(f)} \otimes_{S_{(f)}} (S_f^{\geqslant})^{(d)} \to (M_f^{\geqslant})^{(d)}$ is an isomorphism whenever $f \in S_d$, which is immediate.

Secondly, the isomorphism (8.12.5.2) gives us, by this time applying (8.12.4.3) to the canonical morphism $r = \operatorname{Proj}(\widehat{\alpha}) : \widehat{C}_X \to \widehat{C}$, a canonical r-morphism

$$\mathscr{M}^\square \longrightarrow (\mathscr{M}_X^{\geqslant})^\square. \tag{8.12.5.4}$$

Recall now (8.6.2) that the restrictions of r to the pointed cones $\widehat{E}_X$ and E_X are *isomorphisms* to $\widehat{E}$ and E (respectively). Furthermore:

Proposition (8.12.6). — *The restrictions to $\widehat{E}_X$ and E_X of the canonical r-morphism (8.12.5.4) are isomorphisms*

$$\mathscr{M}^\square|\widehat{E} \xrightarrow{\sim} (\mathscr{M}_X^{\geqslant})^\square|\widehat{E}_X \tag{8.12.6.1}$$

$$\mathscr{M}^\sim|\widehat{E} \xrightarrow{\sim} (\mathscr{M}_X^{\geqslant})^\sim|\widehat{E}_X. \tag{8.12.6.2}$$

Proof. We restrict to the case where Y is affine, as in the proof of (8.6.2) (whose notation we adopt); by reducing to definitions (2.8.8), we have to show that the canonical homomorphism

$$\widehat{M}_{(f)} \otimes_{\widehat{S}_{(f)}} (S_f^{\geqslant})^\wedge_{(f/1)} \longrightarrow (M \otimes_S S_f^{\geqslant})^\wedge_{(f/1)}$$

is an isomorphism; but, by (8.2.3.2) and (8.2.5.2), the left-hand side is canonically identified with $M_f^{\leqslant} \otimes_{S_f^{\leqslant}} (S_f^{\geqslant})^{\leqslant}_{f/1}$, and thus with $M_f^{\leqslant}$, by (8.2.7.2), and the right-hand side with $(M_f^{\geqslant})^{\leqslant}_{f/1}$, and thus also with $M_f^{\leqslant}$, by (8.2.9.2), whence the conclusion concerning (8.12.6.1); (8.12.6.2) then follows from (8.12.6.1) and (8.12.2.1). □

Corollary (8.12.7). — *With the identifications of (8.6.3), the restriction of $(\mathscr{M}_X^{\geqslant})^\square$ to $\widehat{E}_X$ can be identified with $(\mathscr{M}_X^{\leqslant})^\sim$, and the restriction of $(\mathscr{M}_X^{\geqslant})^\square$ to E_x with $\widetilde{\mathscr{M}}_X$.*

Proof. We can restrict to the affine case, and this follows from the identification of $(M_f^{\geqslant})^{\leqslant}_{f/1}$ with $M_f^{\leqslant}$, and of $(M_f^{\geqslant})_{f/1}$ with M_f (8.2.9.2). □

Proposition (8.12.8). — *Under the hypotheses of (8.6.4), the canonical homomorphism (8.12.3.1) is an isomorphism.*

Proof. Taking into account the fact that $\operatorname{Proj}(\mathscr{S}_X^{\geqslant}) \to X$ is an isomorphism (8.6.2), and the iso- II | 195
morphisms (8.12.5.4) and (8.12.6.1), we are led to proving the corresponding proposition for the canonical homomorphism $p_X^*(\mathscr{P}roj_0(\mathscr{M}_X^{\geqslant})) \to (\mathscr{M}_X^{\geqslant})^{\square}|E_X$, or, in other words, we can restrict to the case where $\mathscr{S}_1$ is an invertible $\mathscr{O}_Y$-module, and where $\mathscr{S}$ is generated by $\mathscr{S}_1$. With the notation of (8.12.3), we then have, for some $f \in S_1$, that $S_f^{\leqslant} = S_{(f)}[1/f]$, and the canonical homomorphism $M_{(f)} \otimes_{S_{(f)}} S_f^{\leqslant} \to M_f^{\leqslant}$ is an isomorphism, by the definition of $M_f^{\leqslant}$. □

(8.12.9). Consider now the quasi-coherent $\mathscr{S}$-modules

$$\mathscr{M}_{[n]} = \bigoplus_{m \geqslant n} \mathscr{M}_m$$

and (with the notation of (8.7.2)) the graded quasi-coherent $\mathscr{S}^{\natural}$-module

$$\mathscr{M}^{\natural} = \left(\bigoplus_{n \geqslant 0} \mathscr{M}_{[n]} \right)^{\sim}. \tag{8.12.9.1}$$

We have seen (8.7.3) that there exists a canonical C-isomorphism $h : C_X \xrightarrow{\sim} \operatorname{Proj}(\mathscr{S}^{\natural})$. Furthermore:

Proposition (8.12.10). — *There exists a canonical h-isomorphism*

$$\mathscr{P}roj_0(\mathscr{M}^{\natural}) \xrightarrow{\sim} \widetilde{\mathscr{M}}_X. \tag{8.12.10.1}$$

Proof. We argue as in (8.7.3), this time using the existence of the di-isomorphism (8.2.9.3) instead of (8.2.7.3). We leave the details to the reader. □

8.13. Projective closures of subsheaves and closed subschemes.

(8.13.1). With hypotheses and notation as in (8.12.1), consider a *not-necessarily graded* quasi-coherent sub-$\mathscr{S}$-module $\mathscr{N}$ of $\mathscr{M}$. We can then consider the quasi-coherent $\mathscr{O}_C$-module $\widetilde{\mathscr{N}}$ associated to $\mathscr{N}$, which is a sub-$\mathscr{O}_C$-module of $\widetilde{\mathscr{M}}$. We have seen elsewhere (8.12.2.1) that $\widetilde{\mathscr{M}}$ can be identified with the restriction of $\mathscr{M}^{\square}$ to C. Since the canonical injection $i : C \to \widehat{C}$ is an affine morphism (8.3.2), and *a fortiori* quasi-compact, the *canonical extension* $(\widetilde{\mathscr{N}})^{-}$, the largest sub-$\mathscr{O}_{\widehat{C}}$-module contained in $\mathscr{M}^{\square}$ and inducing $\widetilde{\mathscr{N}}$ on C, is a *quasi-coherent* $\mathscr{O}_{\widehat{C}}$-module (**I**, 9.4.2). We will give a more explicit description by using a graded $\widehat{\mathscr{S}}$-module.

(8.13.2). For this, consider, for every integer $n \geqslant 0$, the homomorphism $\bigoplus_{i \leqslant n} \mathscr{M}_i \to \mathscr{M}$ which, for every open U of Y, sends the family

$$(s_i) \in \bigoplus_{i \leqslant n} \Gamma(U, \mathscr{M}_i)$$

to the section $\sum_i s_i \in \Gamma(U, \mathscr{M})$. Denote by $\mathscr{N}_n'$ the inverse image of $\mathscr{N}$ by this homomorphism, which is a quasi-coherent sub-$\mathscr{S}$-module of $\bigoplus_{i \leqslant n} \mathscr{M}_i$. Now consider the homomorphism $\bigoplus_{i \leqslant n} \mathscr{M}_i \to \widehat{\mathscr{M}} = \mathscr{M}[\mathbf{z}]$ which sends (s_i) to the section $\sum_{i \leqslant n} s_i \mathbf{z}^{n-i} \in \Gamma(U, \widehat{\mathscr{M}_n})$, and let $\mathscr{N}_n$ be the image of $\mathscr{N}_n'$ under this homomorphism; we immediately have that $\overline{\mathscr{N}} = \bigoplus_{n \geqslant 0} \mathscr{N}_n$ is a (quasi-coherent) sub-$\widehat{\mathscr{S}}$-module of $\widehat{\mathscr{M}}$; we say that $\overline{\mathscr{N}}$ is induced from $\mathscr{N}$ by *homogenisation*, via the "homogenising variable" $\mathbf{z}$. We note that, if $\mathscr{N}$ is already a *graded* sub-$\mathscr{S}$-module of $\mathscr{M}$, then $\overline{\mathscr{N}}$ can be identified II | 196
with the direct sum of the components $\widehat{\mathscr{N}_n}$ of degree $n \geqslant 0$ in $\widehat{\mathscr{N}} = \mathscr{N}[\mathbf{z}]$.

Proposition (8.13.3). — *The $\mathscr{O}_{\widehat{C}}$-module $\mathscr{P}roj_0(\overline{\mathscr{N}})$ is the canonical extension $(\widetilde{\mathscr{N}})^{-}$ of $\widetilde{\mathscr{N}}$ to $\widehat{C}$.*

Proof. The question is local on Y and $\widehat{C}$ by the definition of the canonical extension (**I**, 9.4.1). We can thus already suppose that $Y = \operatorname{Spec}(A)$ is affine, with $\mathscr{S} = \widetilde{S}$, $\mathscr{M} = \widetilde{M}$, and $\mathscr{N} = \widetilde{N}$, where N is a non-necessarily-graded sub-S-module of M. Furthermore (8.3.2.6), $\widehat{C}$ is a union of affine opens $\widehat{C}_z = C$ and $\widehat{C}_f = \operatorname{Spec}(S_f^{\leqslant})$ (with f homogeneous in S_+). It thus suffices to show that: (1) the restriction of $\mathscr{P}roj_0(\overline{\mathscr{N}})$ to C is $\widetilde{\mathscr{N}}$; (2) the restriction of $\mathscr{P}roj_0(\overline{\mathscr{N}})$ to each $\widehat{C}_f$ is the canonical extension of the restriction of $\mathscr{N}$ to $C \cap \widehat{C}_f = \operatorname{Spec}(S_f)$ (8.3.2.6). For the first point, note that $\mathscr{P}roj_0(\overline{\mathscr{N}})|C$ can be identified with $(\overline{N}_{(\mathbf{z})})^{\sim}$ (8.3.2.4); but $\overline{N}_{(\mathbf{z})}$ is canonically identified (2.2.5) with the image of $\overline{N}$ in $\widehat{M}/(\mathbf{z}-1)\widehat{M}$, and by the canonical isomorphism of the latter with M (8.2.5), this image can be identified with N, by the definition of $\overline{N}$ given in (8.13.2).

To prove the second point, note that the injection $i : C \cap \widehat{C}_f \to \widehat{C}$ corresponds to the canonical injection $S_f^{\leqslant} \to S_f$ (8.3.2.6); we also have that $\Gamma(\widehat{C}_f, \mathscr{M}^{\square}) = M_f^{\leqslant}$, that $\Gamma(\widehat{C}_f, i_*(\widetilde{\mathscr{N}})) = N$, and, by (8.12.2.1), that $\Gamma(\widehat{C}_f, i_*(i^*(\mathscr{M}^{\square}))) = M_f$. Taking (**I**, 9.4.2) into account, we are thus led to showing that $\overline{N}_{(f)} \subset \widehat{M}_{(f)} = M_f^{\leqslant}$ is canonically identified with the inverse image of N_f under the canonical injection $M_f^{\leqslant} \to M_f$. Indeed, let $d = \deg(f) > 0$, and suppose that an element $(\sum_{k\leqslant md} x_k)/f^m$ of M_f (with $x_k \in M_k$) is of the form y/f^m with $y \in N$. By multiplying y and the x_k by one single suitable f^h, we can already assume that $\sum_{k\leqslant md} x_k = y$. But in the identification of (8.2.5.2), $(\sum_{k\leqslant md} x_k)/f^m$ corresponds to $\sum_{k\leqslant md} x_k \mathbf{z}^{md-k}/f^m$, and this is indeed an element of $\overline{N}_{(f)}$, since $\sum_{k\leqslant md} x_k \in N$; the converse is evident. □

Remark (8.13.4). — (i) The most important case of application of (8.13.3) is that where $\mathscr{M} = \mathscr{S}$, with $\widetilde{\mathscr{N}}$ then being an *arbitrary* quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_C$ (1.4.3), corresponding bijectively to a *closed subprescheme* Z of C. Then the canonical extension $\overline{\mathscr{J}}$ of $\mathscr{J}$ is the quasi-coherent sheaf of ideals of $\mathscr{O}_{\widehat{C}}$ that defines the *closure* $\overline{Z}$ of Z in $\widehat{C}$ (**I**, 9.5.10); Proposition (8.13.3) gives a canonical way of defining $\overline{Z}$ by using a graded ideal in $\widehat{\mathscr{S}} = \mathscr{S}[\mathbf{z}]$.

(ii) Suppose, to simplify things, that Y is affine, and adopt the notation from the proof of (8.13.3). For every non-zero $x \in N$, let $d(x)$ be the largest degree of the homogeneous components x_i of x in M; by definition, $\overline{N}$ is the submodule of $\widehat{M}$ consisting of 0 and elements of the form $h(x,k) = \mathbf{z}^k \sum_{i\leqslant d(x)} x_i \mathbf{z}^{d(x)-i}$ (for integral $k \geqslant 0$); it is thus generated, as a module over $\widehat{S} = S[\mathbf{z}]$, by the elements of the form

$$h(x,0) = \sum_{i\leqslant d(x)} x_i \mathbf{z}^{d(x)-i}.$$

We say that $h(x,0)$ is induced from x by *homogenisation* via the "homogenising variable" $\mathbf{z}$. **II** | 197
But since $h(x,0)$ does not depend additively on x (nor *a fortiori* S-linearly), *we will refrain from believing* (even when $M = S$) that the $h(x,0)$ form a *system of generators* of the graded $\widehat{S}$-module $\overline{N}$ when we let x run over a *system of generators* of the S-module N. This is, however, the case (considered only in elementary algebraic geometry) when N is a *free cyclic* S-module, since, if t is a basis of N, then $h(t,0)$ generates the $\widehat{S}$-module $\overline{N}$.

8.14. Supplement on sheaves associated to graded $\mathscr{S}$-modules.

(8.14.1). Let Y be a prescheme, $\mathscr{S}$ a *positively-graded* quasi-coherent $\mathscr{O}_Y$-algebra, $X = \operatorname{Proj}(\mathscr{S})$, and $q : X \to Y$ the structure morphism (which is separated, by (3.1.3)). Using the notation of (8.12.1), we have defined a functor $\mathscr{M}_X = \mathscr{P}roj(\mathscr{M})$ in $\mathscr{M}$, from the category of graded quasi-coherent $\mathscr{S}$-modules to the category of graded quasi-coherent $\mathscr{S}_X$-modules; it is further clear (3.2.4) that this is an *additive* and *exact* functor, commuting with inductive limits.

Note, furthermore, that it follows immediately from the definition (8.12.1.1) that we have

$$\mathscr{P}roj(\mathscr{M}(n)) = (\mathscr{P}roj(\mathscr{M}))(n) \quad \text{for all } n \in \mathbf{Z}. \tag{8.14.1.1}$$

(8.14.2). We will first extend the canonical homomorphisms λ and μ, defined in (3.2.6), to $\mathscr{S}_X$-modules of the form $\mathscr{P}roj(\mathscr{M})$. For this, note that, for any $m \in \mathbf{Z}$ and $n \in \mathbf{Z}$, we have, by (2.1.2.1), a canonical homomorphism of $\mathscr{O}_X$-modules

$$\lambda_{mn} : \mathscr{P}roj_0((\mathscr{H}om_{\mathscr{S}}(\mathscr{M},\mathscr{N}))(n-m)) \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(\mathscr{P}roj_0(\mathscr{M}(m)), \mathscr{P}roj_0(\mathscr{N}(n))) \tag{8.14.2.1}$$

for any graded quasi-coherent $\mathscr{S}$-modules $\mathscr{M}$ and $\mathscr{N}$. This induces a homomorphism

$$\mu_k : \mathscr{P}roj_0((\mathscr{H}om_{\mathscr{S}}(\mathscr{M},\mathscr{N}))(k)) \longrightarrow (\mathscr{H}om_{\mathscr{S}_X}(\mathscr{P}roj(\mathscr{M}), \mathscr{P}roj(\mathscr{N})))_k$$

given by sending every $u \in \Gamma(U, \mathscr{P}roj_0((\mathscr{H}om_{\mathscr{S}}(\mathscr{M},\mathscr{N}))(k)))$ to the homomorphism $\mu_k(u)$, of degree k, of graded $\mathbf{Z}$-modules $\Gamma(U, \mathscr{P}roj(\mathscr{M})) \to \Gamma(U, \mathscr{P}roj(\mathscr{N}))$ (where U is open in X) which, in each $\Gamma(U, \mathscr{P}roj_0(\mathscr{M}(m)))$, agrees with $\mu_{m,m+k}(u)$; furthermore, by returning to the definition of the μ_{mn} (2.5.12.1), we immediately see that $\mu_k(u)$ is in fact a homomorphism of degree k of graded $\Gamma(U, \mathscr{S}_X)$-modules, and, furthermore, that the μ_k define a homomorphism of *graded* $\mathscr{S}_X$*-modules*

$$\mathscr{P}roj(\mathscr{H}om_{\mathscr{S}}(\mathscr{M}, \mathscr{N})) \longrightarrow \mathscr{H}om_{\mathscr{S}_X}(\mathscr{P}roj(\mathscr{M}), \mathscr{P}roj(\mathscr{N})). \tag{8.14.2.3}$$

Similarly, taking the associativity diagram (2.5.11.4) into account, the homomorphisms (8.14.2.1) give a homomorphism of *graded* $\mathscr{S}_X$*-modules*

$$\lambda : \mathscr{P}roj(\mathscr{M}) \otimes_{\mathscr{S}_X} \mathscr{P}roj(\mathscr{N}) \longrightarrow \mathscr{P}roj(\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N}). \tag{8.14.2.4}$$

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Proposition (8.14.3). — *The homomorphism (8.14.2.4) is bijective; so too is (8.14.2.3) whenever the graded $\mathscr{S}$-module $\mathscr{M}$ admits a finite presentation* (3.1.1).

Proof. The question is clearly local on X and Y; we can thus suppose that $Y = \operatorname{Spec}(A)$ is affine, with $\mathscr{S} = \widetilde{S}$, $\mathscr{M} = \widetilde{M}$, and $\mathscr{N} = \widetilde{N}$, where S is a positively-graded A-algebra, and M and N are graded S-modules. If f is a homogeneous element of S_+, then the homomorphisms (8.14.2.1) and (8.14.2.2), restricted to the affine open $D_+(f)$, correspond to the canonical homomorphisms (2.5.11.1) and (2.5.12.1):

$$M(m)_{(f)} \otimes_{S_{(f)}} N(n)_{(f)} \longrightarrow (M \otimes_S N)(m+n)_{(f)}$$
$$(\operatorname{Hom}_S(M, N))(n-m)_{(f)} \longrightarrow \operatorname{Hom}_{S_{(f)}}(M(m)_{(f)}, N(n)_{(f)}).$$

If we refer to the definitions of these homomorphisms, we thus see (taking (8.2.9.1) into account) that the restriction of (8.14.2.4) to $D_+(f)$ corresponds to the canonical homomorphism

$$M_f \otimes_{S_f} N_f \longrightarrow (M \otimes_S N)_f$$

defined in (**0**, 1.3.4), and we know that this latter homomorphism is an isomorphism. Similarly, the restriction of (8.14.2.3) to $D_+(f)$ corresponds to the canonical homomorphism (**0**, 1.3.5)

$$(\operatorname{Hom}_S(M, N))_f \longrightarrow \operatorname{Hom}_{S_f}(M_f, N_f)$$

taking into account the fact that, since M is of finite type, the module $\operatorname{Hom}_S(M, N)$, the direct sum of the subgroups consisting of *homogeneous* homomorphisms of S-modules (2.1.2), agrees with the set of *all* homomorphisms $M \to N$ of S-modules. The hypothesis that M admits a finite presentation then implies (**0**, 1.3.5) that the canonical homomorphism in question is indeed an isomorphism. ☐

Proposition (8.14.4). — *If U is a quasi-compact open of X, then there exists an integer d such that, for every integer n that is a multiple of d, $\mathscr{O}_X(n)|U$ is invertible, with its inverse being $\mathscr{O}_X(-n)|U$.*

Proof. Since $q(U)$ is quasi-compact, it is covered by a finite number of affine opens V_i, and so every $x \in U$ is contained in some affine open of the form $D_+(f)$, where f is a homogeneous element of degree > 0 of one of the rings $\Gamma(V_i, \mathscr{S})$. Since U is quasi-compact, we can cover it by a finite number of such opens $D_+(f_j)$; let d be a common multiple of the degrees of the f_j. This d satisfies the desired property, by (2.5.17). ☐

(8.14.5). With the hypotheses and notation of (8.14.1), we defined, in (3.3.2), canonical homomorphisms of $\mathscr{O}_Y$-modules

$$\alpha_n : \mathscr{M}_n \longrightarrow q_*(\mathscr{P}roj_0(\mathscr{M}(n))) \qquad (n \in \mathbf{Z}). \tag{8.14.5.1}$$

Generalising the notation of (3.3.1), we set, for every *graded* $\mathscr{S}_X$*-module* $\mathscr{F}$,

$$\mathbf{\Gamma}_*(\mathscr{F}) = \bigoplus_{n \in \mathbf{Z}} q_*(\mathscr{F}_n). \tag{8.14.5.2}$$

In particular, $\mathbf{\Gamma}(\mathscr{S}_X) = \bigoplus_{n \in \mathbf{Z}} q_*(\mathscr{O}_X(n))$ is the graded $\mathscr{O}_Y$-algebra denoted by $\mathbf{\Gamma}_*(\mathscr{O}_X)$ in (3.3.1.2); it is clear that $\mathbf{\Gamma}(\mathscr{F})$ is a *graded* $\mathbf{\Gamma}_*(\mathscr{S}_X)$*-algebra* (**0**, 4.2.2). When we take $\mathscr{M} = \mathscr{S}$ in the homomorphisms (8.14.5.1), we obtain the homomorphism of graded $\mathscr{O}_Y$-algebras **II | 199**

$$\alpha : \mathscr{S} \longrightarrow \mathbf{\Gamma}(\mathscr{S}_X) \tag{8.14.5.3}$$

previously defined in (3.3.2), and which makes $\mathbf{\Gamma}_*(\mathscr{F})$ a *graded* $\mathscr{S}$*-module*; the homomorphisms (8.14.5.1) then define a homomorphism (of degree 0) of *graded* $\mathscr{S}$*-modules*

$$\alpha : \mathscr{M} \longrightarrow \mathbf{\Gamma}_*(\mathscr{P}roj(\mathscr{M})). \tag{8.14.5.4}$$

(8.14.6). In general, for a graded quasi-coherent $\mathscr{S}_X$-module $\mathscr{F}$, it is not certain that the graded $\mathscr{S}$-module $\mathbf{\Gamma}_*(\mathscr{F})$ will necessarily be quasi-coherent. Consider an open X' of X such that the restriction $q' : X' \to Y$ of q to X' is a *quasi-compact* morphism. Since q' is further separated, $q'_*(\mathscr{F}')$ is then a quasi-coherent $\mathscr{O}_Y$-module for every quasi-coherent $\mathscr{O}_{X'}$ module $\mathscr{F}'$ (**I**, 9.2.2, b). We set

$$\mathscr{S}_{X'} = \mathscr{S}_X|X' = \bigoplus_{n\in\mathbf{Z}} \mathscr{O}_X(n)|X' \tag{8.14.6.1}$$

and, for every graded $\mathscr{S}_{X'}$-module $\mathscr{F}'$,

$$\mathbf{\Gamma}'_*(\mathscr{F}') = \bigoplus_{n\in\mathbf{Z}} q'_*(\mathscr{F}'_n). \tag{8.14.6.2}$$

The previous remark then shows that, if $\mathscr{F}'$ is a quasi-coherent $\mathscr{S}_{X'}$-module, then $\mathbf{\Gamma}'_*(\mathscr{F}')$ is a graded *quasi-coherent* $\mathscr{S}$-module (**I**, 9.6.1).

We note also that the canonical injection $j : X' \to X$ is *quasi-compact*, because $q' = q \circ j$ is quasi-compact and q is separated (**I**, 6.6.4, v). Then $\mathscr{F} = j_*(\mathscr{F}')$ is a graded quasi-coherent $\mathscr{S}_X$-module for every graded quasi-coherent $\mathscr{S}_{X'}$-module $\mathscr{F}'$, and it follows from the previous definitions that

$$\mathbf{\Gamma}'_*(\mathscr{F}') = \mathbf{\Gamma}_*(\mathscr{F}). \tag{8.14.6.3}$$

With the same hypotheses on X', for every graded quasi-coherent $\mathscr{S}$-module $\mathscr{M}$, we set

$$\mathscr{P}roj'(\mathscr{M}) = \mathscr{P}roj(\mathscr{M})|X' \tag{8.14.6.4}$$

which is a graded quasi-coherent $\mathscr{S}_{X'}$-module. The canonical homomorphism

$$\mathscr{P}roj(\mathscr{M}) \longrightarrow j_*(\mathscr{P}roj'(\mathscr{M}))$$

(**0**, 4.4.3) thus gives a canonical homomorphism $\mathbf{\Gamma}_*(\mathscr{P}roj(\mathscr{M})) \to \mathbf{\Gamma}'_*(\mathscr{P}roj'(\mathscr{M}))$ of graded $\mathscr{S}$-modules, and, by composition with (8.14.5.4), we obtain a functorial canonical homomorphism (of degree 0) of graded quasi-coherent $\mathscr{S}$-modules

$$\alpha' : \mathscr{M} \longrightarrow \mathbf{\Gamma}'_*(\mathscr{P}roj'(\mathscr{M})). \tag{8.14.6.5}$$

(8.14.7). Keeping the hypotheses on X' from (8.14.6), let $\mathscr{F}'$ be a *graded quasi-coherent $\mathscr{S}_{X'}$-module* such that $\mathscr{P}roj'(\mathbf{\Gamma}'_*(\mathscr{F}'))$ is also a graded *quasi-coherent* $\mathscr{S}_{X'}$-module. We will define a functorial canonical homomorphism (of degree 0) of graded $\mathscr{S}_{X'}$-modules **II** | 200

$$\beta' : \mathscr{P}roj'(\mathbf{\Gamma}'_*(\mathscr{F}')) \longrightarrow \mathscr{F}'. \tag{8.14.7.1}$$

Suppose first of all that $Y = \operatorname{Spec}(A)$ is affine, and that $\mathscr{S} = \widetilde{S}$, where S is a positively-graded A-algebra; then $\mathbf{\Gamma}'_*(\mathscr{F}') = \widetilde{M}$, where $M = \bigoplus n \in \mathbf{Z}\Gamma(X', \mathscr{F}'_n)$ is a graded S-module. Let $f \in S_d$ be such that $D_+(f) \subset X'$; by definition (2.6.2), $\alpha_d(f)$ restricted to $D_+(f)$ is the section of $\mathscr{O}_X(d)$ over $D_+(f)$ corresponding to the element $f/1$ of $(S(d))_{(f)}$, and is thus invertible; thus so too is $\alpha_d(f^n)$ for every $n > 0$. From this, we immediately conclude that we have defined an S_f-homomorphism (of degree 0) of graded modules $\beta_f : M_f \to \Gamma(D_+(f), \mathscr{F}')$ by sending each element $z/f^n \in M_f$ (where $z \in M$) to the section $(z|D_+(f))(\alpha_d(f^n)|D_+(f))^{-1}$ of $\mathscr{F}'$ over $D_+(f)$. Furthermore, we have a commutative diagram corresponding to (2.6.4.1), whence the definition of β' in this case. To pass to the general case, we must consider an A-algebra A', the graded A'-algebra $S' = S \otimes_A A'$, and use the commutative diagram analogous to (2.8.13.2); we leave the details to the reader.

Proposition (8.14.8). — *If X' is an open of $X = \operatorname{Proj}(\mathscr{S})$ such that $q' : X' \to Y$ is quasi-compact, then the homomorphism β' defined in (8.14.7) is bijective.*

Proof. We can clearly restrict to the case where Y is affine, and everything then reduces to proving (with the notation of (8.14.7)) that the homomorphism $\beta_f : M_f \to \Gamma(D_+(f), \mathscr{F}')$ is an isomorphism. But replacing f by one of its powers changes neither $D_+(f)$ nor β_f; since X' is *quasi-compact* by hypothesis, we can always assume, by (8.14.4), that the sheaf $\mathscr{O}_X(d)$ is *invertible*. Since X' is a scheme (because q' is separated), the proposition is then exactly (**I**, 9.3.1). □

Corollary (8.14.9). — *Under the hypotheses of (8.14.8), every graded quasi-coherent $\mathscr{S}_{X'}$-module is isomorphic to a graded $\mathscr{S}_{X'}$-module of the form $\mathscr{P}roj'(\mathscr{M})$, where $\mathscr{M}$ is a graded quasi-coherent $\mathscr{S}$-module. Further, if $\mathscr{F}'$ is of finite type, and if we assume that Y is a quasi-compact scheme, or a prescheme whose underlying space is Noetherian, then we can assume that $\mathscr{M}$ is of finite type.*

Proof. The proof starting from (8.14.8) follows exactly the same route as the proof of (3.4.5) starting from (3.4.4), and we leave the details to the reader. □

Proposition (8.14.10). — *Under the hypotheses of (8.14.7), let $\mathscr{M}$ be a graded quasi-coherent $\mathscr{S}$-module, and $\mathscr{F}'$ a graded quasi-coherent $\mathscr{S}_{X'}$-module; the composite homomorphisms*

(8.14.10.1) $$\mathscr{P}roj'(\mathscr{M}) \xrightarrow{\mathscr{P}roj'(\alpha')} \mathscr{P}roj'(\Gamma'_*(\mathscr{P}roj'(\mathscr{M}))) \xrightarrow{\beta'} \mathscr{P}roj'(\mathscr{M})$$

(8.14.10.2) $$\Gamma'_*(\mathscr{F}') \xrightarrow{\alpha'} \Gamma'_*(\mathscr{P}roj'(\Gamma'_*(\mathscr{F}'))) \xrightarrow{\Gamma'_*(\beta')} \Gamma'_*(\mathscr{F}')$$

are the identity isomorphisms.

Proof. The question is local on Y, and the proof follows as in (2.6.5); we leave the details to the reader. □

Remark (8.14.11). — In chapter III (**III**, 2.3.1), we will see that, when Y is *locally Noetherian*, and $\mathscr{S}$ is a graded quasi-coherent $\mathscr{O}_Y$-algebra *of finite type* (in which case we can take $X' = X$), then **II | 201**
the homomorphism α (8.14.5.4) is **(TN)**-*bijective* for every graded quasi-coherent $\mathscr{S}$-module $\mathscr{M}$ satisfying condition **(TF)**.

Remark (8.14.12). — The situation described in (8.14.4) is a particular case of the following. Let X be a ringed space, and $\mathscr{S}$ a (positively- and negatively-) graded $\mathscr{O}_X$-algebra; suppose that there exists an integer $d > 0$ such that $\mathscr{S}_d$ and $\mathscr{S}_{-d}$ are *invertible*, with the canonical homomorphism

(8.14.12.1) $$\mathscr{S}_d \otimes_{\mathscr{O}_X} \mathscr{S}_{-d} \longrightarrow \mathscr{O}_X$$

being an *isomorphism* (such that $\mathscr{S}_{-d}$ is identified with $\mathscr{S}_d^{-1}$). We then say that the graded $\mathscr{O}_X$-algebra $\mathscr{S}$ is *periodic*, *of period* d. This nomenclature stems from the following property: *under the preceding hypotheses, for every graded $\mathscr{S}$-module $\mathscr{F}$, the canonical homomorphism*

(8.14.12.1) $$\mathscr{S}_d \otimes \mathscr{F}_n \longrightarrow \mathscr{F}_{n+d}$$

is an isomorphism for all $n \in \mathbf{Z}$. Indeed, the question is local on X, and we can assume that $\mathscr{S}_d$ has an *invertible* section s over X, with its inverse s' being a section of $\mathscr{S}_{-d}$. The homomorphism $\mathscr{F}_{n+d} \to \mathscr{S}_d \otimes \mathscr{F}_n$, which sends each section $z \in \Gamma(U, \mathscr{F}_{n+d})$ to the section $(s|U) \otimes (s'|U)z$ of $\mathscr{S}_d \otimes \mathscr{F}_n$ over U, is then the inverse of (8.14.12.2), whence our claim. This induces, for all $k \in \mathbf{Z}$, a canonical isomorphism

$$(\mathscr{S}_d)^{\otimes k} \otimes \mathscr{F}_n \xrightarrow{\sim} \mathscr{F}_{n+kd}.$$

Then *the data of a graded $\mathscr{S}$-module $\mathscr{F}$ is equivalent to the data of $\mathscr{S}_0$-modules $\mathscr{F}_i$ ($0 \leqslant i \leqslant d-1$) and canonical homomorphisms*

$$\mathscr{S}_i \otimes \mathscr{F}_j \longrightarrow \mathscr{F}_{i+j} \qquad \text{for } 0 \leqslant i, j \leqslant d-1$$

(setting $\mathscr{F}_{i+j} = \mathscr{S}_d \otimes_{\mathscr{S}_0} \mathscr{F}_{i+j-d}$ whenever $i + j \geqslant d$). Of course, for theses homomorphisms to give a well-defined $\mathscr{S}$-module structure on the direct sum of the $(\mathscr{S}_d)^{\otimes k} \otimes \mathscr{F}_i$ ($k \in \mathbf{Z}$, $0 \leqslant i \leqslant d-1$), they should satisfy some associativity conditions that we will not explain.

In the case where $d = 1$ (which is the one considered in (3.3)), we can thus say that the category of graded $\mathscr{S}$-modules (resp. quasi-coherent $\mathscr{S}$-modules if X is a prescheme and $\mathscr{S}$ is quasi-coherent) is *equivalent* to the category of arbitrary $\mathscr{S}_0$-modules (resp. quasi-coherent $\mathscr{S}_0$-modules); it is in this way that we can think of the results of this paragraph as generalising those of Âğ3. Furthermore, we see that, under suitable finiteness conditions, the results of this paragraph (along with (8.14.11)) reduces, in some sense, the study of graded quasi-coherent algebras on a prescheme, and graded modules "modulo **(TN)**" on such algebras, to the study of the particular case where the algebras in question are *periodic* (and where condition **(TN)** for $\mathscr{M}$ (3.4.2) thus implies that $\mathscr{M} = 0$).

Remark (8.14.13). — Under the hypotheses of (8.14.1), let d be an integer > 0; we have defined a canonical Y-isomorphism h from X to $X^{(d)} = \operatorname{Proj}(\mathscr{S}^{(d)})$ (3.1.8). For every graded quasi-coherent **II | 202**
$\mathscr{S}$-module $\mathscr{M}$ and every integer k such that $0 \leqslant k \leqslant d-1$, we also have (with the notation of (3.1.1)) a canonical h-isomorphism

(8.14.13.1) $$(\mathscr{P}roj(\mathscr{M}))^{(d,k)} \xleftarrow{\sim} \mathscr{P}roj(\mathscr{M}^{(d,k)}).$$

Suppose, first of all, that $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$, and $\mathscr{M} = \widetilde{M}$, where S is a positively-graded A-algebra, and M a graded S-module. We know, for every $f \in S_e$ ($e > 0$), that h sends $D_+(f)$ to $D_+(f^d)$, and corresponds to the canonical isomorphism $S_{(f^d)} \to S_{(f)}$ (2.2.2). The restriction of (8.14.13.1) to $D_+(f^d)$ then corresponds to the canonical di-isomorphism $M_{f^d} \to M_f$ restricted to the elements of M_{f^d} whose degree is congruent to k (modulo d). We leave to the reader the task of showing that these isomorphisms are compatible with passing from f to some homogeneous multiple fg, and then that there is an analogous compatibility with passing from S to a graded A'-algebra $S' = S \otimes_A A'$, where A' is some A-algebra. In particular, this gives us an h-isomorphism

(8.14.13.2) $$(\mathscr{S}^{(d)})_{X^{(d)}} \xrightarrow{\sim} (\mathscr{S}_X)^{(d)}$$

that respects the multiplicative structures of both the source and the target, and that, thanks to (8.14.13.1), becomes an h-di-isomorphism from a graded $(\mathscr{S}^{(d)})_{X^{(d)}}$-module to a graded $(\mathscr{S}_X)^{(d)}$-module. Similarly, we have an h-isomorphism

(8.14.13.3) $$\mathscr{P}roj_0(\mathscr{S}^{(d,k)}(n)) \xrightarrow{\sim} \mathscr{O}_X(nd+k),$$

which completes the result of (3.2.9, ii).

The isomorphism in (8.14.13.1) immediately induces an isomorphism of graded $\mathscr{S}^{(d)}$-modules

(8.14.13.4) $$\mathbf{\Gamma}_*^{(d)}(\mathscr{P}roj(\mathscr{M}^{(d,k)})) \xrightarrow{\sim} \mathbf{\Gamma}_*((\mathscr{P}roj(\mathscr{M}))^{(d,k)})$$

where $\mathbf{\Gamma}_*^{(d)}$ corresponds to the structure morphism $q^{(d)} : X^{(d)} \to Y$; it can be immediately verified that the canonical homomorphism α (8.14.5.4), and the analogous homomorphism $\alpha^{(d)}$ for $X^{(d)}$, make the following diagram commute:

(8.14.13.5) $$\begin{array}{ccc} & \mathscr{M}^{(d,k)} & \\ {}^{\alpha^{(d)}}\swarrow & & \searrow^{\alpha} \\ \mathbf{\Gamma}_*^{(d)}(\mathscr{P}roj(\mathscr{M}^{(d,k)})) & \xrightarrow{\sim} & \mathbf{\Gamma}_*((\mathscr{P}roj(\mathscr{M}))^{(d,k)}) \end{array}$$

where we proceed by supposing that Y is affine and then calculating the restrictions of the images under $\alpha^{(d)}$ and α of some single element of $M^{(d,k)}$ to the open subsets $D_+(f^d)$ and $D_+(f)$ (using the same notation as above).

Proposition (8.14.14). — *Let Y be a quasi-compact prescheme, $\mathscr{S}$ a graded quasi-coherent $\mathscr{O}_Y$-algebra of finite type, and $\mathscr{M}$ a graded quasi-coherent $\mathscr{S}$-module satisfying condition* **(TF)***; let $X = \operatorname{Proj}(\mathscr{S})$. Then $\mathscr{S}_X$ is a periodic graded $\mathscr{O}_X$-algebra (8.14.12), and there exists some period d of $\mathscr{S}_X$ such that the $(\mathscr{P}roj(\mathscr{M}))^{(d,k)}$* **II** | 203
($0 \leqslant k \leqslant d-1$) are $(\mathscr{S}_X)^{(d)}$-modules of finite type.

Proof. Indeed, (3.1.10) proves that there exists some d such that $\mathscr{S}^{(d)}$ is generated by $\mathscr{S}_d = (\mathscr{S}^{(d)})_1$, with the latter being an $\mathscr{S}_0$-module of finite type. To prove the first claim, we can thus, by (8.14.13.2), restrict to the case where $d = 1$, and the proposition then follows from (3.2.7). Furthermore, taking (8.14.13.1) into account, the second claim is a consequence of (2.1.6, iii) and (3.4.3). □

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