LOCAL STUDY OF SCHEMES AND THEIR MORPHISMS (EGA IV)

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The subjects discussed in the chapter call for the following remarks.

(a) The common property of all the subjects discussed is that they all related to *local* properties of preschemes or morphisms, i.e. considered at a point, or the points of a fibre, or on a (non-specified) neighbourhood of a point or of a fibre. These properties are generally of a *topological*, *differential*, or *dimensional* nature (i.e. bringing the ideas of *dimension* and *depth* into play), and are linked to the properties of the *local rings* at the points considered. One type of problem is the relating, for a given morphisms $f : X \to Y$ and point $x \in X$, of the properties of X at x with those of Y at $y = f(x)$ and those of the fibre $X_y = f^{-1}(y)$ at x. Another is the determining of the topological nature (for example, the constructibility, or the fact of being open or closed) of the set of points $x \in X$ at which X has a certain property, or for which the fibre $X_{f(x)}$ passing through x has a certain property at x. Similarly, we are interested in the topological nature of the set of points $y \in Y$ such that X has a certain property at all the points of the fibre X_y, or those such that this fibre itself has a certain property.

[1]The order and content of §§11–21 are given only as an indication of what the titles will be, and will possibly be modified before their publication. *[Trans.] This was indeed the case: many of §§11–21 ended up having entirely different titles.*

(b) The most important idea for the following chapters is that of *flat morphisms of finite presentation*, as well as the particular cases of *smooth morphisms* and *étale morphisms*. Their detailed study (as well as that of connected questions) really starts in §11.

(c) Sections §§1–10 can be considered as being preliminary in nature, and as developing three types of techniques, used, not only in the other sections of the chapter, but also, of course, in the follow chapters:

(c1) Sections §§1–4 are envisaged as treating the diverse aspects of the idea of *change of base*, above all in relation with the conditions of *finiteness* or *flatness*; we there initiate the technique of *descent*, with its most elementary aspects (the questions of "effectiveness" linked to this technique will be studied in Chapter V).

(c2) Sections §§5–7 are focused on what we may call *Noetherian* techniques, since the preschemes considered are always locally Noetherian, whereas, on the contrary, there is generally no finiteness condition imposed on the *morphisms*; this is essentially due to the fact that the ideas of dimension and depth are hardly manageable except in the case of Noetherian local rings. Recall that §7 constitutes a "delicate (?)" theory of Noetherian local rings, not much used in what follows in the chapter.

(c3) Sections §§8–10 describe, amongst other things, the means of *eliminating the Noetherian hypotheses* on the preschemes considered, by substituting such hypotheses for suitable ones of *finiteness* ("finite presentation") on the *morphisms* considered: the advantage of this substitution is that the latter such hypotheses (those of finiteness on the morphisms) are *stable under base change*, which is not the case for the Noetherian hypotheses on the preschemes. The technique permitting this substitution relies, in some part, on the use of the idea of the *projective limit* of preschemes, thanks to which we can reduce a question to the same question with *Noetherian* hypotheses; on the other hand, it relies on the systematic use of *constructible sets*, which have the double interest of being preserved under taking inverse images (of arbitrary morphisms) and by direct images (of morphisms of finite presentation), and having **IV-1** | 224 manageable topological properties in locally Noetherian preschemes. The same techniques often even allow to restrict to the case of more specific Noetherian rings, for example the **Z**-*algebras of finite type*, and it is here that the properties of "excellent" rings (studied in §7) intervene in a decisive manner. Independently of the question of elimination of Noetherian hypotheses, the techniques of §§8–10, elementary in nature, find constant use in nearly all applications.

§1. Relative finiteness conditions. Constructible sets of preschemes

In this section. we will resume the exposé of "finiteness conditions" for a morphism of preschemes $f : X \to Y$ given in (**I**, 6.3 and 6.6). There are essentially two notions of "finiteness" of a *global* nature on X, that of *quasi-compact* morphism (defined in (**I**, 6.6.1)) and that of a *quasi-separated* morphism; on the other hand, there are two notions of "finiteness" of a *local* nature on X, that of a morphism *locally of finite type* (defined in (**I**, 6.6.2)) and that of a morphism *locally of finite presentation*. By combining these local notions with the preceding global notions, we obtain the notion of a morphism *of finite type* (defined in (**I**, 6.3.1)) and of a morphism *of finite presentation*. For the convenience of the reader, we will give again in this section the properties stated in (**I**, 6.3 and 6.6), referring to their labels in Chapter I for their proofs.

In n^{os}1.8 and 1.9, we complete, in the context of preschemes, and making use of the previous notions of finiteness, the results on constructible sets given in ($\mathbf{0}_{\text{III}}$, §9).

1.1. Quasi-compact morphisms.

Definition (1.1.1). — We say that a morphism of preschemes $f : X \to Y$ is *quasi-compact* if the continous map f from the topological space X to the topological space Y is quasi-compact (**0**, 9.1.1), in other words, if the inverse image $f^{-1}(U)$ of every quasi-compact open subset U of Y is quasi-compact (cf. (**I**, 6.6.1)).

If $\mathfrak{B}$ is a basis for the topology of Y consisting of affine open sets, then for f to be quasi-compact, it is necessary and sufficient that for all $V \in \mathfrak{B}$, $f^{-1}(V)$ is a *finite union of affine open sets*. For example, if Y is affine and X is quasi-compact, *every* morphism $f : X \to Y$ is quasi-compact (**I**, 6.6.1).

If $f : X \to Y$ is a quasi-compact morphism, then it is clear that for every open subset V of Y, the restriction of f to $f^{-1}(V)$ is a quasi-compact morphism $f^{-1}(V) \to V$. Conversely, if (U_α) is an open cover of Y and $f : X \to Y$ is a morphism such that the restrictions $f^{-1}(U_\alpha) \to U_\alpha$ are quasi-compact, then f is quasi-compact. As a result, if **IV-1** | 225 $f : X \to Y$ is an S-morphism of S-preschemes, and if there exists an open cover (S_λ) of S such that the restrictions $g^{-1}(S_\lambda) \to h^{-1}(S_\lambda)$ of f (where g and h are the structure morphisms) are quasi-compact, then f is quasi-compact.

References

[CC] H. Cartan and C. Chevalley, Séminaire de l'École Normale Supérieure, 8th year (1955–56),Géométrie algébrique.